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MINIMAL SIZE OF A GRAPH WITH DIAMETER 2 AND GIVEN MAXIMAL DEGREE, II

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ABSTRACT. Let $F_2(n, \lfloor pn \rfloor)$ be the minimal size of a graph on n vertices with diameter 2 and maximal degree $\lfloor pn \rfloor$. The asymptotic behaviour of $F_2(n, \lfloor pn \rfloor)$ is considered for $2/5 < p < 5/12$.

1. INTRODUCTION

Denote by $H_2(n, \lfloor pn \rfloor)$ the family of undirected graphs of order n, diameter 2 and maximal degree $\lfloor pn \rfloor$ $(0 < p < 1)$ and put

$$F_2(n, \lfloor pn \rfloor) = \min_{G \in H_2(n, \lfloor pn \rfloor)} e(G)$$

where $e(G)$ is the size of G. Denote further by

$$f(p) = \lim_{n \to \infty} F_2(n, \lfloor pn \rfloor) .$$

The function $f(p)$ was introduced in [**1**] and in [**5**] the existence of the limit (conjectured in [**2**]) was proved for all values of p except of a sequence tending to 0. It is also showed in [**5**] that for a given p, $f(p)$ can be determined using linear programming. However, this procedure is too slow to enable us to solve the problem even for relatively large values of p.

In [**2**] the values of $f(p)$ for $p > 1/2$ were determined. Further, in [**4**] it was shown that if a projective plane of order t exists, then $f(p) = t+1$ for $(t+1)/(t^2 + t + 1) < p < 1/t$, hence putting $t = 2$ we get $f(p) = 3$ if $3/7 < p < 1/2$. In [**6**] $f(p)$ is determined for $5/12 < p < 3/7$. Thus for all $p > 5/12$ the values of $f(p)$ are known.

In this paper we determine $f(p)$ for smaller p. In fact, we prove here the following result conjectured in [**5**].

Theorem.

(1) $$f(p) = 8 - 11p \qquad \text{for} \quad 2/5 < p < 5/12 .$$

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We shall use here very often methods and results of [**5**]. In those cases when our assertions can be proved by a slight modification of those in [**5**], the proofs will be omitted here. On the other hand, the following lemma is used here in exactly the same way as in [**5**].

Lemma 1 ([**5**]). *Let a set U with $|U| \geq 8$ be given. Let Z be a system of triples of distinct elements of U. If every element of U is contained in some triple of Z and any two triples of Z intersect then there exist x, $y \in U$ such that every triple of Z contains at least one of x, y (we say that x, y cover Z).*

2. The Main Inequality

We prove now that if p fulfils (1) and n is sufficiently large then for every $G \in H_2(n, \lfloor pn \rfloor)$ we have

$$e(G) \geq 8n - 11\lfloor pn \rfloor - 8\left\{4\binom{192}{3} + 128\right\}\sqrt{n}\,. \tag{2}$$

Let $I = 4\binom{192}{3} + 128$. We shall proceed indirectly: suppose there exists a graph $G_0 \in H_2(n, \lfloor pn \rfloor)$ with

$$e(G_0) < 8n - 11\lfloor pn \rfloor - 8I\sqrt{n}\,. \tag{3}$$

Denote by V the set of all vertices of G_0, and by A the set of vertices of degree at least $\sqrt{n}$. According to (3) we have

$$|A| < 16\sqrt{n}\,. \tag{4}$$

Denote further by B the set of vertices of degree at most 7 adjacent to 3 vertices of A (due to (1), no vertex of degree less than 3 exists in G_0), and by C the set of such vertices adjacent to at least 4 vertices of A, and finally, let $D = V - A - B - C$. If $x \in D$ then $8 \leq \deg x \leq \sqrt{n}$.

The proof of the following lemma is straightforward (and very similar to that of Lemma 2 in [**5**]).

Lemma 2.

$$2e(G_0) \geq 8n - 2|B| - 128\sqrt{n}\,.$$

Let now E be the set of vertices of degree at least $n/12$. By (3) we have

$$|E| < 192\,. \tag{5}$$

According to (1) every vertex of B is adjacent to 3 vertices of E. Let abc be the set of vertices of B adjacent to vertices a, b, $c \in E$ (a set of the form abc will be called a triple-set). B consists of at most $\binom{192}{3}$ triple-sets. Let F be the union of triple-sets from B with cardinalities at least $4\sqrt{n}$. Then

$$|B| < |F| + 4\binom{192}{3}\sqrt{n}\,. \tag{6}$$

Denote by T the system of neighbours of triple-sets in F and by W the set of vertices occurring in T. Now we shall state some lemmas.

Lemma 3. *Any two triples of T have a common element.*

The proof is straightforward and very similar to that of Lemma 3 in [**5**].

Lemma 4. *Let J be the set of vertices x having the following property: there is some triple-set abc in F such that x is not adjacent to any of the vertices a, b, c. Then $|J| \le 4\binom{192}{3}\sqrt{n}$.*

The proof follows from the obvious fact that to any triple-set abc there exist at most $4\sqrt{n}$ vertices adjacent to neither a, b, c (see also the proof of Lemma 4 in [**5**]).

Lemma 5. *T is covered by two vertices.*

Proof. Assume the opposite. Then by Lemma 4 each of at least $n - 4\binom{192}{3}\sqrt{n}$ vertices must be adjacent to at least 3 vertices of W. However, by Lemma 1 and Lemma 2 we get $|W| \le 7$, a contradiction with (1). □

In what follows we shall use the notation $\lfloor pn \rfloor = k$.
First of all, from (3), (6) and Lemma 2 follows

$$(7) \quad 8n - 2|F| - 2I\sqrt{n} \le 16n - 22k - 16I\sqrt{n}, \quad \text{thus } |F| \ge 11k - 4n + 7I\sqrt{n}\,.$$

Let now $M = V - A - J - H$ where H is a set of cardinality less than $\sqrt{n}$ which will be specified later. From (4) and Lemma 4 we have

$$m = |M| > n - I\sqrt{n}\,. \tag{8}$$

Thus (7) can be rewritten in the form

$$|F| \ge 11k - 4m + 3I\sqrt{n}\,. \tag{9}$$

Further considerations will be restricted to the vertices of M. The following two lemmas will be of some use later.

Lemma 6. *If a, b, c_1, ..., c_r are distinct vertices of W, and there exist p vertices of M adjacent to both a and b then*

$$|abc_1 \cup \dots \cup abc_r| + rp \le r(3k - m)\,.$$

Proof. For every $i \in \{1, \dots, r\}$ the number of edges incident to vertices a, b, c_i is at least $m + p + abc_i$ ($\le 3k$), and thus the assertion follows. □

From the obvious inequality $|abc_1| \le p$ we get, by taking $r = 1$:

Corollary.

$$|abc_i| \le (3k - m)/2\,.$$

Lemma 7. *There exist less than* $(5m-11k)/2$ *vertices of* M *having at least 5 neighbours in* $V-M$.

Proof. If it is not the case then $e(G_0) \geq 3m + 2(5m-11k)/2 = 8m-11k$, a contradiction with (3). □

Now we shall prove that (3) leads to a contradiction. First of all, if all triples of T contain a fixed vertex then we have $|F| \leq k$, a contradiction with (9) for $k \geq 2n/5$ (see (1)). Assume now that T is covered by two vertices x, y (see Lemma 5). Denote by X, Y and Z, respectively, the union of all triple-sets of F adjacent only to x, only to y, and to both x and y, respectively.

We have to distinguish several cases depending on the form of the sets X, Y and Z.

Case 1. xuv, $xwt \subset X$ and yuw, $yvt \subset Y$ (or yuv, $ywt \subset Y$, or yut, $yvw \subset Y$) where all included vertices are distinct.

Let $K = \{x,y\}$, $L = \{u,v,w,t\}$. Further let S, Q, and R, respectively, be the set of vertices of M adjacent to exactly one vertex, exactly two vertices, and no vertex, respectively, of K. We can easily derive the following inequalities:

(10) $$2m + |S| \leq 6k, \qquad \text{i.e. } |S| \leq 6k - 2m;$$

(11) $$2|Q| + |S| \leq 2k, \qquad \text{i.e. } |Q| \leq m - 2k;$$

(12) $$|R| = m - |S| - |Q| \geq 2m - 4k\,.$$

Suppose $Z = xyz_1 \cup \cdots \cup xyz_i$ and let (xyx_1) $[(xyx_2)]$ be the set of all vertices of M adjacent to x, y, x_1 (to x, y, x_2, respectively). The vertices of R are adjacent to every z_j, hence by (12) we get

(13) $$(xyz_j) + 2m - 4k \leq k, \qquad \text{i.e. } (xyz_j) \leq 5k - 2m\,.$$

Now we need to consider several subcases.

Case 1a. If there exist at least 3 vertices $z_j \notin L$ then a vertex of R is adjacent to 2 vertices of L, to at least three vertices z_j but this is by (12) a contradiction to Lemma 7.

Case 1b. There exists at most one $z_j \notin L$. Because all the remaining triple-sets of F contain 3 vertices of $L \cup K$, we have $|F - xyz_i| \leq 6k - 2m$, and by (13) we get a contradiction with (9). (In this case for $|F| = 11k - 4n - 3$ we obtain the extremal graph – see Section 3.)

Case 1c. Now we have the most complicated case when Z contains exactly 2 triple-sets xyz_1 and xyz_2 such that z_1, $z_2 \notin K \cup L$. Denote by N, O, and P, respectively, the set of all vertices of M adjacent to at least two vertices, to one vertex, and to no vertex, respectively, of L. Then

$$2|N| + |O| \leq 4k\,,$$
$$|O| + |P| \leq |Q|\,,$$
$$-2(|N| + |O| + |P|) = -2m.$$

Adding these inequalities, and taking into account (11), we have

$$|P| \geq 2m - 4k - |Q| \geq m - 2k\,.$$

Hence, by (13)

$$|P| - |(xyz_1)| - |(xyz_2)| \geq 5m - 12k\,.$$

Now put $p = 5/12 - \varepsilon$, $0 < \varepsilon < 1/60$ (see (1)). Then, from (8), $5m - 12k \geq 5(n - I\sqrt{n}) - 12(5/12 - \varepsilon)n = 12\varepsilon n - 5I\sqrt{n}$ which for sufficiently large n is greater than $10\varepsilon n$. Hence there exist at least $10\varepsilon n$ vertices of M not adjacent to any vertex of L, thus adjacent to both x and y, but not adjacent to z_1, z_2. By (3), among these vertices there exists a vertex b of degree less than $16/(10\varepsilon)$. Let $N(b)$ be the set of neighbours of b, and put $H = N(b)$ (H is the above-mentioned set.) Now, each vertex of R is adjacent to at least two vertices of L, to the vertices z_1, z_2 and to some vertex of H. Thus each vertex of R is adjacent to at least 5 vertices of $V - M$ which by (1) and (12) contradicts Lemma 7.

Thus if the conditions xuv, $xwt \subset X$, yuw, $yvt \subset Y$ (or $yuv, ywt \subset Y$, or yut, $yvw \subset Y$) are satisfied, then we always get a contradiction. Assume now that these conditions are not satisfied. Say, X does not contain triple-sets of the form xuv, xwt. Then the following cases can occur:

(a) $X = xuv \cup xuw \cup xvw$,
(b) all the triple-sets of X are adjacent to a further fixed vertex of W,
(c) $X = xuv$.

Now consider these cases.

Case 2. Suppose $X = xuv \cup xuw \cup xvw$. The vertices of M not adjacent to x are adjacent to at least two vertices of the set $\{u, v, w\}$. Hence $2(m - k) + 2|X| \leq 3k$, i.e. $|X| \leq (5k - 2m)/2$, a contradiction with (9), because all the remaining vertices of F are adjacent to y.

Case 3. Suppose $X = xtx_1 \cup \cdots \cup xtx_q$, $q \geq 2$. Again we have to distinguish several subcases.

Case 3a. $yx_ix_j \subset Y$ for some $i, j \in \{1, \ldots, q\}$. Then obviously $q = 2$. Let $U = xtx_1 \cup xtx_2 \cup yx_1x_2$. Then every vertex of M is adjacent to at least two vertices of the set $\{x, t, y, x_1, x_2\} = W_0$ and every vertex of U is adjacent to 3 vertices of W_0. Thus $2m + |U| \leq 5k$, i.e. $|U| \leq 5k - 2m$ which is a contradiction to (9) because all remaining vertices of F are adjacent to Y.

Case 3b. Suppose $Y = yty_1 \cup \cdots \cup yty_r$ and put $Z = xyz_1 \cup \cdots \cup xyz_s$. Assume that among the vertices $x_1, \ldots, x_p, y_1, \ldots, y_r, z_1, \ldots, z_s$ there exist u distinct vertices $w_1, \ldots, w_u$ different from x, y, t, and put $W_1 = \{x, y, t, w_1, \ldots, w_u\}$. Since F is not covered by a single vertex, every vertex of M is adjacent to at least two vertices of W_1 and the vertices of F to 3 vertices of W_1. Thus $2m + |F| \leq (3 + u)k$ which contradicts (9) if $u \leq 3$. Hence assume $u > 3$.

Denote by L the set of all vertices of M not adjacent to any vertex of the set $\{x, y, t\}$. Every vertex of F is adjacent to two vertices of this set, hence by (9) we have

$$(14) \qquad 3k + |L| \geq m + |F|, \quad \text{i.e. } |L| \geq m - 3k + |F| \geq 8k - 3m\,.$$

The vertices of L are adjacent to all vertices w_i, thus for $u = 4$ we have

$$2m + |F| + 2(m - 3k + |F|) \leq 7k, \quad \text{i.e. } |F| \leq (13k - 4m)/3$$

which is, by (1), a contradiction to (9).

For $u \geq 5$ all vertices of L are adjacent to at least 5 vertices of W_1 which by (14) contradicts Lemma 7.

Case 4. The last case is $X = xuv$. If $Z = \emptyset$ then F is covered also by y and u, and we may proceed as in the case 3a. Similarly, if all vertices of F are adjacent to u (or v) then we get again the case 3b. Hence we may assume

$$F = xuv \cup xyx_1 \cup \cdots \cup xyx_i \cup uyu_1 \cup \cdots \cup uyu_j \cup vyv_1 \cup \cdots \cup vyv_q$$

where i, j, q are different from 0. Now we show that all the vertices x_a, u_b, v_c are distinct. Indeed, suppose, for example, that $x_1 = u_1$ and consider $U = xuv \cup xyx_1 \cup uyx_1$. Every vertex of M is adjacent to two vertices of $W_3 = \{x, y, u, v, x_1\}$ and every vertex of U is adjacent to 3 vertices of W_3, thus $|U| \leq 5k - 2m$. However, all remaining vertices of F are adjacent to y, and so we get a contradiction with (9). Now, if $i = j = q = 1$ then by Corollary of Lemma 6 we get a contradiction to (9). Hence suppose $i > 1$ (as F is covered also by the couples u, y and v, y, we may proceed in the remaining cases similarly). Distinguish now two subcases.

Case 4a. Suppose $j = q = 1$. Consider first the case $i \geq 3$. By (9) and by Corollary to Lemma 6 we have

$$|xyx_1 \cup \cdots \cup xyx_i| \geq 11k - 4m - (9k - 3m)/2\,.$$

Thus the number of vertices adjacent neither to x nor to y is at least $m - 2k + [11k - 4m - (9k - 3m)/2] = (9k - 3m)/2$. However, each such vertex is adjacent to at least 5 vertices of $V - M$, a contradiction to Lemma 7 (see (1)).

Assume now $i = 2$. Let s be the number of vertices adjacent to y but not to x_1 nor to x_2. Each such vertex is adjacent to at least one of vertices x, u, v. So let s_1, s_2, and s_3, respectively, be the number of such vertices adjacent to x, u, v, respectively. Obviously,

$$(15) \qquad s_1 + s_2 + s_3 \geq s\,.$$

Now according to Lemma 6 we have

$$(16) \qquad |uyu_1| \leq (3k - m - s_2)/2, \qquad |vyv_1| \leq (3k - m - s_3)/2\,.$$

Consider now the number of vertices adjacent to x. By (16),

$$|xuv| + |xyx_1| + |xyx_2| \geq |F| - 3k + m + (s_2 + s_3)/2\,. \tag{17}$$

Each vertex of $M-F$ must be adjacent to at least one of vertices x, y, x_1. However, there exist at most $k - |xyx_1|$ such vertices adjacent to x_1 and $s - s_1$ such vertices adjacent to y but not to x. So, by (17), the total number of vertices adjacent to x is at least

$$[|F| - 3k + m + (s_2 + s_3)/2] + [m - |F| - k + |xyx_1| - s + s_1] \leq k\,.$$

Thus by (15) we get

$$|xyx_1| \leq 5k - 2m + (s_1 + s_2)/2\,.$$

Hence, according to (16), we get

$$|uyu_1| + |vyv_1| + |xyx_1| \leq 8k - 3m\,,$$

thus, by Corollary of Lemma 6, $|F| \leq 11k - 4m$, a contradiction to (9).

Case 4b. Assume that at least two of the numbers i, j, q are greater than 1, say, $i \geq 2$, $j \geq 2$ (in the remaining cases we may proceed similarly). Now we introduce some notation. Let $W_3 = \{x, u, v, y\}$, and let X_1, U_1, and V_1 be the set of all vertices of M adjacent only to x, only to u, and only to v (and to no other vertices of W_3), respectively. Let further XU, XV, and UV be the set of vertices adjacent only to x and u, only to x and v, and only to u and v (and to no other vertices of W_3), respectively. Finally, let XUV be the set of vertices adjacent to x, u, and v but not to y. Then

$$|X_1| + |U_1| + |V_1| + |XU| + |XV| + |UV| + |XUV| \geq m - k\,. \tag{18}$$

The number of vertices adjacent to x is

$$k \geq |XUV| + |XU| + |XV| + |X_1| + |xyx_1| + \cdots + |xyx_i|\,. \tag{19}$$

The number of vertices adjacent to u is

$$k \geq |XUV| + |XU| + |UV| + |U_1| + |uyu_1| + \cdots + |uyu_j|\,. \tag{20}$$

Adding (18), (19) and (20) and using Lemma 6 gives

$$\begin{aligned} |V_1| &\geq m - 3k + |XU| + |F| - (|vyv_1| + \cdots + |vyv_p|) \\ &\geq m - 3k + |F| - (3k - m) = 2m - 6k + |F|\,. \end{aligned}$$

However, all vertices of V_1 are adjacent to vertices v, x_1, x_2, u_1, u_2, thus $e(G) \geq 4m - |F| + (2m - 6k + |F|) = 6m - 6k$ which, by (1), contradicts (3).

We have seen that (3) leads to a contradiction in all cases, and so for any graph $G \in H_2(n, \lfloor pn \rfloor)$ we have

$$e(G) \geq 8n - 11\lfloor pn \rfloor - 8I\sqrt{n}\,. \tag{21}$$

3. Proof of the Theorem

Consider the graph G_1 consisting of

(a) nine vertices $a, b, c, d, e, f, g, h, i$ of high degrees;
(b) the edges $ab, ag, af, bg, bd, ce, cg, ch, df, di, eg, eh, fi, gi, gh$;
(c) triple-sets acd, aef, abg, bcf, bed;
(d) groups $abih$, $dfgh$, $ceig$ of vertices of degree 4 adjacent to vertices involved in these 4-tuples.

In case $3k - n$ is even, the cardinalities of these triple-sets and groups are (in other case proceed similarly):

$$\begin{aligned} |acd| &= |aef| = |bcf| = |bed| = (3k-n)/2, \\ |abg| &= 5k - 2n - 3, \\ |abih| &= 3n - 7k, \\ |ceig| &= |dfgh| = n - 2k - 3\,. \end{aligned}$$

Then the vertices a, b, c, d, e, f are of degree k, g is of degree $k-3$ and deg $h =$ deg $i = 4n - 9k$ which is less than k for $p > 2/5$ and n sufficiently large. It is easy to check that $G_1 \in H_2(n, \lfloor pn \rfloor)$ for such p and n and $e(G_1) = 8n - 11k - 18$. Hence, by (21), if $G \in H_2(n, \lfloor pn \rfloor)$ where p satisfies (1) and n is sufficiently large, we get

$$8n - 11\lfloor pn \rfloor - 8I\sqrt{n} \le F_2(n, \lfloor pn \rfloor) \le 8n - 11\lfloor pn \rfloor - 18\,,$$

and the assertion of Theorem follows.

Remark. The structure of extremal graphs in a similar problem for graphs of diameter 3 was determined in [**3**]. The author hopes to find a characterization of extremal graphs in general for our case in a future paper. The "kernel system" of the extremal graph is a uniquely determined hypergraph in general.

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