Acta Math. Univ. Comenianae
Vol. LXII, 2(1993), pp. 229–236

ON CONTINUITY AND MONOTONICITY OF DARBOUX TRANSFORMATIONS

R. J. PAWLAK

ABSTRACT. In this paper we consider the problems connected with the continuity of Darboux transformations and the monotonicity of the restrictions of these transformations. We show that it becomes possible to give answers to many questions concerning these problems if our considerations are confined to the family of c-functions which is defined in the paper.

In paper [**DG**] (1875), the first example of a discontinuous Darboux function was given. Since then, there have appeared many papers devoted to the studies of the properties of these functions. The proving of a series of interesting properties for real Darboux transformations of a real variable became a cause of the search for a generalization of the notion of a Darboux function to the case of transformations defined and taking their values in more general spaces. Different ways of the generalizations can be found, among others, in papers [**BB**], [**GK**], [**PR**], and the specification and discussion of many of them — in paper [**JJ**]. The definitions presented below are analogous to those of Darboux($\mathcal{B}$) transformations and weakly connected ones considered in papers [**BB**] and [**GK**]. However, what differentiates one from another is that we free ourselves from the strictly defined classes of sets, considered in these papers.

So, let $\mathcal{L}$ be some family of connected sets (in the sequel, unless otherwise stated, $\mathcal{L}$ will always denote a fixed family of connected sets) in a topological space X and let $f\colon X \to Y$.
We say that f is a **Darboux**$^*(\mathcal{L})$ **function** if $\overline{f(C)}$ is a connected set for any $C \in \mathcal{L}$.
We say that f is a **Darboux**$(\mathcal{L})$ **function** if $f(C)$ is a connected set for any $C \in \mathcal{L}$.

It will be convenient to our considerations to adopt the following definition, too:
We say that $\mathcal{B}$ is an $\mathcal{L}$-base of the topological space X if $\mathcal{B}$ is an open base of

Received June 18, 1992; revised September 16, 1993.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 26A15, 26A48.
Supported by KBN research grant PB 691/2/91.

this space and, for any $U \in \mathcal{B}$ and any $x, y \in U$, there exists $C \in \mathcal{L}$ such that $x, y \in C \subset U$.

Many problems studied by mathematicians in connection with the papers concerning Darboux transformations were related to the question of the continuity and the monotonity of these functions. In order to make it possible to carry out the studies on this subject, it was necessary to confine the considerations to some narrower families of functions (e.g. [**KU**], [**WD**], [**GZ**], [**HT**], [**PH**] or [**PJ**]). In this situation, it seems essential to find as wide a family of transformations as possible, whose properties would enable one to obtain results analogous (or stronger) to those included in the papers cited above. To accomplish this aim, we shall adopt the following definition (the symbol M^d stands for the derived set of a set M).

Definition. Let $f\colon X \to Y$ where X and Y are arbitrary topological spaces. We say that f is a c-function if, for any subset A of the space X and each $x \in A^d$, there exists a set $B \subset A$ such that $x \in B^d$ and $(f(B))^d \subset \{f(x)\}$.

Throughout the paper, we adopt the classical symbols and notations. The adoption of the way of defining a Darboux function justifies the adoption of certain modifications of the notations applied, among others, in papers [**GK**] and [**GZ**]. So, let $\mathcal{L}$ be some family of connected sets in X and let f be a function defined on X. Then $Y_{\mathcal{L}}(f) = \{\alpha \in f(X)\colon f^{-1}(\alpha) \in \mathcal{L}\}$ and $S_{\mathcal{L}}(f) = f^{-1}(Y_{\mathcal{L}}(f))$. Of course, if $\mathcal{L}$ is the family of all connected subsets (continua) of the space X, and $S_{\mathcal{L}}(f) = X$, then f is weakly monotone (Morrey monotone) ([**GK**]). We say that $f\colon X \to Y$ is $\mathcal{L}$-pseudo monotone if $f(X)$ is a connected set and, for any $\alpha \in Y$, any $x, y \in f^{-1}(\alpha)$ and any neighbourhoods U, V and W of the points x, y and α, respectively, there exists a set $C \in \mathcal{L}$ such that $U \cap C \neq \emptyset \neq V \cap C$ and $f(C) \subset W$. Of course, every $\mathcal{L}$-weakly monotone function (i.e. such that $S_{\mathcal{L}}(f) = X$) is $\mathcal{L}$-pseudo monotone. There exist, however, continuous c-functions being C-pseudo monotone and K-pseudo monotone which are not C-weakly monotone (C — the family of all connected sets, K — the family of all continua).

We say that a function $f\colon X \to Y$ is relatively proper if $\overline{f^{-1}(K)}$ is a compact set for any compact set $K \subset Y$ ([**GK**], [**GZ**]).

As far back as the XIXth century, many mthematicians thought the Darboux property to be equivalent to the continuity of functions ([**BA**, Chapter I]). Later, additional conditions under which Darboux transformations (also in more abstract spaces) are continuous were sought for (e.g. [**HT**], [**JJ**], [**KU**], [**PH**], [**WD**]). These considerations comprised mainly metric spaces because, in the case of topological ones, the situation became considerably complicated. Therefore it seems essential to ask about the possibility of obtaining results analogous (or stronger) to those contained in the papers cited above, in a more general case.

A partial solution to this question is the following

Theorem 1. *Let X be a Hausdorff space with an $\mathcal{L}$-base and let Y be a locally compact space; moreover, let X and Y be first-countable. Then a c-function $f\colon X \to Y$ is Darboux$^*(\mathcal{L})$ if and only if f is continuous.*

Proof. The **sufficiency** of the above condition is obvious.

Necessity. Suppose to the contrary that a c-function f is Darboux$^*(\mathcal{L})$ and there exists a point $t_0 \in X$ such that f is discontinuous at t_0.

Let Z be a compact set included in Y such that $f(t_0) \in \operatorname{Int}(Z)$ and

$$f(U) \setminus Z \neq \emptyset \quad \text{for any neighbourhood } U \text{ of } f(t_0). \tag{1}$$

Let P be a compact set such that $f(t_0) \in \operatorname{Int}(P) \subset P \subset \operatorname{Int}(Z)$. Assume that $\mathcal{B}$ is an $\mathcal{L}$-base of X, and

$$\mathcal{B}(t_0) = \{U \in \mathcal{B} : t_0 \in U\}.$$

We infer that

$$\begin{aligned} &f(U) \cap (V \setminus P) \neq \emptyset \quad \text{for any } U \in \mathcal{B}(t_0) \text{ and for any open set } V, \\ &\text{such that } P \subset V \subset \operatorname{Int}(Z). \end{aligned} \tag{2}$$

Indeed. Assume that (2) does not take place. Then there exist $U \in \mathcal{B}(t_0)$ and an open set V such that $P \subset V \subset \operatorname{Int}(Z)$ and $f(U) \cap (V \setminus P) = \emptyset$. By virtue of (1), there exists $x \in U$ such that $f(x) \notin Z$. Let $C \in \mathcal{L}$ be such that $t_0, x \in C$. Therefore $f(C) \cap (V \setminus P) = \emptyset$ and $f(C) \cap P \neq \emptyset \neq f(C) \setminus V$. This means that $\overline{f(C)}$ is not a connected set, which contradicts our assumption that f is Darboux$^*(\mathcal{L})$. So, (2) is true.

Let $\Xi = \{(U, V) : U \in \mathcal{B}(t_0) \wedge P \subset \operatorname{Int}(V) = V \subset \operatorname{Int}(Z)\}$. Define the directing relation $\leq$ in Ξ in t he following way:

$$(U, V) \leq (U', V') \Leftrightarrow U \supset U' \wedge V \supset V'.$$

Now, we define the net $\{x_\xi\}_{\xi \in \Xi}$ in the following way: for each $\xi = (U, V) \in \Xi$, let x_ξ denote an arbitrary element of the set U satisfying $f(x_\xi) \in V \setminus P$ (by (2), it is possible).

Let α_0 be a cluster point of the net $\{f(x_\xi)\}_{\xi \in \Xi}$. We shall show that

$$\alpha_0 \in P \setminus \operatorname{Int}(P). \tag{3}$$

Let us first prove that $\alpha_0 \in P$. Assume to the contrary that $\alpha_0 \notin P$. Thus, by the regularity of Y, there exist open sets V_0 and V_1 such that $P \subset V_0 \subset \operatorname{Int}(Z)$, $\alpha_0 \in V_1$ and $V_0 \cap V_1 = \emptyset$. Put $\xi_0 = (U_0, V_0)$ where U_0 is an arbitrary set from $\mathcal{B}(t_0)$. It is easy to see that $f(x_\xi) \in V_0$ for each $\xi \geq \xi_0$, which contradicts the fact that α_0 is a cluster point of $\{f(x_\xi)\}_{\xi \in \Xi}$. This proves that $\alpha_0 \in P$. Note that $\alpha_0 \notin \operatorname{Int} P$ because, in the opposite case, by the definition of $\{x_\xi\}_{\xi \in \Xi}$, there exists a neighbourhood of α_0 disjoint from $\{f(x_\xi) : \xi \in \Xi\}$, which is impossible. Condition (3) is proved.

By virtue of (3), we have

$$\alpha_0 \notin \{f(x_\xi) : \xi \in \Xi\}.$$

Let $\mathcal{B}^*(t_0)$ be a countable open base of X at t_0 and let $\mathcal{R}(\alpha_0)$ be a countable base of Y at α_0 consisting of open sets such that their closures are included in $\mathrm{Int}\,(Z)$.

Write the above bases down in the following way:

$$\mathcal{B}^*(t_0) = \{U_1, U_2, \dots\},$$
$$\mathcal{R}(\alpha_0) = \{V_1, V_2, \dots\}.$$

We may assume that $V_1 \supset V_2 \supset \dots$.

Let n be a fixed positive integer. From the above cosiderations we infer that there exists a point

$$z_n \in (U_n \setminus \{t_0\}) \cap f^{-1}(V_n \setminus \{\alpha_0\}).$$

Put $A_1 = \{z_n : n = 1, 2, \dots\}$. Of course, $t_0 \in A_1^d$. So, let B be an arbitrary subset of A_1 such that $t_0 \in B^d$. Thus there exists a sequence $\{k_n\}_{n=1}^{\infty}$ consisting of positive integers such that $z_{k_n} \in B$ (for $n = 1, 2, \dots$). It is easy to see that $\alpha_0 \in (f(B))^d$ and (by (3) and the fact that $f(t_0) \in \mathrm{Int}\,(P)$) $\alpha_0 \neq f(t_0)$, which contradicts the assumption that f is a c-function.

The contradiction obtained ends the proof. □

In paper [**GK**], the following problem was posed: under what assumptions with respect to X and f is a connected function $f : X \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of all real numbers) monotone (i.e. the preimage of an arbitrary connected set is a connected set) or weakly monotone (i.e. each level $f^{-1}(\alpha)$ is a connected set) relatively to the closure of the union of all connected levels? Some partial answer can be found in [**GZ**], [**PJ**], [**PRJ**], [**RP**]. The adoption of our definitions and notations allows one to write this problem down in a little more general form:

Under what assumptions with respect to X and f is a Darboux ($\mathcal{L}$) function $f : X \to \mathbb{R}$ monotone or weakly monotone relatively to $\overline{S_{\mathcal{L}}(f)}$?

The theorem below constitutes a partial answer to the problem thus formulated. It deserves attention that, unlike the theorems in the works cited above, we assume relatively little about the domains of the transformations considered and adopt the weakest version of the Darboux property.

In the sequel, we shall assume that the family $\mathcal{L}$ contains all continua and possesses the following property: for any $A \in \mathcal{L}$, if $x \in \overline{A}$, then $A \cup \{x\} \in \mathcal{L}$.

Theorem 2. *Let X be a T_5-space with an $\mathcal{L}$-base $\mathcal{B}$ and let $f\colon X \to \mathbb{R}$ be a c-function Darboux($\mathcal{L}$), relatively proper and $\mathcal{L}$-pseudo monotone. Then $f\big|_{\overline{S_{\mathcal{L}}(f)}}$ is weakly monotone.*

Proof. Assume to the contrary that:

(i) there exists $\alpha \in \mathbb{R}$ such that $f^{-1}(\alpha) \cap \overline{S_{\mathcal{L}}(f)}$ is a nonempty set which is not connected.

By the above, we may show that:

(ii) there exists $\varepsilon > 0$ such that
either $(\alpha - \varepsilon, \alpha) \cap Y_{\mathcal{L}}(f) = \emptyset$ or $(\alpha, \alpha + \varepsilon) \cap Y_{\mathcal{L}}(f) = \emptyset$.

Indeed, suppose that (ii) does not take place. Then there exist two sequences $\{\alpha_n^-\}_{n=1}^{\infty}, \{\alpha_n^+\}_{n=1}^{\infty} \subset Y_{\mathcal{L}}(f)$ such that

$$\lim_{n\to\infty} \alpha_n^- = \alpha = \lim_{n\to\infty} \alpha_n^+ \quad \text{and} \quad \alpha_i^- < \alpha_{i+1}^-, \quad \alpha_i^+ > \alpha_{i+1}^+ \quad (i = 1, 2, \dots).$$

Let n be a fixed positive integer. We shall show that

$$f^{-1}([\alpha_n^-, \alpha_n^+]) \quad \text{is the continuum.} \tag{4}$$

First, we shall prove that

$$f^{-1}([\alpha_n^-, \alpha_n^+]) \quad \text{is closed.} \tag{5}$$

Consider $x \in X \setminus f^{-1}([\alpha_n^-, \alpha_n^+])$. We shall now show that:

$$\text{there exists } U \in \mathcal{B} \text{ such that } x \in U \text{ and } U \cap (f^{-1}(\alpha_n^-) \cup f^{-1}(\alpha_n^+)) = \emptyset\,. \tag{6}$$

Suppose that (6) does not hold. Then either $x \in \overline{f^{-1}(\alpha_n^-)}$ or $x \in \overline{f^{-1}(\alpha_n^+)}$. Assume, for instance, that $x \in \overline{f^{-1}(\alpha_n^-)}$. Thus $f^{-1}(\alpha_n^-) \cup \{x\} \in \mathcal{L}$, which contradicts the asumption that f is Darboux($\mathcal{L}$). The contradiction obtained proves (6).

By virtue of the assumption that f is Darboux($\mathcal{L}$) and by (6), we may observe that $x \notin \overline{f^{-1}([\alpha_n^-, \alpha_n^+])}$, which proves (5) (by the free choice of x).

We shall now show that:

$$\text{the set } f^{-1}([\alpha_n^-, \alpha_n^+]) \text{ is connected.} \tag{7}$$

Suppose that $f^{-1}([\alpha_n^-, \alpha_n^+])$ is not connected; then $f^{-1}([\alpha_n^-, \alpha_n^+]) = P \cup Q$ where P and Q are nonempty, closed and disjoint sets. Then, according to the $\mathcal{L}$-pseudomonotonicity of f, $f(P) \cap f(Q) = \emptyset$. Now, we shall show that

$$\overline{f(P)} \cap f(Q) = \emptyset\,. \tag{8}$$

Indeed, in the opposite case, there exists $\beta_n \in f(P)$, $(n = 1, 2, \dots)$ and $\beta \in f(Q)$ such that $\beta = \lim_{n\to\infty} \beta_n$ and $\beta_n \neq \beta_m$ for $n \neq m$. Let $p_n \in P \cap f^{-1}(\beta_n)$ for $n = 1, 2, \dots$. From the assumption that f is relatively proper and from the closedness of P we infer that there exists a subnet $\{q_\sigma\}_{\sigma\in S}$ of $\{p_n\}_{n=1}^{\infty}$ converging to some $q_0 \in P$. It is easy to see that $q_0 \in \{q_\sigma : \sigma \in S\}^d$. Let P_1 be an arbitrary subset of $\{q_\sigma : \sigma \in S\}$ such that $q_0 \in P_1^d$. Therefore $f(P_1)$ contains some subsequence of $\{\beta_n\}_{n=1}^{\infty}$ and, consequently, $\beta \in (f(P_1))^d$. Of course, $\beta \neq f(q_0)$, which is impossible because f is a c-function. The contradiction obtained proves equality (8).

Similarly we can show that $f(P) \cap \overline{f(Q)} = \emptyset$.

Moreover, we infer that $[\alpha_n^-, \alpha_n^+] = f(P) \cup f(Q)$. This situation is impossible according to (8) and the above remark. This contradiction ends the proof of (7).

By virtue of (5), (7) and the fact that f is relatively proper, we infer that (4) takes place.

So, by (4), $f^{-1}(\alpha) = \bigcap_{n=1}^{\infty} f^{-1}([\alpha_n^-, \alpha_n^+])$ is a continuum ([**ER**, Corollary 6.1.2, p. 437]), which contradicts (i). The contradiction obtained ends the proof of (ii).

By virtue of (i), we have $f^{-1}(\alpha) \cap \overline{S_{\mathcal{L}}(f)} = T \cup D$ where T and D are separated sets. Let U_T and U_D be open sets such that $T \subset U_T$, $D \subset U_D$ and $U_T \cap U_D = \emptyset$ ([**ER**, Theorem 2.1.7, p. 97]). Moreover, let $t_0 \in T$ and $d_0 \in D$. Of course, $t_0, d_0 \notin S_{\mathcal{L}}(f)$, so there exist nets $\{t_\sigma\}_{\sigma\in\Sigma}$ and $\{d_\delta\}_{\delta\in\Delta}$ consisting of elements of $S_{\mathcal{L}}(f)$, such that $t_0 = \lim_{\sigma\in\Sigma} t_\sigma$ and $d_0 = \lim_{\delta\in\Delta} d_\delta$. We shall show that

$$\lim_{\sigma\in\Sigma} f(t_\sigma) = \alpha = \lim_{\delta\in\Delta} f(d_\delta). \tag{9}$$

Since the proofs of both the equalities are analogous, therefore we shall show only the first of them.

Assume to the contrary that α is not a limit of the net $\{f(t_\sigma)\}_{\sigma\in\Sigma}$. Thus there exists a real number $\eta > 0$ such that, for each $\sigma_0 \in \Sigma$, there exists $\sigma \geq \sigma_0$ such that $f(t_\sigma) \notin (\alpha - \eta, \alpha + \eta)$. Put $\Sigma' = \{\sigma \in \Sigma : f(t_\sigma) \notin (\alpha - \eta, \alpha + \eta)\}$. Of course, $\{f(t_\sigma)\}_{\sigma\in\Sigma'}$ is a subnet of $\{f(t_\sigma)\}_{\sigma\in\Sigma}$. Consider the following sets: $\Sigma'' = \{\sigma \in \Sigma' : f(t_\sigma) \leq \alpha - \eta\}$, $\Sigma''' = \{\sigma \in \Sigma' : f(t_\sigma) \geq \alpha + \eta\}$.

Let us remark that (we assume that Σ is directed by $\leq$)

(10)
either,

for each $\sigma' \in \Sigma'$, there exists $\sigma'' \in \Sigma''$ such that $\sigma'' \geq \sigma'$

or,

for each $\sigma' \in \Sigma'$, there exists $\sigma''' \in \Sigma'''$ such that $\sigma''' \geq \sigma'$.

Assume, for instance, that (10) takes place for Σ''. Thus Σ'' is a directed set and $\{t_\sigma\}_{\sigma\in\Sigma''}$ is a subnet of $\{t_\sigma\}_{\sigma\in\Sigma}$ and, of course, $\{f(t_\sigma)\}_{\sigma\in\Sigma''}$ is a subnet of $\{f(t_\sigma)\}_{\sigma\in\Sigma}$.

Consider the following cases:

1^0. $\xi^- = \sup\{f(t_\sigma) : \sigma \in \Sigma"\} \in \{f(t_\sigma) : \sigma \in \Sigma"\}$.

Then $\xi^- \in Y_{\mathcal{L}}(f)$ and, as is easy to show, $f^{-1}((\xi^-, +\infty))$ is a neighbourhood of t_0 such that $t_\sigma \notin f^{-1}((\xi^-, +\infty))$ for any $\sigma \in \Sigma"$, which leads to a contradiction.

2^0. $\xi^- = \sup\{f(t_\sigma) : \sigma \in \Sigma"\} \notin \{f(t_\sigma) : \sigma \in \Sigma"\}$.

Let $\Pi = \{(U, n) : U \in \mathcal{B} \wedge t_0 \in U \wedge n \in \mathbb{N}\}$ ($\mathbb{N}$ denotes the set of all positive integers). Let us define the relation $\ll$ directing Π as follows:

$$(U, n) \ll (V, m) \Leftrightarrow U \supset V \wedge n \leq m.$$

Define a net $\{\xi_p\}_{p\in\Pi}$ in the following way: for each $p = (U, n) \in \Pi$, give ξ_p such that $\xi_p \in U$ and $f(\xi_p) \in (\xi^- - \frac{1}{n}, \xi^-)$. We infer that

$$t_0 = \lim_{p\in\Pi} \xi_p \quad \text{and} \quad \xi^- = \lim_{p\in\Pi} f(\xi_p).$$

Of course, $t_0 \in \{\xi_p : p \in \Pi\}^d$. Let B_1 be an arbitrary subset of $\{\xi_p : p \in \Pi\}$ such that $t_0 \in B_1^d$. We may notice that $f(B_1) \cap (\xi^- - \frac{1}{n}, \xi^-) \neq \emptyset$ (for $n = 1, 2, \dots$), which means that $\xi^- \in f(B_1)^d$, which contradicts the fact that f is a c- function.

The proof of (9) is thus finished.

So, let W be an arbitrary element from $\mathcal{B}$ containing d_0 and such that $W \subset U_D$. According to (ii), assume $(\alpha, \alpha + \varepsilon) \cap Y_{\mathcal{L}}(f) = \emptyset$. By (9), one may assume that $f(t_\sigma) < \alpha$ for $\sigma \in \Sigma$ and infer that $f(W)$ is a nondegenerate interval such that $(\alpha - \varepsilon, \alpha) \cap f(W) \neq \emptyset$. Consequently, there exists $\sigma_0 \in \Sigma$ such that, for each $\sigma \geq \sigma_0$, $f(t_\sigma) \in f(W)$. Since $f^{-1}(f(t_\sigma))$ is a connected set non-disjoint from U_T and U_D, there exists $h_\sigma \in f^{-1}(t_\sigma) \setminus (U_T \cup U_D)$, $(\sigma \geq \sigma_0)$. By the fact that f is relatively proper, there exists a cluster point $h_0 \in X$ of $\{h_\sigma\}_{\sigma\geq\sigma_0}$. Let $\{g_{\sigma^*}\}_{\sigma^*\in\Sigma^*}$ be a subnet of $\{h_\sigma\}_{\sigma\geq\sigma_0}$, converging to h_0. Of course, $h_0 \notin U_T \cup U_D$, and so, $h_0 \notin f^{-1}(\alpha)$ (because $h_0 \in \overline{S_{\mathcal{L}}(f)}$). Let $Z = \{g_{\sigma^*} : \sigma^* \in \Sigma^*\}$. Therefore $h_0 \in Z^d$. Let S be an arbitrary subset of Z such that $h_0 \in S^d$.

Now, we shall show that

$$f(S) \cap \left(\alpha - \frac{1}{n}, \alpha\right) \neq \emptyset \quad \text{for any} \quad n \in \mathbb{N}. \tag{11}$$

First, we shall prove that

$$f(h_0) > \alpha. \tag{12}$$

Indeed, in the opposite case, between $f(h_0)$ and α there lies some element $\tau \in Y_{\mathcal{L}}(f)$. Thus, as is easy to see, $f^{-1}((-\infty, \tau))$ is a neighbourhood of h_0, and this is impossible because h_0 is a cluster point of $\{h_\sigma\}_{\sigma\geq\sigma_0}$.

Let us return to the proof of (11). Assume that there exists n_0 such that $f(S) \cap (\alpha - \frac{1}{n_0}, \alpha) = \emptyset$. Thus, by (12), it is easy to indicate a neighbourhood of h_0 which is disjoint from S, which is impossible.

From (11) we infer that $\alpha \in f(S)^d$, which contradicts the fact that f is a c-function.

Thus the proof of Theorem 2 is finished.

References

[**BA**] Bruckner A. M., *Differentiation of Real Functions*, Springer Verlag, 1978.

[**BB**] Bruckner A. M. and Bruckner J. B., *Darboux transformations*, Trans. Amer. Math. Soc. **128** (1967), 103–111.

[**DG**] Darboux G., *Memoire sur les fonctions discontinues*, Ann. Sci. Scuola Norm. Sup. **4** (1875), 57–112.

[**ER**] Engelking R., *General Topology*, Warszawa, 1977.

[**GK**] Garg K. M., *Properties of connected functions in terms of their levels*, Fund. Math. **47** (1977), 17–36.

[**GZ**] Grande Z., *Les ensambles de niveau et la monotonic d'une fonction*, Fund. Math. **102** (1979), 9–12.

[**HT**] Hamlett T. R., *Compact maps, connected maps and continuity*, J. London Math. Soc. **10** no. 2 (1975), 25–26.

[**JJ**] Jędrzejewski J. M., *Własności funkcji związane z pojęciem spójności*, Acta Univ. Lodz. (1984), 1–84.

[**KU**] Klee V. L. and Utz W. R., *Some remarks on continuous transformations*, Proc. Amer. Math. Soc. **5** (1954), 182–184.

[**PH**] Pawlak H., *On some condition equivalent to the continuity of closed functions*, Dem. Math. **17.3** (1984), 723–732.

[**PJ**] Pawlak H. and Pawlak R.J., *On some properties of closed functions in terms of their levels*, Comm. Math. **26** (1986), 81–87.

[**PR**] Pawlak R.J., *On Zahorski classes of functions of two variables*, Rev. Roum. de Math. Pures et Appl. **35.1** (1990), 53–71.

[**PRJ**] ———, *On the continuity and monotonicity of restrictions of connected functions*, Fund. Math. **114** (1981), 91–107.

[**RP**] Pawlak R. J. and Rychlewicz A., *On K. M. Garg's problem in respect to Darboux functions*, Acta Univ. Lodz. Folia Math. **4** (1991), 91–107.

[**WD**] White D. J., *Functions preserving compactness and connectedness*, J. London Math. Soc. **3** (1971), 767–768.

R. J. Pawlak, Institute of Mathematics, Łódż University, Banacha 22, 90-238 Łódż, Poland