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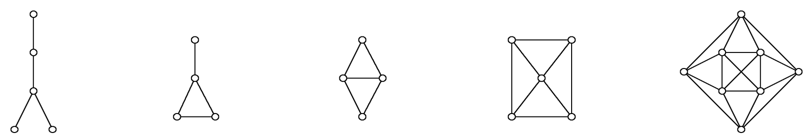
A NOTE ON THE RADIUS OF ITERATED LINE GRAPHS

M. KNOR

Abstract. We prove that almost all i-iterated line graphs are selfcentric with radius $i+2$. This generalizes the well-known result that almost all graphs are selfcentric with radius two.

Introduction

Let G be a graph. Then by its line graph $L(G)$ we mean a graph whose nodes are the edges of G, and two nodes are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. We remark that if G has no edges, then $L(G)$ is an empty graph. The i-iterated line graph of G, the $L^i(G)$, is $L(L^{i-1}(G))$ where $L^0(G)=G$ and $i \geq 1$. For an example of iterated line graphs see Figure 1.



G $L(G)$ $L^2(G)$ $L^3(G)$ $L^4(G)$

Figure 1.

By $d(G)$ and $r(G)$ we denote the radius and the diameter of D, respectively. Let G be a graph different from a path, a cycle, and a claw $K_{1,3}$. Then, as proved in [**2**], there are numbers d_G, i_G, c_G, and c'_G, such that

$$d(L^i(G)) = d_G + i \quad \text{for every} \quad i \geq i_G;$$
$$i - \sqrt{2\log_2 i} + c_G \leq r(L^i(G)) \leq i - \sqrt{2\log_2 i} + c'_G \quad \text{for every} \quad i \geq 0\,.$$

These results imply that if G is not a path, a cycle, and a claw, then there is a number s_G such that $d(L^i(G)) > r(L^i(G))$ for every $i \geq s_G$, i.e., the $L^i(G)$ is not selfcentric. In contrast with this we show that almost all i-iterated line graphs are selfcentric of radius $i+2$.

As a model of random graphs we use the well-established model of Erdős and Rényi, see [**3**, the model A]. In this model the node set of the graph is fixed, and

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each pair of nodes is joined by an edge with probability p, or left unjoined with probability $1-p$. A property is said to hold for **almost all graphs** if the limit of the probability that a random graph has the property is 1.

Result

We will identify edges in a graph G with the corresponding nodes in $L(G)$. Hence, if u and v are two adjacent nodes in G then by uv we mean an edge in G, as well as the node in $L(G)$ corresponding to the edge uv. This notation enables us to consider a node in $L^i(G)$, $i \geq 2$, as a pair of adjacent nodes in $L^{i-1}(G)$, either of these is a pair of adjacent nodes from $L^{i-2}(G)$, and so on. Furthermore, we can define each node in $L^i(G)$ using only edges of G, and such a definition will be called the **recursive definition of v in G**.

Let G be a graph and v be a node in $L^i(G)$, $i \geq 1$. By the j-butt $B_j(v)$ of the node v in $L^i(G)$ we mean a subgraph of $L^{i-j}(G)$ induced by the edges involved into the recursive definition of v. The butt we will abbreviate to $B(v)$ if $i = j$. We have:

Lemma 1 [**2**]. *Let H be a subgraph of a graph G. Then H is an i-butt for some node in $L^i(G)$ if and only if H is a connected graph with at most i edges, distinct from any path with less than i edges.*

The distance $d_G(H, J)$ between two subgraphs H and J of a graph G equals to the length of a shortest path in G joining a node from H to a node from J. The following lemma enables us to compute distances between nodes in iterated line graphs:

Lemma 2 [**2**]. *Let G be a connected graph, and let u and v be distinct nodes in $L^i(G)$. Then*

(i) $d_{L^i(G)}(u, v) = i + d_G(B_i(u), B_i(v))$ *if the i-butts of v and u are edge-disjoint.*

(ii) $d_{L^i(G)}(u, v) = \max\{t : t\text{-butts of } u \text{ and } v \text{ are edge-disjoint}\}$ *if i-butts of u and v have a common edge.*

For the diameter and the radius of line graphs we have:

Lemma 3 [**1**]. *Let G be a connected graph such that $L(G)$ is not empty. Then*

$$d(G) - 1 \leq d(L(G)) \leq d(G) + 1 \quad \textit{and}$$
$$r(G) - 1 \leq r(L(G)) \leq r(G) + 1\,.$$

Let H consists of two node-disjoint triangles. Since almost all graphs contain a prescribed graph as an induced subgraph, see [**3**, p. 14], the H is an induced subgraph of almost all graphs. Thus, $d(L^i(G)) \geq i + 2$ for almost all graphs G, by Lemma 1 and Lemma 2. From the other side for almost all graphs G we have $d(G) = 2$, see [**3**, p. 14]. Thus, by Lemma 3 $d(L^i(G)) \le i+2$ for almost all graphs G, and hence $d(L^i(G)) = i+2$ for almost all graphs. It means that the following theorem implies that almost all i-iterated line graphs are selfcentric:

Theorem 4. *Let $i \ge 0$. Then $r(L^i(G)) = i+2$ for almost all graphs G.*

Proof. By $V(G)$ is denoted the node set of G; and by $e_G(u)$ we denote the eccentricity of the node u in G, i.e., $e_G(u) = \max\{d_G(u,v) : v \in V(G)\}$.

Let G be a graph on n nodes, n is sufficiently large, in which each edge appears with probability p, $0 < p < 1$. We give an upper bound for the probability $P(r(L^i(G)) \le i+1)$, i.e. that the radius of $L^i(G)$ does not exceed $i+1$.

Let H be a subgraph of G on m nodes. Then $V(H)$ can be partitioned into $\lfloor \frac{m}{3} \rfloor$ sets, each consisting of at least three nodes. Thus, for the probability P_H that H contains no triangle we have $P_H \le (1-p^3)^{\lfloor \frac{m}{3} \rfloor}$.

Let $u \in V(L^i(G))$ such that $e_{L^i(G)}(u) \le i+1$. The $B(u)$ contains at most $i+1$ nodes, by Lemma 1. Let $S \supseteq V(B(u))$ such that $|S| = i+1$. Since $e_{L^i(G)}(u) \le i+1$, there is no $v \in V(L^i(G))$ such that $d_G(B(u), B(v)) \ge 2$, by Lemma 2. In particular, there is no triangle T in G such that $d_G(S,T) \ge 2$. Let $v \in V(G) \setminus S$. Then the probability that $d_G(S,v) \ge 2$ equals $(1-p)^{i+1}$. Thus, we have:

$$P(e_{L^i(G)}(u) \le i{+}1) \le \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \Big(1-(1-p)^{i+1}\Big)^{n-i-1-j} \Big((1-p)^{i+1}\Big)^j (1-p^3)^{\lfloor \frac{j}{3} \rfloor}$$

(here j denotes the number of nodes v such that $d_G(S,v) \ge 2$). Further,

$$\begin{aligned} P(e_{L^i(G)}(u) \le i+1) &< \frac{1}{(1-p^3)} \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} \Big(1-(1-p)^{i+1}\Big)^{n-i-1-j} \Big((1-p)^{i+1}\Big)^j \sqrt[3]{1-p^3}^{\,j} \\ &= \frac{1}{(1-p^3)} \Big(1-(1-p)^{i+1} + (1-p)^{i+1}\sqrt[3]{1-p^3}\Big)^{n-i-1} \\ &= \frac{1}{(1-p^3)} a_i^{n-i-1}. \end{aligned}$$

Since $(1-(1-p)^{i+1} + (1-p)^{i+1}) = 1$ and $0 < \sqrt[3]{1-p^3} < 1$, we have $0 < a_i < 1$.

Since each $B(u)$, $u \in V(L^i(G))$, is contained in a subgraph of G induced by $i+1$ nodes, we have $P(r(L^i(G)) \le i+1) < \frac{1}{(1-p^3)}\binom{n}{i+1} a_i^{n-i-1}$. Clearly $\lim_{n\to\infty} \frac{1}{(1-p^3)}\binom{n}{i+1} a_i^{n-i-1} = 0$, and hence $r(L^i(G)) \ge i+2$ for almost all graphs G. Since $r(G) = 2$ for almost all graphs G, see [**3**, p. 14], by Lemma 3 we have $r(L^i(G)) \le i+2$ for almost all graphs G. □

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M. Knor, Department of Mathematics, Faculty of Civil Engineering, Slovak Technical University, Radlinského 11, 813 68 Bratislava, Slovakia