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ON THE ORDER–COMPLETION OF ADDITIVE CONJOINT STRUCTURES

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Abstract. Measurement theory provides additive conjoint structures for additive representations of empirical data. Roughly, an additive conjoint structure is a product of (quasi)ordered sets with some properties connecting the different factors of the product. Well-known Debreu's Theorem says that every additive conjoint structure can be embedded in a vector space over the real numbers. This embedding yields a completion of the additive conjoint structure where every factor becomes a complete lattice. This paper introduces a synthetical way of constructing this completion without using real numbers.

1. Introduction

Measurement theory is the theory of representing empirical data by relations on ordered or algebraic structures. **Additive conjoint structures** are basic for the additive representation of multidimensional data in measurement theory (cf. [**3**]). Additive representations reflect certain kinds of ordinal dependencies between attributes of the data. In this frame, the question of order-completions is motivated by the "standard case" of embedding the rational numbers into the reals. Before we start, we shall make a short note concerning two different approaches for the constructions of such representations or embeddings:

1. The **analytical** approach: For finding an embedding of a structure **M** into a similar structure **N** with some additional properties, we consider a well-known structure (e.g. the real numbers), construct **N** from this structure, and embed **M**.
2. The **synthetical** approach: We construct the structure **N** directly from **M**, using internal properties of **M**.

Both approaches have their typical advantages. The analytical approach enables us to represent the original structure in a well understood environment. An important advantage is that we can use the most often powerful language of this well-known structure, e.g. we can calculate with real numbers. Contrary to this,

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the advantage of the synthetical approach is that we can come to a deeper understanding of the original structure by using its own language and internal properties. U. Wille presented in [**6**] a broad discussion of the role of the synthetical approach for measurement theory.

We start now to introduce the basic notion of additive conjoint structures by giving some definitions (cf. [**3**, Chap. 6]).

Definition 1.1. Let M be a set. A binary relation $\lesssim$ on M is called a **quasi-order** if the following conditions hold for all $x, y, z \in M$:

(1) $x \lesssim x$ (**reflexivity**),
(2) If $x \lesssim y$ and $y \lesssim z$ then $x \lesssim z$. (**transitivity**).

The quasi-order is called **linear** if also the condition

(3) $x \lesssim y$ or $y \lesssim x$ (**connectivity**)

holds. $(M, \lesssim)$ is called a **(linear) quasi-ordered set**.

Hence, a (linear) quasi-order is almost a (linear) order, but it lacks anti-symmetry. Observe that connectivity already implies reflexivity.

Definition 1.2. Let $M_1, \dots, M_n$ be sets, and let $\lesssim$ be a (linear) quasi-order on $\prod_{i=1}^n M_i$. For any $x := (x_1, \dots, x_n) \in \prod_{i=1}^n M_i$ and any subset $J := \{j_1, \dots, j_m\}$ of the index set $I := \{1, \dots, n\}$, let $x_J := (x_{j_1}, \dots, x_{j_m})$ denote the **projection** of x in $\prod_{j \in J} M_j$. The **induced relation** $\lesssim_J$ on $\prod_{j \in J} M_j$ is defined by

$$a \lesssim_J b \iff \text{there are } x, y \in \prod_{i=1}^n M_i : x \lesssim y, a = x_J, b = y_J, \text{ and } x_{I \setminus J} = y_{I \setminus J}\,.$$

The (linear) quasi-order $\lesssim$ is called **independent** if

$$a \lesssim_J b \implies x \lesssim y \quad \text{for all } x, y \in \prod_{i=1}^n M_i \text{ with } a = x_J, b = y_J, \text{ and } x_{I \setminus J} = y_{I \setminus J}$$

holds for all $J \subseteq I$.

In other words, the definition of the induced relation says that $a \lesssim_J b$ holds for $a, b \in \prod_{j \in J} M_j$ if there exists a simultaneous extension of a and b to elements $a', b' \in \prod_{i=1}^n M_i$ with $a' \lesssim b'$. Then indepence says that is does not matter how we extend a and b. In fact, this enables us to prove that the induced relation is again a (linear) quasi-order.

Lemma 1.1. *Let $\lesssim$ be an independent (linear) quasi-order and let $J \subseteq I$. Then $\lesssim_J$ is a (linear) quasi-order on $\prod_{j \in J} M_j$, which is called the* **induced (linear) quasi-order** *on $\prod_{j \in J} M_j$.*

Proof. Let $a, b, c \in \prod_{j \in J} M_j$ be arbitrary. If $a' \in \prod_{i=1}^n M_i$ is an arbitrary extension of a, then $a' \lesssim a'$ and therefore $a \lesssim_J a$, i.e., $\lesssim_J$ is reflexive.

If $a \lesssim_J b$ and $b \lesssim_J c$ then there are extensions $a', b', b'', c' \in \prod_{i=1}^n M_i$ with $a' \lesssim b'$, $b'' \lesssim c'$, and the required projection properties. Since $\lesssim$ is independent, we can choose $b'' = b'$, and by transitivity of $\lesssim$ we conclude $a' \lesssim c'$. This implies $a \lesssim_J c$, hence $\lesssim_J$ is transitive.

In the linear case, for any extensions $a', b' \in \prod_{i=1}^n M_i$ with $a'_J = a$, $b'_J = b$ and $a'_{I \setminus J} = b'_{I \setminus J}$ we have $a' \lesssim b'$ or $b' \lesssim a'$. Hence, $a \lesssim_J b$ or $b \lesssim_J a$ holds, and $\lesssim_J$ is connected. □

Definition 1.3. Let $M_1, \ldots, M_n$, and $\lesssim$ be as in Definition 1.2. Further, let $\sim$ denote the equivalence relation $\lesssim \cap \gtrsim$. Then $\lesssim$ satisfies **restricted solvability** provided that for every $i \in I$, whenever

$$(x_1, \ldots, \underline{x}_i, \ldots x_n) \lesssim (y_1, \ldots, y_i, \ldots, y_n) \lesssim (x_1, \ldots, \overline{x}_i, \ldots x_n),$$

then there is $x_i \in M_i$ such that

$$(x_1, \ldots, x_i, \ldots, x_n) \sim (y_1, \ldots, y_i, \ldots, y_n).$$

The equivalence relation $\sim$ indicates which elements cannot be distinguished by $\lesssim$. The meaning of restricted solvability is shown in Figure 1, where the factor M_i is drawn horizontally and the other factors are drawn vertically. The quasi-order $\lesssim$ becomes greater from the lower left to the upper right corner; the diagonal line is an equivalence class of $\sim$.

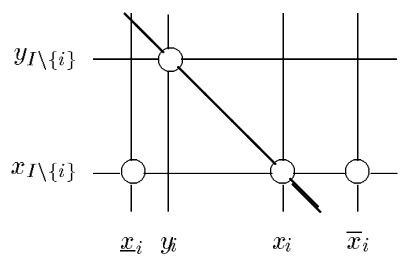

Figure 1. Restricted solvability.

Definition 1.4. A factor M_i is called **essential** for $\lesssim$ if $\lesssim_{\{i\}} \neq \gtrsim_{\{i\}}$.

A factor is essential if the induced quasi-order is not already an equivalence relation. In the linear case, this means that it still contains some strict comparability.

Definition 1.5. For any $i \in I$ and any set N of consecutive integers (positive or negative, finite or infinite), a subset $\{x_l \mid l \in N\}$ of M_i is called a **standard sequence** on the factor M_i if there are

$$(y_1, \ldots y_{i-1}, y_{i+1}, \ldots, y_n) \not\sim_{I \setminus \{i\}} (z_1, \ldots z_{i-1}, z_{i+1}, \ldots, z_n)$$

such that

$$(y_1, \dots y_{i-1}, x_l, y_{i+1}, \dots, y_n) \sim (z_1, \dots z_{i-1} x_{l+1}, z_{i+1}, \dots, z_n)$$

holds for all $l, l+1 \in N$.
A standard sequence $\{x_l \mid l \in N\}$ on the factor M_i is **strictly upper (lower) bounded** if there is $u \in M_i$ such that $x_l <_{\{i\}} u$ ($u <_{\{i\}} x_l$) for all $l \in N$. Here $<$ denotes the relation $\lesssim \setminus \sim$, and $<_{\{i\}}$ is the induced quasi-order of $<$ on M_i (observe $<_{\{i\}} = \lesssim_{\{i\}} \setminus \sim_{\{i\}}$). Furthermore, $\{x_l \mid l \in N\}$ is **strictly bounded** if it is strictly upper and strictly lower bounded.

Figure 2 shows a typical standard sequence. Standard sequences are introduced in order to create a notion of equidistance. Hence, strictly bounded standard sequences will be used to model Archimedian properties.

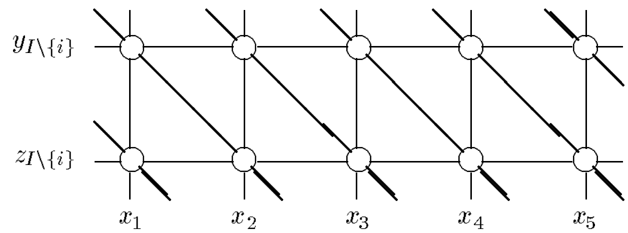

Figure 2. A standard sequence $\{x_1, x_2, x_3, x_4, x_5\}$.

Lemma 1.2. *Let $\lesssim$ be an independent linear quasi-order and let $\{x_l \mid l \in N\}$ be a standard sequence on a factor M_i. Then either $x_l <_{\{i\}} x_{l+1}$ holds for all $l \in N$ or $x_{l+1} <_{\{i\}} x_l$ holds for all $l \in N$.*

Proof. Since $\{x_l \mid l \in N\}$ is a standard sequence, there are

$$(y_1, \dots y_{i-1}, y_{i+1}, \dots, y_n) \not\sim_{I\setminus\{i\}} (z_1, \dots z_{i-1}, z_{i+1}, \dots, z_n)$$

with the properties asserted in Definition 1.5. We have to distinguish two cases for the comparability of these two elements. If

$$(y_1, \dots y_{i-1}, y_{i+1}, \dots, y_n) <_{I\setminus\{i\}} (z_1, \dots z_{i-1}, z_{i+1}, \dots, z_n)$$

holds, then we conclude

$$(y_1, \dots y_{i-1}, x_l, y_{i+1}, \dots, y_n) < (z_1, \dots z_{i-1}, x_l, z_{i+1}, \dots, z_n)$$

for all $l \in N$. By the definition of a standard sequence, this implies

$$(z_1, \dots z_{i-1}, x_{l+1}, z_{i+1}, \dots, z_n) < (z_1, \dots z_{i-1}, x_l, z_{i+1}, \dots, z_n)$$

and therefore $x_{l+1} <_{\{i\}} x_l$. Since $\lesssim$ is independent, $\lesssim_{\{i\}}$ is a linear quasi-order. Thus, we can conclude that $x_l <_{\{i\}} x_{l+1}$ cannot hold. In the second case, we have

$$(z_1, \dots z_{i-1}, z_{i+1}, \dots, z_n) <_{I\setminus\{i\}} (y_1, \dots y_{i-1}, y_{i+1}, \dots, y_n),$$

which analogously implies $x_l <_{\{i\}} x_{l+1}$ and not $x_{l+1} <_{\{i\}} x_l$ for all $l \in N$. $\square$

Now we are ready to define our basic structure. We follow the definition which is given in [**3**].

Definition 1.6. Let $M_1, \dots, M_n$, $n \geq 3$, be nonempty sets and let $\lesssim$ be a linear quasi-order on $\prod_{i=1}^n M_i$. Then $((M_i)_{i=1}^n, \lesssim)$ is an n**-component additive conjoint structure** if

(1) $\lesssim$ is independent,
(2) $\lesssim$ satisfies restricted solvability,
(3) every factor M_i is essential for $\lesssim$, and
(4) every strictly bounded standard sequence is finite.

Typical examples are discussed in the next section. With regard to the purpose of this paper, we can sketch our program as follows: Find for every factor M_i a (minimal) complete lattice L_i such that M_i can be embedded into L_i, and $((M_i)_{i=1}^n, \lesssim)$ can be embedded into $((L_i)_{i=1}^n, \sqsubseteq)$ for an appropriate linear quasi-order $\sqsubseteq$. We close this section with the introduction of the notion of embedding we need.

Definition 1.7. Let $(M, \lesssim)$ and $(P, \sqsubseteq)$ be quasi-ordered sets. A map $\varphi\colon M \longrightarrow P$ is called a **quasi-embedding** of $(M, \lesssim)$ into $(P, \sqsubseteq)$ if

$$x \lesssim y \iff \varphi(x) \sqsubseteq \varphi(y)$$

holds for all $x, y \in M$.

Observe that a quasi-embedding does not need to be injective. However, $\varphi(x) = \varphi(y)$ implies always $x \sim y$, i.e., any quasi-embedding identifies at most the equivalent elements.

2. Debreu's Theorem

A prototype example of an n-component additive conjoint structures is given by the real numbers: $((\mathbb{R})_{i=1}^n, \lesssim)$ is an n-component additive conjoint structure if we define

$$(1) \qquad (x_1, \dots, x_n) \lesssim (y_1, \dots, y_n) \iff x_1 + \dots + x_n \leq y_1 + \dots + y_n,$$

where $\leq$ is the usual linear order on $\mathbb{R}$. Similar examples can be obtained by replacing $\mathbb{R}$ by the rational numbers $\mathbb{Q}$, the integers $\mathbb{Z}$, or the unit interval $[0, 1]$. All these examples may serve as motivations for the name "additive conjoint structure": If we interprete the n factors as representations of n different attributes, then these attributes are mutually dependent. The dependence can be expressed by the "additive" inequality above.

In fact, $((\mathbb{R})_{i=1}^n, \lesssim)$ is not only a prototype. Debreu's Theorem ([**1**]) says that every n-component additive conjoint structure can be embedded into $((\mathbb{R})_{i=1}^n, \lesssim)$. We cite Debreu's Theorem in the way it is stated in [**3**].

Theorem 2.1 [Debreu's Theorem]. *If $((M_i)_{i=1}^n, \lesssim)$, $n \geq 3$, is an n-component additive conjoint structure then there exist functions $\varphi_i \colon M_i \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$ such that for all $x_i, y_i \in M_i$ the equivalence*

$$(x_1, \ldots, x_n) \lesssim (y_1, \ldots, y_n) \iff \sum_{i=1}^{n} \varphi_i(x_i) \leq \sum_{i=1}^{n} \varphi_i(y_i)$$

holds. If $\psi_1, \ldots, \psi_n$ is another such family of functions, then there exist real numbers $r > 0$ and s_i, $i = 1, \ldots, n$, with

$$\psi_i(x_i) = r\varphi_i(x_i) + s_i.$$

In terms of Definition 1.7, $(\varphi_1, \ldots, \varphi_n)$ is a quasi-embedding of $((M_i)_{i=1}^n, \lesssim)$ into $((\mathbb{R})_{i=1}^n, \lesssim)$. In the introdution, the differences between the analytical and the synthetical approach for the construction of such quasi-embeddings were mentioned. Clearly, Debreu's Theroem follows the analytical approach. The advantage of this approach concerning representations of additive conjoint structures is that we can calculate with real numbers and use the usual linear order instead of making considerations using the rather complex properties of the original linear quasi-order $\lesssim$ on $\prod_{i=1}^n M_i$. Contrary to this, a synthetical construction of an embedding may lead to a deeper understanding of the complex properties of $\lesssim$ and their dependencies.

From Debreu's Theorem we obtain an analytical solution of our completion problem. For a given n-component additive conjoint structure $((M_i)_{i=1}^n, \lesssim)$, $n \geq 3$, we define

$$L_i := \bigl\{x \in \mathbb{R} \mid \text{there is } X \subseteq \varphi_i(M_i) \text{ such that } x = \sup X \text{ or } x = \inf X\bigr\}.$$

In other words, L_i is the "complete sublattice" of $\mathbb{R}$ which is generated by $\varphi_i(M_i)$. The quotes in the previous sentence indicate that L_i might not be a complete lattice since it may fail to have a least or greatest element. If $\sqsubseteq$ denotes the restriction of the linear quasi-order defined in (1) to $\prod_{i=1}^n L_i$, then $((L_i)_{i=1}^n, \sqsubseteq)$ is the desired order-completion. The aim of this paper is to present a synthetical solution of the completion problem. Before we start to do this, we must discuss the little failure of completeness which was already mentioned above.

Suppose that M_i already equals $\mathbb{R}$ for all i. With the above approach, L_i equals $\mathbb{R}$, too. Hence, $(L_i, \leq)$ is not a complete lattice because it has no least or greatest element. It turns out that this is not a lack of the construction but a basic consequence of the Archimedian properties of additive conjoint structures. Hence, we cannot go beyond this point and must allow that every factor in our completion may or may not have a least or greatest element. The following proposition gives a precise formulation.

Proposition 2.1. *Let $((M_i)_{i=1}^n, \lesssim)$, $n \geq 3$, be an n-component additive conjoint structure. If there is a strictly lower (upper) bounded infinite standard sequence on a factor M_i, then the quasi-ordered set $(M_i, \lesssim_{\{i\}})$ does not have a greatest (least) element.*

Proof. Assume that there exists a strictly lower bounded infinite standard sequence $\{x_l \mid l \in \mathbb{N}\}$ on the factor M_i with strict lower bound x. Let us conclude first that $x_l <_{\{i\}} x_{l+1}$ for all $l \in \mathbb{N}$. Otherwise, we would have $x <_{\{i\}} x_l <_{\{i\}} x_1$ for all $k \geq 2$ by Lemma 1.2 But then $\{x_l \mid l \geq 2\}$ would be an infinite strictly bounded standard sequence, what cannot be the case by Definition 1.6.

Suppose now that $(M_i, \lesssim_{\{i\}})$ has a greatest element 1. Then we must have $x_l <_{\{i\}} 1$, because otherwise 1 could not be the greatest element. Thus, $x <_{\{i\}} x_l <_{\{i\}} < 1$ holds for all $l \in \mathbb{N}$. Again, we would have an infinite strictly bounded standard sequence, in contradiction to the assumption. □

3. Solvability and Thomsen Conditions

We start now with our synthetical construction of the order-completion of an additive conjoint structure. In order to prepare this, we must say some more words about solvability and Thomsen Conditions. The notion of restricted solvability can be generalized in several ways. One possibility is to specify less than $n-1$ coordinates of the element we are searching for, as it is formulated in the following lemma.

Lemma 3.1. *Let $((M_i)_{i=1}^n, \lesssim)$, $n \geq 3$, be an n-component additive conjoint structure. Let $J \subseteq \{1, \dots, n\}$ and let $x, y, z \in \prod_{i=1}^n M_i$ such that $x \lesssim y \lesssim z$ and $x_J = z_J$. Then there is $w \in \prod_{i=1}^n M_i$ with $w_J = x_J$ and $y \sim w$.*

Proof. If $J = \emptyset$ then we can choose $w := y$. If $J = \{1, \dots, n\}$ we have $x = z$ and therefore $x \sim y$. Thus, we can choose $w := x$. For the other cases, w.l.o.g., let $J := \{1, \dots, j\}$ for some $j \in \{1, \dots, n-1\}$. Further, let

$$x =: (x_1, \dots, x_n) \quad \text{and} \quad z =: (x_1, \dots, x_j, z_{j+1}, \dots, z_n).$$

Now, we consider the elements

$$\begin{aligned}&(x_1, \dots, x_j, x_{j+1}, \dots, x_n),\\&(x_1, \dots, x_j, x_{j+1}, \dots, x_{n-1}, z_n),\\&\qquad\dots\\&(x_1, \dots, x_j, z_{j+1}, \dots, z_n).\end{aligned}$$

Since $\lesssim$ is connected, all these elements are comparable, and $x \lesssim y \lesssim z$ implies that there exists some $k \in \{j+1, \dots, n\}$ with

$$\begin{aligned}(x_1, \dots, x_j, x_{j+1}, \dots, x_{k-1}, x_k, z_{k+1}, \dots, z_n) \lesssim y\\ \lesssim (x_1, \dots, x_j, x_{j+1}, \dots, x_{k-1}, z_k, z_{k+1}, \dots, z_n).\end{aligned}$$

By restricted solvability, we get some $w_k \in M_k$ such that

$$y \sim (x_1, \dots, x_j, x_{j+1}, \dots, x_{k-1}, w_k, z_{k+1}, \dots, z_n),$$

what completes the proof. $\square$

The following generalization is also often considered in measurement theory.

Definition 3.1. The linear quasi-order $\lesssim$ on $((M_i)_{i=1}^n, \lesssim)$ satisfies **general solvability** if for every $i \in \{1, \dots n\}$ and all $x, y \in \prod_{i=1}^n M_i$ there is some $w \in \prod_{i=1}^n M_i$ with $x_{I\setminus\{i\}} = w_{I\setminus\{i\}}$ and $y \sim w$.

The "prototype" $((\mathbb{R})_{i\in I}, \lesssim)$ also satisfies general solvability. In [**4**], a synthetical proof of the following embedding theorem is given.

Theorem 3.1. *For $n \geq 3$, every n-component additive conjoint structure has a quasi-embedding into an n-component additive conjoint structure which satisfies general solvability.*

For the proof of this theorem, we need the validity of the Thomsen Condition, which also can be shown synthetically (see [**3**, p. 306f]). The geometrical meaning of the condition is shown in Figure 3.

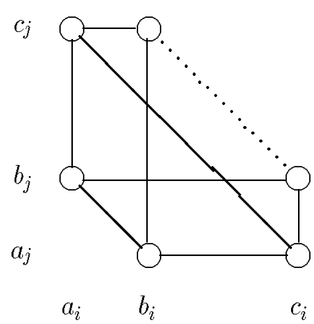

Figure 3. The Thomsen condition.

Proposition 3.1. *Let $((M_i)_{i=1}^n, \precsim)$, $n \geq 3$, be an n-component additive conjoint structure. Let $i \neq j \in \{1, \dots n\}$. Then $((M_i)_{i=1}^n, \precsim)$ satisfies the Thomsen Condition on the factors M_i and M_j, i.e., for any $a_i, b_i, c_i \in M_i$ and $a_j, b_j, c_j \in M_j$ the equivalences $(a_i, b_j) \sim_{\{i,j\}} (b_i, a_j)$ and $(a_i, c_j) \sim_{\{i,j\}} (c_i, a_j)$ imply $(b_i, c_j) \sim_{\{i,j\}} (c_i, b_j)$.*

For the synthetical construction of the order-completion of an additive conjoint structure, we need a generalized version of the Thomsen Condition, which is no longer restricted to two factors. As notation we introduce that $(x_J, y_{I \setminus J})$ for $x_J \in \prod_{j \in J} M_j$ and $y_{I \setminus J} \in \prod_{j \in I \setminus J} M_j$ denotes the unique element $z \in \prod_{i \in I} M_i$ with $z_J = x_J$ and $z_{I \setminus J} = y_{I \setminus J}$.

Proposition 3.2. *Let $((M_i)_{i=1}^n, \precsim)$, $n \geq 3$, be an n-component additive conjoint structure. $((M_i)_{i=1}^n, \precsim)$ satisfies the* **generalized Thomsen Condition***, i.e., for every nonempty $J \subset I := \{1, \dots, n\}$ and all $a_J, b_J, c_J \in \prod_{j \in J} M_j$ and $a_{I \setminus J}, b_{I \setminus J}, c_{I \setminus J} \in \prod_{j \in I \setminus J} M_j$ the equivalences $(a_J, b_{I \setminus J}) \sim (b_J, a_{I \setminus J})$ and $(a_J, c_{I \setminus J}) \sim (c_J, a_{I \setminus J})$ imply $(b_J, c_{I \setminus J}) \sim (c_J, b_{I \setminus J})$.*

Proof. We will give the proof for an additive conjoint structure with general solvability. This is sufficient because of Theorem 3.1, since we can embed the original structure into one which satisfies general solvability. If the generalized Thomsen Condition holds there, it must hold in the original structure because we have a quasi-embedding.

If $J = \emptyset$ or $J = I$, the proposition is trivial. In all other cases, w.l.o.g., we may assume that $J = \{1, \dots, j\}$ with $1 \leq j < n$. We define $a_J =: (a_1, \dots, a_j)$, $b_J =: (b_1, \dots, b_j)$, $c_J =: (c_1, \dots, c_j)$, $a_{I \setminus J} =: (a_{j+1}, \dots, a_n)$, $b_{I \setminus J} =: (b_{j+1}, \dots, b_n)$, and $c_{I \setminus J} =: (c_{j+1}, \dots, c_n)$. The assumptions of the proposition read then as

$$
\begin{aligned}
(a_1, \dots, a_j, b_{j+1}, \dots, b_n) &\sim (b_1, \dots, b_j, a_{j+1}, \dots, a_n), \\
(a_1, \dots, a_j, c_{j+1}, \dots, c_n) &\sim (c_1, \dots, c_j, a_{j+1}, \dots, a_n).
\end{aligned} \tag{2}
$$

Within two steps, we will reduce now the general Thomsen Condition to the two-dimensional Thomsen Condition. By general solvability, there exist $b_n', c_n' \in M_n$ such that

$$
\begin{aligned}
(b_1, \dots, b_j, c_{j+1}, \dots, c_n) &\sim (b_1, \dots, b_j, a_{j+1}, \dots, a_{n-1}, c_n'), \\
(c_1, \dots, c_j, b_{j+1}, \dots, b_n) &\sim (c_1, \dots, c_j, a_{j+1}, \dots, a_{n-1}, b_n').
\end{aligned}
$$

Using independence, we obtain also

$$
\begin{aligned}
(a_1, \dots, a_j, b_{j+1}, \dots, b_n) &\sim (a_1, \dots, a_j, a_{j+1}, \dots, a_{n-1}, b_n'), \\
(a_1, \dots, a_j, c_{j+1}, \dots, c_n) &\sim (a_1, \dots, a_j, a_{j+1}, \dots, a_{n-1}, c_n').
\end{aligned}
$$

Analogously, we can find $b_1', c_1' \in M_1$ such that

$$\begin{aligned}
(a_1, \dots, a_j, b_{j+1}, \dots, b_n) &\sim (a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, b_n'),\\
(a_1, \dots, a_j, c_{j+1}, \dots, c_n) &\sim (a_1, a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, c_n'),\\
(b_1, \dots, b_j, a_{j+1}, \dots, a_n) &\sim (b_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, a_n),\\
(b_1, \dots, b_j, c_{j+1}, \dots, c_n) &\sim (b_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, c_n'),\\
(c_1, \dots, c_j, a_{j+1}, \dots, a_n) &\sim (c_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, a_n),\\
(c_1, \dots, c_j, b_{j+1}, \dots, b_n) &\sim (c_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, b_n').
\end{aligned}$$

Considering this together with the equivalences (2), we can apply Proposition 3.1 for the factors M_1 and M_n. We obtain

$$\begin{aligned}
(b_1, \dots, b_j, c_{j+1}, \dots, c_n) &\sim (b_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, c_n')\\
&\sim (c_1', a_2, \dots, a_j, a_{j+1}, \dots, a_{n-1}, b_n')\\
&\sim (c_1, \dots, c_j, b_{j+1}, \dots, b_n)
\end{aligned}$$

what completes the proof. □

4. Dedekind-MacNeille Completions

In order to construct the order-completion of an additive conjoint structure, we start with the construction of a complete lattice from a quasi-ordered set. The background of this construction comes from **Formal Concept Analysis** (see [**5**]), but in this paper we elaborate the theory only with respect to our needs. For the entire framework and the proofs, the reader is referred to [**2**] and [**5**]. For any quasi-ordered set $(M, \lesssim)$ and any $A \subseteq M$, we define

$$\begin{aligned}
A^{\lesssim} &:= \{m \in M \mid a \lesssim m \quad \text{for all } a \in A\} \quad \text{and}\\
A^{\gtrsim} &:= \{m \in M \mid a \gtrsim m \quad \text{for all } a \in A\}
\end{aligned}$$

Further, $\mathfrak{D}(M, \lesssim)$ denotes the set of all pairs (A, B) with $A, B \subseteq M$ and $A^{\lesssim} = B$, $B^{\gtrsim} = A$.

Theorem 4.1. *Let $(M, \lesssim)$ be a quasi-ordered set. An order relation $\leq$ is defined by*

$$(A_1, B_1) \leq (A_2, B_2) \quad :\Longleftrightarrow \quad A_1 \subseteq A_2 \quad (\Longleftrightarrow \quad B_1 \supseteq B_2)$$

for $(A_1, B_1), (A_2, B_2) \in \mathfrak{D}(M, \lesssim)$. Then $\underline{\mathfrak{D}}(M, \lesssim) := (\mathfrak{D}(M, \lesssim), \leq)$ is a complete lattice, called the **Dedekind-MacNeille Completion** *of $(M, \lesssim)$. Infima and* *suprema in* $\underline{\mathfrak{D}}(M,\lesssim)$ *are given by*

$$\bigwedge_{t\in T}(A_t,B_t)=\left(\bigcap_{t\in T}A_t,\left(\bigcup_{t\in T}B_t\right)^{\gtrsim\lesssim}\right)\quad \textit{and}$$

$$\bigvee_{t\in T}(A_t,B_t)=\left(\left(\bigcup_{t\in T}A_t\right)^{\lesssim\gtrsim},\bigcap_{t\in T}B_t\right).$$

The name "Dedekind-MacNeille Completion" is motivated by the following theorem.

Theorem 4.2. *For every quasi-ordered set* $(M,\lesssim)$*, a quasi-embedding* ι *of* $(M,\lesssim)$ *into* $(\underline{\mathfrak{D}}(M,\lesssim)$ *is defined by* $\iota(m):=(m^{\gtrsim},m^{\lesssim})$ *for all* $m\in M$*. If* φ *is also a quasi-embedding of* $(M,\lesssim)$ *into some complete lattice* $\mathbf{L}$*, then there is a lattice embedding* ψ *of* $\underline{\mathfrak{D}}(M,\lesssim)$ *into* $\mathbf{L}$ *such that* $\varphi=\psi\circ\iota$*.*

This theorem says that, in the sense of embeddings, $\underline{\mathfrak{D}}(M,\lesssim)$ is the smallest complete lattice in which $(M,\lesssim)$ can be quasi-embedded. The elements of this lattice are the **Dedekind cuts** of the quasi-ordered set. We finish this section by stating some technical facts we need in the further development.

Lemma 4.1. *Let* $(M,\lesssim)$ *be a quasi-ordered set and let* $A,B\subseteq M$*. Then*

(1) $A\subseteq A^{\lesssim\gtrsim}$ *and* $B\subseteq B^{\gtrsim\lesssim}$,
(2) $A^{\lesssim}=A^{\lesssim\gtrsim\lesssim}$ *and* $B^{\gtrsim}=B^{\gtrsim\lesssim\gtrsim}$,
(3) $A\subseteq B$ *implies* $B^{\lesssim}\subseteq A^{\lesssim}$ *and* $B^{\gtrsim}\subseteq A^{\gtrsim}$,
(4) $A\subseteq B$ *implies* $A^{\lesssim\gtrsim}\subseteq B^{\lesssim\gtrsim}$ *and* $A^{\gtrsim\lesssim}\subseteq B^{\gtrsim\lesssim}$,
(5) $(A^{\lesssim\gtrsim},A^{\lesssim})$ *and* $(B^{\gtrsim},B^{\gtrsim\lesssim})$ *are always elements of* $\underline{\mathfrak{D}}(M,\lesssim)$*.*

5. A Synthetical Completion of Additive Conjoint Structures

Now, we start to construct the order-completion of an additive conjoint structure. We will assume that always $I:=\{1,\dots,n\}$, $n\geq 3$, and that $\mathbf{M}:=((M_i)_{i\in I},\lesssim)$ is an n-component additive conjoint structure. Our aim is to introduce a linear quasi-order on the product of the Dedekind-MacNeille Completions of the factors, such that we obtain a new order-complete n-component additive conjoint structure in which $\mathbf{M}$ can be quasi-embedded. Proposition 2.1 already shows that we have to be careful with the least and greatest elements of the Dedekind-MacNeille Completions. If there are infinite standard sequences on a factor M_i, then we will have problems with the least or the greatest element of $\underline{\mathfrak{D}}(M_i,\lesssim_{\{i\}})$. On the other hand, if there are no such sequences, it makes sense to include the least and greatest elements of the completion. As an example, let us consider the case where each factor M_i is the open unit interval $(0,1)$. As the completion of an M_i we want to have the closed unit interval $[0,1]$, which clearly makes it neccessary to include the least and the greatest elements of the Dedekind-MacNeille Completion. In order to cover this problem, we will complete the structure in two steps. In the first step, we will add all “internal” infima and suprema and will worry only about the greatest elements of the factors. By dualizing our argumentation, we will add the least elements in a second step, whenever this is possible.

For every $i \in I$, let $M_i^{\uparrow} := \mathfrak{D}(M_i, \precsim_{\{i\}}) \setminus \{(\emptyset, M_i)\}$ if every strictly lower bounded standard sequence on the factor M_i is finite; else, let $M_i^{\uparrow} := \mathfrak{D}(M_i, \precsim_{\{i\}}) \setminus \{(\emptyset, M_i), (M_i, \emptyset)\}$. With this definition we achieve that we have no greatest element $(M_i, \emptyset)$ in $(M_i^{\uparrow}, \precsim_{\{i\}})$ which may cause trouble. In either case, a least element is there if and only if $(M_i, \precsim_{\{i\}})$ already has a least element, since $(\emptyset, M_i)$ is the least element of $\mathfrak{D}(M_i, \precsim_{\{i\}})$ if and only if $(M_i, \precsim_{\{i\}})$ has no least element. A binary relation $\underset{\sim}{\sqsubset}$ on $\prod_{i=1}^n M_i^{\uparrow}$ is defined by

$$((A_i, B_i))_{i=1}^n \underset{\sim}{\sqsubset} ((C_i, D_i))_{i=1}^n \quad :\Longleftrightarrow \quad \left(\prod_{i=1}^n A_i\right)^{\precsim\succsim} \subseteq \left(\prod_{i=1}^n C_i\right)^{\precsim\succsim}.$$

Further, let $\mathbf{M}^{\uparrow} := ((M_i^{\uparrow})_{i=1}^n, \underset{\sim}{\sqsubset})$. Our aim is to prove that $\mathbf{M}^{\uparrow}$ is an n-component additive conjoint structure in which $\mathbf{M}$ can be quasi-embedded. As a first step, we make the relation $\underset{\sim}{\sqsubset}$ accessible on a more elementary level.

Lemma 5.1. *For $((A_i, B_i))_{i=1}^n, ((C_i, D_i))_{i=1}^n \in \prod_{i=1}^n M_i^{\uparrow}$, the following are equivalent:*

(1) $((A_i, B_i))_{i=1}^n \underset{\sim}{\sqsubset} ((C_i, D_i))_{i=1}^n$.
(2) *Let $(z_1, \ldots, z_n) \in \prod_{i=1}^n M_i$ be arbitrary. If $(c_1, \ldots, c_n) \precsim (z_1, \ldots, z_n)$ holds for all $(c_1, \ldots, c_n) \in \prod_{i=1}^n C_i$, then $(a_1, \ldots, a_n) \precsim (z_1, \ldots, z_n)$ holds for all $(a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$.*

Proof. By definition and Lemma 4.1, condition (1) is equivalent to

$$\left(\prod_{i=1}^n C_i\right)^{\precsim} \subseteq \left(\prod_{i=1}^n A_i\right)^{\precsim},$$

which is precisely the assertion of (2). □

Lemma 5.2. $\underset{\sim}{\sqsubset}$ *is a linear quasi-order.*

Proof. Obviously, $\underset{\sim}{\sqsubset}$ is transitive. Since $\precsim$ was linear, the complete lattice $\underline{\mathfrak{D}}(\prod_{i=1}^n M_i, \precsim)$ is a linear ordered set. Thus, $\underset{\sim}{\sqsubset}$ is connected. □

Lemma 5.3. *Let $((A_i, B_i))_{i=1}^n \sqsubset ((C_i, D_i))_{i=1}^n$. Then there is some $(c_1, \ldots, c_n) \in \prod_{i=1}^n C_i$ such that $(a_1, \ldots, a_n) < (c_1, \ldots, c_n)$ holds for all $(a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$.*

Proof. Suppose, for all $(c_1, \ldots, c_n) \in \prod_{i=1}^n C_i$ there is some $(a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$ with $(c_1, \ldots, c_n) \precsim (a_1, \ldots, a_n)$. By Lemma 5.1 we get $((C_i, D_i))_{i=1}^n \underset{\sim}{\sqsubset}$

$((A_i,B_i))_{i=1}^n$. Since $\sqsubseteq$ is a quasi-order by Lemma 5.2, this contradicts the assumption. □

Now we are ready to start the proof that $\mathbf{M}^\uparrow$ is an additive conjoint structure.

Lemma 5.4. *$\sqsubseteq$ is independent.*

Proof. First we consider $J \subseteq I$ with $|J| = n-1$; w.l.o.g., let $J := \{2,\dots,n\}$. Further, let $((A_i,B_i))_{i=2}^n \sqsubseteq_J ((C_i,D_i))_{i=2}^n$. Then there is $(E,F) \in M_1^\uparrow$ such that

$$((E,F),(A_2,B_2),\dots,(A_n,B_n)) \sqsubseteq ((E,F),(C_2,D_2),\dots,(C_D,D_n)). \tag{3}$$

Suppose there would be $(G,H) \in M_1^\uparrow$ with

$$((G,H),(A_2,B_2),\dots,(A_n,B_n)) \not\sqsubseteq ((G,H),(C_2,D_2),\dots,(C_D,D_n)). \tag{4}$$

By Lemma 5.1, there would be $(u_1,\dots,u_n) \in \prod_{i=1}^n M_i$ and $(g,a_2,\dots,a_n) \in G \times \prod_{i=2}^n A_i$ such that $(\tilde g,c_2,\dots,c_n) \lesssim (u_1,\dots,u_n) < (g,a_2,\dots,a_n)$ holds for all $(\tilde g,c_2,\dots,c_n) \in G \times \prod_{i=2}^n C_i$. Especially, we have

$$(g,c_2,\dots,c_n) \lesssim (u_1,\dots,u_n) < (g,a_2,\dots,a_n)$$

for all $(c_2,\dots,c_n) \in \prod_{i=2}^n C_i$. By Definition 1.3, we may assume $u_1 = g$. From the independence of $\lesssim$, we conclude

$$(x,c_2,\dots,c_n) \lesssim (x,u_2,\dots,u_n) < (x,a_2,\dots,a_n)$$

for all $(x,c_2,\dots,c_n) \in M_1 \times \prod_{i=2}^n C_i$. Furthermore,

$$(c_2,\dots,c_n) \lesssim_J (u_2,\dots,u_n) <_J (a_2,\dots,a_n)$$

holds for all $(c_2,\dots,c_n) \in \prod_{i=2}^n C_i$. Now, we fix $e \in E$. We want to prove that there is $\bar e \in E$ with $(e,a_2,\dots,a_n) \lesssim (\bar e,u_2,\dots,u_n)$. By (3) and Lemma 5.1, there is $(\tilde e,\tilde c_2,\dots,\tilde c_n) \in E \times \prod_{i=2}^n C_i$ with $(e,u_2,\dots,u_n) < (\tilde e,\tilde c_2,\dots,\tilde c_n)$, because otherwise $(e,u_2,\dots,u_n)$ would be an upper bound of $E \times \prod_{i=2}^n C_i$, but not of $E \times \prod_{i=2}^n A_i$. Since we have $(\tilde c_2,\dots,\tilde c_n) \lesssim_J (u_2,\dots,u_n)$, this implies $e <_{\{1\}} \tilde e$ and therefore $(e,a_2,\dots,a_n) < (\tilde e,a_2,\dots,a_n)$. Thus, $(e,a_2,\dots,a_n)$ cannot be an upper bound of $E \times \prod_{i=2}^n A_i$. With the same argument as before, we conclude from this that there is $(\bar e,\bar c_2,\dots,\bar c_n) \in E \times \prod_{i=2}^n C_i$ with $(e,a_2,\dots,a_n) < (\bar e,\bar c_2,\dots,\bar c_n)$. This implies $(e,a_2,\dots,a_n) \lesssim (\bar e,u_2,\dots,u_n)$ because of $(\bar c_2,\dots,\bar c_n) \lesssim_J (u_2,\dots,u_n)$.

Now we are able to construct a stricly lower bounded standard sequence in E. For this, let $e_0 \in E$ be arbitrary. By the above, there is $\bar e_0 \in E$ such that

$$(e_0,u_2,\dots,u_n) < (e_0,a_2,\dots,a_n) \lesssim (\bar e_0,u_2,\dots,u_n),$$

and by restricted solvability we get $e_1 \in M_1$ with $(e_0, a_2, \dots, a_n) \sim (e_1, u_2, \dots, u_n)$. This implies $e_0 <_{\{1\}} e_1 \lesssim_{\{1\}} \bar{e}_0$. Since $e_1 \lesssim_{\{1\}} \bar{e}_0$, we have $e_1 \in E$. From e_1 we construct $e_2 \in E$ in the same fashion, etc., and get an infinite sequence

$$e_0 <_{\{1\}} e_1 <_{\{1\}} e_2 <_{\{1\}} \dots$$

in E. We claim that the sequence $\{e_l \mid l \in \mathbb{N}\}$ is a strictly lower bounded standard sequence on the factor M_1. In order to see this, consider the elements $y := (a_2, \dots, a_n)$ and $z := (u_2, \dots, u_n)$. Now we distinguish two cases:

(i) If $F \neq \emptyset$, there is $f \in F$ with $e_l \lesssim_{\{1\}} f$ for all $l \in \mathbb{N}$. In fact, we have $e_l <_{\{1\}} f$, because $e_{l_0} \sim_{\{1\}} f$ for some $l_0 \in \mathbb{N}$ would imply $f <_{\{1\}} e_{l_0+1}$. Hence, $\{e_l \mid l \in \mathbb{N}\}$ is a strictly bounded standard sequence on the factor M_1, and therefore finite, in contradiction to the construction of the sequence.

(ii) If $F = \emptyset$, then $\{e_l \mid l \in \mathbb{N}\}$ has to be finite by the definition of $M_1^{\uparrow}$.

Thus, in every case, (4) leads to a contradiction, what completes the proof of the case $|J| = n - 1$.

Now, let $|J| < n - 1$, w.l.o.g., let $J := \{j+1, \dots, n\}$ for some $j \geq 2$. Let $((A_i, B_i))_{i=j+1}^n \sqsubseteq_J ((C_i, D_i))_{i=j+1}^n$. Then there is $((E_i, F_i))_{i=1}^j \in \prod_{i=1}^j M_i^{\uparrow}$ where

$$\begin{aligned}((E_1, F_1), \dots, (E_j, F_j), (A_{j+1}, B_{j+1}), \dots, (A_n, B_n))\\ \sqsubseteq ((E_1, F_1), \dots, (E_j, F_j), (C_{j+1}, D_{j+1}), \dots, (C_n, D_n)).\end{aligned}$$

The first part of the proof yields now that

$$\begin{aligned}((G_1, H_1), (E_2, F_2), \dots, (E_j, F_j), (A_{j+1}, B_{j+1}), \dots, (A_n, B_n))\\ \sqsubseteq ((G_1, H_1), (E_2, F_2), \dots, (E_j, F_j), (C_{j+1}, D_{j+1}), \dots, (C_n, D_n))\end{aligned}$$

holds for all $(G_1, H_1) \in M_1^{\uparrow}$. By induction, we get

$$\begin{aligned}((G_1, H_1), \dots, (G_j, H_j), (A_{j+1}, B_{j+1}), \dots, (A_n, B_n))\\ \sqsubseteq ((G_1, H_1), \dots, (G_j, H_j), (C_{j+1}, D_{j+1}), \dots, (C_n, D_n))\end{aligned}$$

for all $((G_i, H_i))_{i=1}^j \in \prod_{i=1}^j M_i^{\uparrow}$. This proves that $\sqsubseteq$ is independent. $\square$

Before we continue to prove that $\mathbf{M}^{\uparrow}$ is an additive conjoint structure, we consider the link between $\mathbf{M}$ and $\mathbf{M}^{\uparrow}$. It will turn out in the further development of the proof that it is convenient to do this right now.

Lemma 5.5. *For all $i = 1, \dots, n$, let $A_i \subseteq M_i$ be such that $(A_i^{\lesssim_{\{i\}} \gtrsim_{\{i\}}}, A_i^{\lesssim_{\{i\}}}) \in M_i^{\uparrow}$. Then $(z_1, \dots, z_n) \in \prod_{i=1}^n M_i$ is an upper bound of $\prod_{i=1}^n A_i$ with respect to $\lesssim$ if and only if $(z_1, \dots, z_n)$ is an upper bound of $\prod_{i=1}^n A_i^{\lesssim_{\{i\}} \gtrsim_{\{i\}}}$.*

Proof. Let $(z_1, \dots, z_n)$ be an upper bound of $\prod_{i=1}^n A_i$. First we prove that $(z_1, \dots, z_n)$ is also an upper bound of $A_1^{\lesssim_{\{1\}} \gtrsim_{\{1\}}} \times \prod_{i=2}^n A_i$. Suppose that this is not the case. Then there would be $(a_2,\dots,a_n) \in \prod_{i=2}^n A_i$ and $a \in A_1^{\lesssim_{\{1\}}\gtrsim_{\{1\}}}$ such that

$$(\tilde{a},\tilde{a}_2,\dots,\tilde{a}_n) \lesssim (z_1,\dots,z_n) < (a,a_2,\dots,a_n)$$

for all $(\tilde{a},\tilde{a}_2,\dots,\tilde{a}_n) \in \prod_{i=1}^n A_i$. By restricted solvability, we would have $z \in M_1$ with $(z,a_2,\dots,a_n) \sim (z_1,\dots,z_n)$. This implies $\tilde{a} \lesssim_{\{1\}} z <_{\{1\}} a$ for all $\tilde{a} \in A_1$, i.e., $z \in A_1^{\lesssim_{\{1\}}}$ and therefore $a \notin A_1^{\lesssim_{\{1\}}\gtrsim_{\{1\}}}$, what is a contradiction. Thus, $(z_1,\dots,z_n)$ must be an upper bound of $A_1^{\lesssim_{\{1\}}\gtrsim_{\{1\}}} \times \prod_{i=2}^n A_i$. Analogously, we prove now that $(z_1,\dots,z_n)$ is an upper bound of $A_1^{\lesssim_{\{1\}}\gtrsim_{\{1\}}} \times A_2^{\lesssim_{\{2\}}\gtrsim_{\{2\}}} \times \prod_{i=3}^n A_i$, etc., until we get that $(z_1,\dots,z_n)$ is an upper bound of $\prod_{i=1}^n A_i^{\lesssim_{\{i\}}\gtrsim_{\{i\}}}$.

The converse assertion is an immediate consequence of $\prod_{i=1}^n A_i \subseteq \prod_{i=1}^n A_i^{\lesssim_{\{i\}}\gtrsim_{\{i\}}}$ (cf. Lemma 4.1). $\square$

The next lemma defines the quasi-embedding what we are searching for.

Lemma 5.6. *Let $\iota^{\uparrow}\colon \prod_{i=1}^n M_i \longrightarrow \prod_{i=1}^n M_i^{\uparrow}$ be defined by $\iota^{\uparrow}((x_i)_{i=1}^n) := ((x_i^{\gtrsim_{\{i\}}}, x_i^{\lesssim_{\{i\}}}))_{i=1}^n$. Then $\iota^{\uparrow}$ is a quasi-embedding of $\mathbf{M}$ into $\mathbf{M}^{\uparrow}$.*

Proof. In $\mathbf{M}$, we have $(x_i)_{i=1}^n \lesssim (y_i)_{i=1}^n$ if and only if every upper bound of $(y_i)_{i=1}^n$ is also an upper bound of $(x_i)_{i=1}^n$. By Lemma 5.5 and Lemma 5.1, this is equivalent to $((x_i^{\gtrsim_{\{i\}}}, x_i^{\lesssim_{\{i\}}}))_{i=1}^n \sqsubseteq ((y_i^{\gtrsim_{\{i\}}}, y_i^{\lesssim_{\{i\}}}))_{i=1}^n$. $\square$

The following lemma shows that the restriction of the quasi-order $\sqsubseteq$ to a factor is precisely the order of the Dedekind-MacNeille Completion of that factor. This means that our construction is compatible with the completions and that, in the end, the factors will be complete lattices as we desire (modulo the restrictions for least and greatest elements).

Lemma 5.7. *$(A,B) \sqsubseteq_{\{i\}} (C,D)$ is equivalent to $A \subseteq C$ for all (A,B), $(C,D) \in M_i^{\uparrow}$.*

Proof. We suppose, w.l.o.g., that $i=1$. If $A=C$, then obviously $(A,B) \sim_{\{1\}} (C,D)$ holds. Hence, let $A \subset C$. Then there are $b \in B$ and $c \in C$ such that $a \lesssim_{\{1\}} b <_{\{1\}} c$ for all $a \in A$. For any $(z_2,\dots,z_n) \in \prod_{i=2}^n M_i$, this implies

$$(a,z_2,\dots,z_n) \lesssim (b,z_2,\dots,z_n) < (c,z_2,\dots,z_n)$$

for all $a \in A$. Thus, $(b,z_2,\dots,z_n)$ is an upper bound of $A \times \prod_{i=2}^n z_i^{\gtrsim_{\{i\}}}$ with respect to $\lesssim$, but not of $C \times \prod_{i=2}^n z_i^{\gtrsim_{\{i\}}}$. By Lemma 5.5, we conclude

$$\begin{aligned}((A,B),&(z_2^{\gtrsim_{\{2\}}}, z_2^{\lesssim_{\{2\}}}),\dots,(z_n^{\gtrsim_{\{n\}}}, z_n^{\lesssim_{\{n\}}}))\\ &\sqsubset ((C,D),(z_2^{\gtrsim_{\{2\}}}, z_2^{\lesssim_{\{2\}}}),\dots,(z_n^{\gtrsim_{\{n\}}}, z_n^{\lesssim_{\{n\}}}))\end{aligned}$$

and therefore $(A,B) \sqsubset_{\{1\}} (C,D)$.

Using contraposition, we get also that $(A,B) \sqsubseteq_{\{1\}} (C,D)$ implies $A \subseteq C$, because $\sqsubseteq$ is linear. $\square$

Lemma 5.8. *Every factor $M_i^{\uparrow}$ is essential for $\underset{\sim}{\sqsubset}$.*

Proof. This is an immediate consequence of Lemma 5.7, since every factor M_i is essential for $\lesssim$. $\square$

Lemma 5.9. *$\underset{\sim}{\sqsubset}$ satisfies restricted solvability.*

Proof. We prove restricted solvability on the factor $M_1^{\uparrow}$. For this, let (A,B), $(C,D) \in M_1^{\uparrow}$, $((E_i,F_i))_{i=2}^n \in \prod_{i=2}^n M_i^{\uparrow}$ and $((G_i,H_i))_{i=1}^n \in \prod_{i=1}^n M_i^{\uparrow}$ be such that

$$\begin{aligned}((A,B),(E_2,F_2),\dots,(E_n,F_n)) &\underset{\sim}{\sqsubset} ((G_i,H_i))_{i=1}^n \\ &\underset{\sim}{\sqsubset} ((C,D),(E_2,F_2),\dots,(E_n,F_n)).\end{aligned}$$

If we have $((A,B),(E_2,F_2),\dots,(E_n,F_n)) \sim ((G_i,H_i))_{i=1}^n$, there is nothing to prove; likewise, if $((C,D),(E_2,F_2),\dots,(E_n,F_n)) \sim ((G_i,H_i))_{i=1}^n$. Hence, we assume that

$$\begin{aligned}((A,B),(E_2,F_2),\dots,(E_n,F_n)) &\sqsubset ((G_i,H_i))_{i=1}^n \\ &\sqsubset ((C,D),(E_2,F_2),\dots,(E_n,F_n))\end{aligned} \tag{5}$$

holds. Let us define

$$K := \Big\{k \in M_1 \;\Big|\; \forall (e_i)_{i=2}^n \in \prod_{i=2}^n E_i \; \exists (g_i)_{i=1}^n \in \prod_{i=1}^n G_i : \\ (k, e_2, \dots e_n) \lesssim (g_1,\dots,g_n)\Big\}.$$

We prove now that K is nonempty. By Lemma 5.3, we conclude from (5) that there is $(g_1,\dots,g_n) \in \prod_{i=1}^n G_i$ such that $(a,e_2,\dots,e_n) < (g_1,\dots,g_n)$ holds for all $(a,e_2,\dots,e_n) \in A \times \prod_{i=2}^n E_i$. Hence, $A \subseteq K$, and K is nonempty, because A is nonempty by the definition of $M_1^{\uparrow}$. Furthermore, there is $(c,\tilde{e}_2,\dots,\tilde{e}_n) \in C \times \prod_{i=2}^n E_i$ with $(\tilde{g}_1,\dots,\tilde{g}_n) < (c,\tilde{e}_2,\dots,\tilde{e}_n)$ for all $(\tilde{g}_1,\dots,\tilde{g}_n) \in \prod_{i_1}^n G_i$. This implies $c \in K^{\lesssim_{\{1\}}}$, i.e., also $K^{\lesssim_{\{1\}}}$ is nonempty. Therefore, we have $(K^{\lesssim_{\{1\}}\gtrsim_{\{1\}}}, K^{\lesssim_{\{1\}}}) \in M_1^{\uparrow}$.

It remains to prove that $((K^{\lesssim_{\{1\}}\gtrsim_{\{1\}}}, K^{\lesssim_{\{1\}}}),(E_2,F_2),\dots,(E_n,F_n)) \sim ((G_i,H_i))_{i=1}^n$ holds. If $(z_1,\dots,z_n) \in \prod_{i=1}^n M_i$ is an upper bound of $\prod_{i=1}^n G_i$, then $(z_1,\dots,z_n)$ is an upper bound of $K \times \prod_{i=2}^n E_i$ by definition of K. Conversely, let $(z_1,\dots,z_n)$ be an upper bound of $K \times \prod_{i=2}^n E_i$ and suppose that it would not be an upper bound of $\prod_{i=1}^n G_i$. Then there would be some $(g_1,\dots,g_n) \in \prod_{i=1}^n G_i$ such that

$$(k,e_2,\dots,e_n) \lesssim (z_1,\dots,z_n) < (g_1,\dots,g_n) < (c,\tilde{e}_2,\dots,\tilde{e}_n) \tag{6}$$

holds for all $(k,e_2,\dots,e_n) \in K \times \prod_{i=2}^n E_i$ and for $(c,\tilde{e}_2,\dots,\tilde{e}_n)$ from the preceeding paragraph. Now, we distinguish two cases:

(i) If there is $(\bar{k}, \bar{e}_2, \dots, \bar{e}_n) \in K \times \prod_{i=2}^n E_i$ such that $(z_1, \dots, z_n) \sim (\bar{k}, \bar{e}_2, \dots, \bar{e}_n)$, then also $(\bar{k}, e_2, \dots, e_n) \lesssim (\bar{k}, \bar{e}_2, \dots, \bar{e}_n)$ holds for all $(e_2, \dots, e_n) \in \prod_{i=2}^n E_i$, i.e., $(\bar{e}_2, \dots, \bar{e}_n)$ is the greatest element of $\prod_{i=2}^n E_i$. Thus, we have $(c, \tilde{e}_2, \dots, \tilde{e}_n) \lesssim (c, \bar{e}_2, \dots, \bar{e}_n)$ and therefore

$$(\bar{k}, \bar{e}_2, \dots, \bar{e}_n) < (g_1, \dots, g_n) < (c, \bar{e}_2, \dots, \bar{e}_n)$$

because of (6). By restricted solvability, we get some $\tilde{k} \in M_1$ such that $(g_1, \dots, g_n) \sim (\tilde{k}, \bar{e}_2, \dots, \bar{e}_n)$. But then $(\tilde{k}, e_2, \dots, e_n) \lesssim (g_1, \dots, g_n)$ holds for all $(e_2, \dots e_n) \in \prod_{i=2}^n E_i$, which implies $\tilde{k} \in K$. By (6) we get $(\tilde{k}, \bar{e}_2, \dots, \bar{e}_n) \lesssim (z_1, \dots, z_n)$, in contradiction to $(z_1, \dots, z_n) < (g_1, \dots, g_n) \sim (\tilde{k}, \bar{e}_2, \dots, \bar{e}_n)$.

(ii) If $(k, e_2, \dots, e_n) < (z_1, \dots, z_n)$ holds for all $(k, e_2, \dots, e_n) \in K \times \prod_{i=2}^n E_i$, then we set $(e_{2,0}, \dots, e_{n,0}) := (\tilde{e}_2, \dots, \tilde{e}_n)$. By restricted solvability, we conclude from (6) that there is $k_0 \in M_1$ such that $(k_0, e_{2,0}, \dots, e_{n,0}) \sim (z_1, \dots, z_n)$. By assumption, we have $k_0 \notin K$. Hence, there is $(\bar{e}_2, \dots, \bar{e}_n) \in \prod_{i=2}^n E_i$ with $(g_1, \dots, g_n) < (k_0, \bar{e}_2, \dots, \bar{e}_n)$. Now Lemma 3.1 implies the existence of $(e_{2,1}, \dots, e_{n,1}) \in \prod_{i=2}^n E_i$ such that $(k_0, e_{2,1}, \dots, e_{n,1}) \sim (g_1, \dots, g_n)$. Thus, we get $(e_{2,0}, \dots, e_{n,0}) <_{\{2,\dots,n\}} (e_{2,1}, \dots, e_{n,1})$. Again using (6), we obtain $k_1 \in M_1 \setminus K$ with $(k_1, e_{2,1}, \dots, e_{n,1}) \sim (z_1, \dots, z_n)$. Similar as above, we construct $(e_{2,2}, \dots, e_{n,2})$, etc., and eventually obtain infinite sequences $\{k_l \mid l \in \mathbb{N}_0\}$ and $\{(e_{2,l}, \dots, e_{n,l}) \mid l \in \mathbb{N}_0\}$ such that the conditions

$$k_0 >_{\{1\}} k_1 >_{\{1\}} k_2 >_{\{1\}} \dots,$$

$$(e_{2,0}, \dots, e_{n,0}) <_{\{2,\dots,n\}} (e_{2,1}, \dots, e_{n,1}) <_{\{2,\dots,n\}} (e_{2,2}, \dots, e_{n,2}) <_{\{2,\dots,n\}} \dots,$$

$$(k_l, e_{2,l}, \dots, e_{n,l}) \sim (k_{l+1}, e_{2,l+1}, \dots, e_{n,l+1}) \quad \text{for all } l \in \mathbb{N}, \tag{7}$$

$$(k_l, e_{2,l+1}, \dots, e_{n,l+1}) \sim (k_{l+1}, e_{2,l+2}, \dots, e_{n,l+2}) \quad \text{for all } l \in \mathbb{N} \tag{8}$$

hold (see Figure 4). We may interprete the configuration as a multi-dimensional standard sequence. In fact, we can construct an infinite upper bounded standard sequence on the factor M_1 as follows: $\lesssim$ satisfies the generalized Thomsen Condition (Proposition 3.2). Thus, from (7) and (8) we conclude $(k_{l+1}, e_{2,l}, \dots, e_{n,l}) \sim (k_{l+2}, e_{2,l+1}, \dots, e_{n,l+1})$ for all $l \in \mathbb{N}_0$. By induction, this implies $(k_l, e_{2,0}, \dots, e_{n,0}) \sim (k_{l+1}, e_{2,1}, \dots, e_{n,1})$ for all $l \in \mathbb{N}_0$. Hence, $\{k_l \mid l \in \mathbb{N}\}$ is an infinite strictly upper bounded standard sequence on the factor M_1 (with strict upper bound k_0). But since $k_l \in M_1 \setminus K$ for all $l \in \mathbb{N}_0$ and $K \neq \emptyset$, this standard sequence is also strictly lower bounded and has to be finite, in contradiction to the construction.

We have proved that the upper bounds of $K \times \prod_{i=2}^n E_i$ are exactly the upper bounds of $\prod_{i=1}^n G_i$. By Lemma 5.5, this is equivalent to $((K^{\lesssim_{\{1\}}\gtrsim_{\{1\}}}, K^{\lesssim_{\{1\}}}), (E_2, F_2), \dots, (E_n, F_n)) \sim ((G_i, H_i))_{i=1}^n$. Hence, $\sqsubseteq$ satisfies restricted solvability on $M_1^{\uparrow}$, and the other factors are done by symmetry. □

Now we finish the proof that $\mathbf{M}^{\uparrow}$ is an additive conjoint structure.

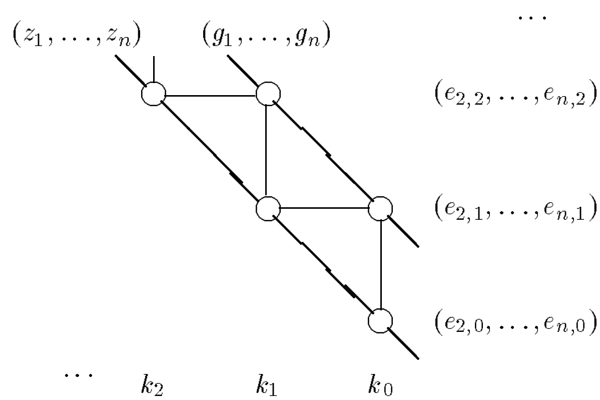

Figure 4. A Multi-dimensional standard sequence.

Lemma 5.10. *Every strictly bounded standard sequence on a factor $M_i^{\uparrow}$ is finite.*

Proof. Suppose, there would be an infinite strictly bounded standard sequence $\{(A_l, B_l) \mid l \in N\}$ on a factor $M_k^{\uparrow}$, where N is an infinite subsequent subset of $\mathbb{Z}$. Let (A^-, B^-) denote a strict lower and (A^+, B^+) denote a strict upper bound of the sequence. W.l.o.g., let us assume that $k = 1$ and $0 \in N$. By definition, there are $((C_i, D_i))_{i=2}^n \not\sim_{\{2,\ldots,n\}} ((E_i, F_i))_{i=2}^n$ in $\prod_{i=2}^n M_k^{\uparrow}$ such that

$$((A_l, B_l), (C_2, D_2), \ldots, (C_n, D_n)) \sim ((A_{l+1}, B_{l+1}), (E_2, F_2), \ldots, (E_n, F_n)) \tag{9}$$

holds for all $l \in N$. We may assume that $((C_i, D_i))_{i=2}^n \sqsupset_{\{2,\ldots,n\}} ((E_i, F_i))_{i=2}^n$. This implies $(A_l, B_l) \sqsubset_{\{1\}} (A_{l+1}, B_{l+1})$ for all $l, l+1 \in N$. Furthermore, there are $(z_2, \ldots, z_n) \in \prod_{i=2}^n M_i$ and $(c_2, \ldots, c_n) \in \prod_{i=2}^n C_i$ such that $(e_2, \ldots, e_n) \lesssim_{\{2,\ldots,n\}} (z_2, \ldots, z_n) <_{\{2,\ldots,n\}} (c_2, \ldots, c_n)$ holds for all $(e_2, \ldots, e_n) \in \prod_{i=2}^n E_i$. By assumption, we have

$$((A_l, B_l), (E_2, F_2), \ldots, (E_n, F_n)) \sqsubset ((A^+, B^+), (E_2, F_2), \ldots, (E_n, F_n)).$$

Thus, there is $(a^+, \bar{e}_2, \ldots, \bar{e}_n) \in A^+ \times \prod_{i=2}^n E_i$ with $(a_l, e_2, \ldots, e_n) < (a^+, \bar{e}_2, \ldots, \bar{e}_n)$ for all $(a_l, e_2, \ldots, e_n) \in A_l \times \prod_{i=2}^n E_i$ and with $(a_l, c_2, \ldots, c_n) < (a^+, \bar{e}_2, \ldots, \bar{e}_n)$ for all $(a_l, c_2, \ldots, c_n) \in A_l \times \prod_{i=2}^n C_i$. Then this implies $(a_l, e_2, \ldots, e_n) < (a^+, z_2, \ldots, z_n)$ and $(a_l, c_2, \ldots, c_n) < (a^+, z_2, \ldots, z_n)$. We distinguish two cases (which may occur both the same time).

(i) If N is infinitely increasing (i.e., $\mathbb{N} \subseteq N$), we choose some $a_0 \in A_0 \setminus A^-$. Then $(a_0, z_2, \ldots, z_n) < (a_0, c_2, \ldots, c_n) < (a^+, z_2, \ldots, z_n)$ holds. By restricted solvability, there is $a_1 \in M_1$ such that $(a_1, z_2, \ldots, z_n) \sim (a_0, c_2, \ldots, c_n)$. This implies $(a_1, e_2, \ldots, e_n) \lesssim (a_0, c_2, \ldots, c_n)$ for all $(e_2, \ldots, e_n) \in \prod_{i=2}^n E_i$. Now, by (9) and Lemma 5.1, we get $a_1 \in A_1$. Analogously, we construct $a_2 \in A_2$, etc. By choosing any $a^- \in A^-$, we obtain an infinite strictly bounded standard sequence

$$a^- < a_0 < a_1 < a_2 < \cdots < a^+$$

on the factor M_1, in contradiction to the assumption.

(ii) If N is infinitely decreasing, we choose some $a_0 \in A_0 \setminus A_{-1}$. Then there is $b_{-1} \in B_{-1}$ with $b_{-1} < a_0$. Therefore

$$\begin{aligned}(\tilde{a}_{-1}, e_2, \ldots, e_n) &\lesssim (b_{-1}, e_2, \ldots, e_n) < (a_0, e_2, \ldots, e_n)\\ &\lesssim (a_0, z_2, \ldots, z_2) < (a_0, c_2, \ldots, c_n)\end{aligned}$$

holds for all $(\tilde{a}_{-1}, e_2, \ldots, e_n) \in A_{-1} \times \prod_{i=2}^{n} E_i$. This implies

$$(\tilde{a}_{-2}, c_2, \ldots, c_n) \lesssim (b_{-1}, e_2, \ldots, e_n) < (a_0, z_2, \ldots, z_2) < (a_0, c_2, \ldots, c_n)$$

for all $\tilde{a}_{-2} \in A_{-2}$, since $A_{-2} \subset A_{-1}$. Thus, there is $a_{-1} \in M_1$ with $(a_{-1}, c_2, \ldots, c_n) \sim (a_0, z_2, \ldots, z_n)$, and we have $a_{-1} \notin A_{-2}$. Analogously, we construct $a_{-2} \notin A_{-3}$, etc. For $a^- \in A^-$ and $a^+ \in A^+ \setminus A_0$, we obtain an infinite strictly bounded standard sequence

$$a^- < \cdots < a_{-2} < a_{-1} < a_0 < a^+$$

on the factor M_1, which again yields a contradiction. □

Now we have shown that $\mathbf{M}^{\uparrow}$ is an n-component order-complete additive conjoint structure, where all factors have greatest elements, as far as possible. It is obvious that we can dualize all assertions and proofs in this section, and obtain an order-complete additive conjoint structure $\mathbf{M}^{\downarrow}$ where all factors have least elements, as far as possible.

6. The Completion Theorem

It remains to collect together the results we have so far. For any n-component additive conjoint structure $\mathbf{M} := ((M_i)_{i=1}^n, \lesssim)$ and any $k \in \{1, \ldots, n\}$, let us define $N_k^{\uparrow}$ and $N_k^{\downarrow}$ as follows:

(1) If there is an infinite strictly lower bounded standard sequence on the factor M_k, let $N_k^{\uparrow} := \{(M_k, \emptyset)\}$; otherwise, let $N_k^{\uparrow}$ be empty.
(2) If there is an infinite strictly upper bounded standard sequence on the factor M_k, let $N_k^{\downarrow} := \{(\emptyset, M_k)\}$; otherwise, let $N_k^{\downarrow}$ be empty.

Furthermore, let $\overline{M}_k := \mathfrak{D}(M_k, \lesssim_{\{k\}}) \setminus (N_k^{\uparrow} \cup N_k^{\downarrow})$. The "complete" lattice-order on $\overline{M}_k$ we denote by $\leq_k$.

Theorem 6.1. *Let* $\mathbf{M} := ((M_i)_{i=1}^n, \lesssim)$, $n \geq 3$, *be an* n*-component additive conjoint structure. Then a linear quasi-order* $\sqsubseteq\!\!\!\sim$ *on* $\prod_{i=1}^n \overline{M}_i$ *can be constructed such that*

(1) $\overline{\mathbf{M}} := ((\overline{M}_i)_{i=1}^n, \sqsubseteq\!\!\!\sim)$ *is an* n*-component additive conjoint structure.*

(2) *A quasi-embedding of* $\mathbf{M}$ *into* $\overline{\mathbf{M}}$ *is given by* $\iota((x_i)_{i=1}^n) := ((x_i^{\gtrsim_{\{i\}}}, x_i^{\lesssim_{\{i\}}}))_{i=1}^n$.

(3) $(\overline{M}_k, \sqsubseteq\!\!\!\sim_{\{k\}})$ *is isomorphic to* $(\overline{M}_k, \leq_k)$.

(4) *Every factor* $(\overline{M}_k, \sqsubseteq\!\!\!\sim_{\{k\}})$ *has a greatest (least) element if and only if every strictly lower (upper) bounded standard sequence on* M_k *is finite.*

Proof. By construction, there is a bijective map σ of $\prod_{i=1}^n M_i^{\uparrow\downarrow}$ onto $\prod_{i=1}^n \overline{M}_i$. By defining $\sqsubseteq\!\!\!\sim := \sigma(\lesssim)$, this map σ becomes an isomorphism of $\mathbf{M}^{\uparrow\downarrow}$ onto $\overline{\mathbf{M}}$. This proves (1), because by the Lemmata 5.2, 5.4, 5.8, 5.9, and 5.10, we know that $\mathbf{M}^{\uparrow}$ and, therefore, also $\mathbf{M}^{\uparrow\downarrow}$ are n-component additive conjoint structures. Since $\iota = \sigma \circ \iota^{\downarrow} \circ \iota^{\uparrow}$, we get (2) from Lemma 5.6, whereas Lemma 5.7 immediately yields (3).

During the construction of $M_k^{\uparrow}$, we obtain a greatest element in $(M_k^{\uparrow}, \sqsubseteq\!\!\!\sim_{\{k\}})$ if every strictly lower bounded standard sequence on M_k is finite or if $(M_k, \lesssim_{\{k\}})$ already has a greatest element. Furthermore, $(M_k^{\uparrow}, \sqsubseteq\!\!\!\sim_{\{k\}})$ has a least element if and only if $(M_k, \lesssim_{\{k\}})$ has a least element. The dual arguments hold for $M_k^{\uparrow\downarrow}$, if we take into consideration that, by the proof of Lemma 5.10, a strictly upper bounded standard sequence exists on $M_k^{\uparrow}$ if and only if such a sequence exists on M_k. Together with Proposition 2.1, this yields (4). □

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