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GRAPHS RELATED TO DIAMETER AND CENTER

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ABSTRACT. A graph is said to be an L-graph if all its paths of diametral length contain a central vertex of G. Using an earlier result we show that any graph can be embedded to an L-graph of radius a and diameter b, where $a \leq b \leq 2a$. We show that the known bounds of the number of edges and the maximum degree of the graphs of diameter $d \geq 2$ are sharp for L-graphs, too. Then we estimate the minimum degree of L-graphs. Finally we estimate the number of central vertices in L-graphs; all bounds are best possible.

1. Introduction

We consider here non-empty, finite and connected graphs, without loops and multiple edges. Basic notions are according to [**2**] and [**3**] and we recall some of them now. Let $d_G(u, v)$ denote the distance between the vertices u, v of a graph $G = (V, E)$. The eccentricity $e_G(u)$ of $u \in V(G)$ is the distance to a node farthest from u, i.e. $e_G(u) = \max\{d_G(u, v) | v \in V\}$.

The **radius** of $G, r(G)$, is the minimum eccentricity and the **diameter** of G, $d(G)$, is the maximum eccentricity. A vertex with minimum eccentricity is called **central vertex** and the set of all central vertices is **center** of G, denoted by $C(G)$. A graph is self-centered if every its node is in the center. A **diametral path** of G is any path of length $d(G)$ between two vertices whose distance is $d(G)$. The induced subgraph on the subset S of $V(G)$ is denoted by $\langle S \rangle$. The **join** of two graphs G_1, G_2 is denoted by $G_1 + G_2$. If x is a real number, then $\lfloor x \rfloor$ denotes the largest integer not exceeding x. The graph P_n will denote the path of n vertices.

The paper [**3**] introduces F-graphs, L-graphs and L'-graphs in the following way:

1. A graph G is an F-**graph** if its center $C(G)$ contains at least two vertices and the distance between any two vertices of $C(G)$ equals $r(G)$.
2. A graph G is an L-**graph** if all its diametral paths contain a central vertex; a graph G is an L'-**graph** if none of its diametral path contains a center vertex.

In [**3**] the authors discuss the possible application of these graphs and present their basic properties. The short paper [**4**] poses the problem of further study of

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these graphs and moreover, of two other classes of graphs (the so called S-graphs and D-graphs). Next we will study the L-graphs and exceptionally also the L'-graphs.

The short paper [**1**] distinguishes three types of L-graphs:

1. G is an L_1-graph if all its diametral paths contain all its central vertices.
2. G is an L_3-graph if G is an L-graph and no diametral path of G contains all central vertices of G.
3. G is an L_2-graph if G is an L-graph, but neither an L_1-graph nor an L_3-graph.

(Thus G is an L_2-graph if it contains at least one diametral path containing all central vertices and at least one diametral path containing at least one but not all central vertices).

In this paper we generalize the existence theorems of [**3**]. We show that any graph G can be embedded into an L-graph with radius a and diameter b, where $a \leq b \leq 2a$. We prove an analogous result for L'-graphs, too. We show that the known bounds on the number of edges and the maximum degree of graphs of diameter $d \geq 2$ hold for L-graphs, too (see [**2**]). We estimate the minimum degree of graphs of diameter $d \geq 2$ and we show that the estimation holds for L-graphs, too.

Then we study the existence and basic properties of L_1-graphs, L_2-graphs, and L_3-graphs. We give bounds for the number of central vertices of L_1-graphs, L_2-graphs, and L_3-graphs that are sharp. (This problem was given in [**1**]).

2. Existence Theorems

The paper [**3**] presents several examples of L-graphs, e.g. complete graphs, self-centered graphs, etc. Moreover, the following existence results are proved there:

Lemma 1 [**3**]. *Given positive integers a and b with $a \leq b \leq 2a$, there exists an L-graph G with $r(G) = a$ and $d(G) = b$.*

Next we will generalize this lemma. Let G and Q be disjoint graphs and let $u \in V(G)$. We say that a graph H is a substitution of Q into G in place of u, if the vertex set $V(H) = (V(G) - \{u\}) \cup V(Q)$ and the edge set $E(H)$ consists of all edges of the graphs $G - \{u\}$ and Q and, moreover, every vertex of Q is joined with every vertex from the neighbourhood of u in G.

Lemma 2. *Let G and Q be disjoint graphs. Let $r(G) \geq 2$ and $u \in V(G)$. Let H be a substitution of Q into G in place of u. Then:*

(a) *$r(H) = r(G), d(H) = d(G)$ and Q is an induced subgraph of H.*
(b) *if G is an L-graph, $u \in C(G)$, then H is an L-graph.*
(c) *if G is an L'-graph, $u \notin C(G)$, then H is an L'-graph.*

Proof. (a) Directly from the construction of H it follows that Q is an induced subgraph of H. If $w \in V(H) - V(Q)$, i.e. $w \in V(G) - \{u\}$, then $d_H(w,x) = d_G(w,u)$ for every $x \in V(Q)$, and $d_H(w,y) = d_G(w,y)$ for every $y \notin V(Q)$. So $e_H(w) = e_G(w)$. If $w \in V(Q)$, then $e_H(w) = e_G(u)$. Thus we get $r(H) = r(G)$ and $d(H) = d(G)$. Part (a) follows.

(b) We show that H is an L-graph by contradiction. Let P be a diametral path in H, not containing a central vertex of H. From the construction of H it follows that $C(H) = (C(G) - \{u\}) \cup V(Q)$. Then P does not contain any vertex from $V(Q)$, all vertices of P lie in G and P is a diametral path not containing any central vertex of G. This is impossible and part (b) holds.

(c) We show by contradiction that H is also an L'-graph. Let $P(x,y)$, where $x, y \in V(H)$, be a diametral path of H containing a central vertex c of H. Directly from the construction of H in this case it follows that $C(H) = C(G)$. If $P(x,y)$ contains no vertex from $V(Q)$ then it is also a diametral path in G, which is a contradiction. Thus the path $P(x,y)$ contains at least one vertex $t \in V(Q)$. From $r(G) \geq 2$ it follows that $d(G) \geq 3$, because if $d(G) = 2$ G would be self-centered. Then at least one of the vertices x, y does not belong to $V(Q)$. Since $P(x,y)$ is a shortest path, it cannot contain two vertices from $V(Q)$. Therefore $P(x,y)$ has exactly one vertex from $V(Q)$. If we replace the vertex t of Q by $u \in V(G)$, then we obtain a new path P' which has the same length as P, belongs to the L'-graph G and contains $c \in C(H) = C(G)$. This is impossible and part (c) follows. □

Let Q be an arbitrary graph and a, b be natural numbers such that $a \leq b \leq 2a$. We study the existence of an L-graph H of radius a, diameter b, and containing Q as an induced subgraph.If $a = b = 1$, then H is a complete graph and cannot contain an arbitrary graph Q as an induced subgraph.

Theorem 3. *Let H be an L-graph of radius one and diameter two. Then H has the form $K_p + G$, where $p = |C(H)|$, each of the components of G is a complete graph and the number of components of G is at least two.*

Proof. Since $r(H) = 1$, any central vertex is connected to all other vertices and $\langle C(G) \rangle$ is a complete graph. Let K be a component of the induced subgraph $G = \langle V(H) - C(H) \rangle$. If there exist two vertices $x, y \in V(K)$ such that $d_K(x,y) = 2$ then the graph H contains a diametral path of length two containing no central vertex, which is a contradiction. Thus K is a complete graph. Since $d(G) = 2$, G has at least two components. This completes the proof of the theorem. □

The other cases are settled by the following theorem.

Theorem 4. *Let Q be a graph and let a, b be positive integers such that $2 \leq a \leq b \leq 2a$. Then there exists an L-graph H of radius a, diameter b, and containing Q as an induced subgraph.*

Proof. According to Lemma 1, there exists an L-graph G of radius a, diameter b, where a, b are natural numbers, such that $a \leq b \leq 2a$. Let $2 \leq a$ and $c \in C(G)$. Let H be a substitution of Q into G in place of c. Then, according to Lemma 2(a), $r(H) = a$, $d(H) = b$ and Q is an induced subgraph of H. The graph H is also an L-graph by Lemma 2(b). The proof is complete. □

Next we prove an analogous theorem for L'-graphs. In [**3**] it is proved that if G is an L'-graph, then $r(G) + 2 \leq d(G) \leq 2r(G) - 1$. Moreover, this paper contains the following existence results.

Lemma 5 [**3**]**.** *For each pair of positive integers a and b, there exists an L'-graph with radius a and diameter b if and only if $a + 2 \leq b \leq 2a - 1$.*

The proof of Lemma 5 gives constructions of the required L'-graphs. The following theorem is a generalization of Lemma 5.

Theorem 6. *Let Q be a graph and let a, b be natural numbers such that $a+2 \leq b \leq 2a-1$. Then there exists an L'-graph H of radius a, diameter b and containing Q as an induced subgraph.*

Proof. It is clear that the smallest values of radius a and diameter b for an L'-graph are $a = 3$, $b = 5$. According to Lemma 5, there exists an L'-graph G of radius a and diameter b for adequate a, b. Let $z \notin c(G)$ and let the graph H be a substitution of Q into G in place of z. Then, according to Lemma 2(a), it is $r(H) = a$, $d(H) = b$ and Q is an induced subgraph of H. The graph H is also an L'-graph, by Lemma 2c). The theorem follows. □

3. Bounds on the Basic Parameters of L-graphs

Ore proved an upper bound on the number of edges of graphs of diameter $d \geq 2$, see [**2**, p. 106]. Bosák, Rosa and Znám proved an upper bound on the maximum degree of graphs of diameter $d \geq 2$, see [**2**, p. 106]. We show that these estimations are best possible also for L-graphs. Then we estimate the minimum degree of graphs of diameter $d \geq 2$ and we show that the obtained bound is sharp for L-graphs, too.

Theorem 7. *Let G be a graph with p vertices, q edges, maximum degree $\Delta(G)$ and diameter $d \geq 2$. If G is an L-graph, then*

$$\text{(a)} \qquad p - 1 \leq q \leq d + \frac{1}{2}(p - d - 1)(p - d + 4)$$

$$\text{(b)} \qquad 2 \leq \Delta(G) \leq p - d + 1.$$

Proof. (a) The inequality $p - 1 \leq q$ holds because G is a connected graph. The equality holds for all trees that are L-graphs. Ore proved that for all graphs of diameter $d \geq 2$ we have $q \leq d + \frac{1}{2}(p-d-1)(p-d+4)$, see [**2**]. The graphs depicted in Fig. 1 show that equality holds for L-graphs, too.

(b) If the diameter $d \geq 2$, then G contains at least one path of length d and then $\Delta(G) \geq 2$. Bosák, Rosa and Znám proved that for all graphs of diameter $d \geq 2$ we have $\Delta(G) \leq p - d + 1$, see [**2**]. Fig. 1 exhibits L-graphs attaining this bound.

This completes the proof of the theorem. $\square$

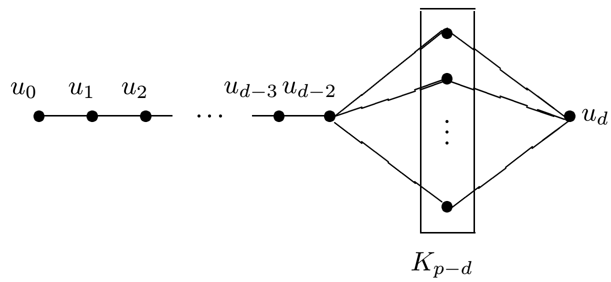


Figure 1. L-graphs that attain the upper bounds.

Next we estimate the minimum degree of a graph G with p vertices and diameter d and then we exhibit L-graphs that attain the obtained bounds.

If $d(G) = 1$, then G is a complete graph and $\delta(G) = p - 1$. If $d(G) = 2$, then $\delta(G) \leq p - 2$ and this bound is attained at the graphs $K_p - e$, where e is an arbitrary edge of K_p. The other cases are handles in the following theorem.

Theorem 8. *Let G be a graph with p vertices, diameter $d \geq 3$ and minimal degree $\delta = \delta(G)$. Then*

$$1 \leq \delta(G) \leq \left\lfloor \frac{p-d+2m-1}{m+1} \right\rfloor,$$

where $m = \lfloor \frac{d}{3} \rfloor$ and these bounds are sharp.

Proof. It is clear that the lower bound holds and is sharp. We shall derive the upper bound.

Let the vertex u of G be such that $e(u) = d$. Let us denote $D_i = D_i(u) = \{x \in V(G) \mid d(u,x) = i\}$, $|D_i| = a_i$ for $i = 0, 1, \dots, d$. The following three inequalities hold and will be useful later:

$$\delta \leq a_1$$
$$\delta \leq \deg(t) \leq a_{i-1} + a_i + a_{i+1} - 1, \quad \text{for } 2 \leq i \leq d-1, t \in D_i$$
$$\delta \leq \deg(t) \leq a_{d-1} + a_d - 1, \quad \text{for } t \in D_d.$$

It is obvious that

$$p = 1 + a_1 + \sum_{i=2}^{d-2} a_i + a_{d-1} + a_d$$

and then

$$a_{d-1} + a_d = p - 1 - a_1 - \sum_{i=2}^{d-2} a_i \leq p - \delta - 1 - \sum_{i=2}^{d-2} a_i.$$

Next we will estimate $\sum\limits_{i=2}^{d-2} a_i$.

(a) Let $d = 3m$, $m = 1, 2, 3, \dots$. Then we have

$$\sum_{i=2}^{d-2} a_i = (a_2 + a_3 + a_4) + \cdots + (a_{d-4} + a_{d-3} + a_{d-2}).$$

(If $d = 3$ then we put $\sum_{i=2}^{d-2} = 0$.) Since we have $m-1$ brackets and every bracket is, according to the above, greater or equal to $\delta + 1$, we have

$$\sum_{i=2}^{d-2} a_i \geq (m-1)(\delta + 1).$$

(b) Let $d = 3m + 1$, $m = 1, 2 \dots$. Then,

$$\sum_{i=2}^{d-2} a_i = (a_2 + a_3 + a_4) + \cdots + (a_{d-5} + a_{d-4} + a_{d-3}) + a_{d-2}.$$

From the fact that $a_{d-2} \geq 1$ and according to the preceding arguments we obtain

$$\sum_{i=2}^{d-2} a_i \geq (m-1)(\delta + 1) + 1.$$

(c) Let $d = 3m + 2$, $m = 1, 2 \dots$. Then,

$$\sum_{i=2}^{d-2} a_i = (a_2 + a_3 + a_4) + \cdots + (a_{d-6} + a_{d-5} + a_{d-4}) + a_{d-3} + a_{d-2}.$$

From the facts $a_{d-3} \geq 1$, $a_{d-2} \geq 1$ and from the arguments in part (a) we have:

$$\sum_{i=2}^{d-2} a_i \geq (m-1)(\delta + 1) + 2.$$

These three inequalities can be joined into a single one:

$$\sum_{i=2}^{d-2} a_i \geq (m-1)(\delta+1) + d - 3m$$

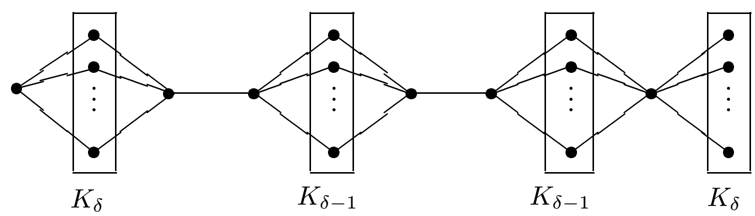
where $d \geq 3, m = \lfloor \frac{d}{3} \rfloor$.

From the above we have

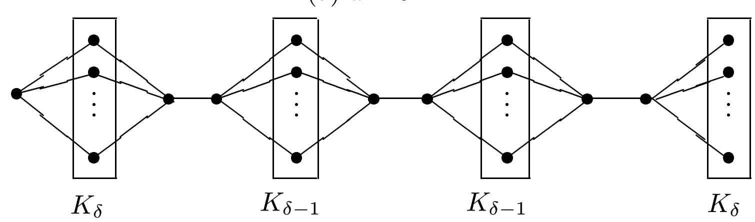
$$\begin{aligned} \delta \leq a_{d-1} + a_d - 1 &\leq p - \delta - 1 - \sum_{i=2}^{d-2} a_i - 1 \\ &\leq p - \delta - 1 - (m-1)(\delta+1) - d + 3m - 1. \end{aligned}$$

Then $2\delta + (m-1)\delta \leq p - d + 2m - 1$ and finally $\delta \leq \lfloor \frac{p-d+2m-1}{m+1} \rfloor$.

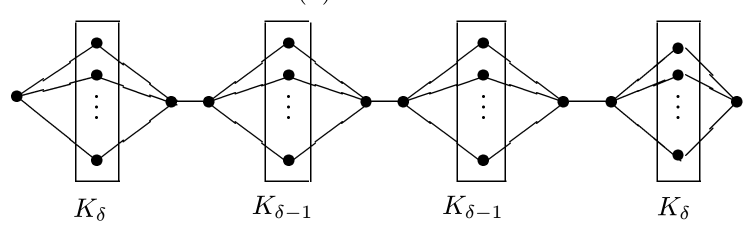
This upper bound for $\delta(G)$ is attained at the graphs in Fig. 2 where $\delta \geq 2$; Fig. 2a presents the graphs for $d = 3m, m \geq 2$, Fig. 2b the graphs for $d = 3m+1$, $m \geq 2$ and Fig. 2c the graphs for $d = 3m+2, m \geq 2$. This completes the proof of the theorem. □



(a) $d = 9$



(b) $d = 10$



(c) $d = 11$

Figure 2. Illustrations for the upper bound of $\delta(G)$.

Corollary. *Let G be an L-graph with p vertices, diameter $d \geq 2$ and minimum degree $\delta(G)$. Then $\delta(G) \leq \lfloor \frac{p-d+2m-1}{m+1} \rfloor$, where $m = \lfloor \frac{d}{3} \rfloor$ and there are L-graphs for which the equality holds.*

Proof. This upper bound holds also for L-graphs, because the graphs in Fig. 2 are L-graphs with $\delta = \lfloor \frac{p-d+2m-1}{m+1} \rfloor$. □

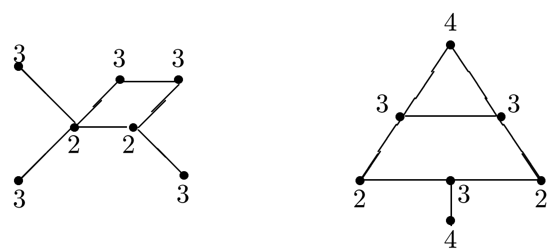
3. Special Classes of L-graphs

The short paper [**1**] introduces L_1-, L_2-, and L_3-graphs and poses the problem of bounds on the cardinality of their centers. We list the basic properties of these classes of graphs and also estimate the number of vertices in their centers.

We begin with L_1-graphs.

Remark 9.

(a) All trees are L_1-graphs. (Any tree has either one or two central vertices, see [**2**]. In both cases one can verify the remark.)
(b) If G is an L-graph with one central vertex, then G is an L_1-graph.
(c) In [**3**] it is proved that if $\langle C(G) \rangle$ is a bridge of G, then G is an L-graph. The short paper [**1**] notes that then G is an L_1-graph.
(d) If a graph G contains two central vertices, then G can be either an L_1-graph (see part (a)) or an L_2-graph (see Fig. 3a) or an L_3-graph (see Fig. 3b).



(a) (b)

Figure 3. L-graphs and their eccentricities.

Next we will estimate the cardinality of the center of an L_1-graph G of diameter d.

Let $d \geq 2$. Then G is not self-centered because otherwise G would contain at least one circuit C_i, $i \geq 3$ and no diametral path in G can contain all vertices of C_i. If $|C(G)| > d-1$ then for any diametral path $P(x,y)$ at least one of the vertices x, y belongs to $C(G)$ and G is self-centered. Thus $|C(G)| \leq d-1$ and this bound is attained at the path P_3 (with 3 vertices) for $d = 2$ and the path P_4 for $d = 3$. For $d \geq 4$ we give a better estimation.

Theorem 10. *Let G be an L_1-graph of diameter $d \geq 4$. Then $1 \leq |C(G)| \leq d - 3$.*

Proof. It is clear that the lower bound $1 \leq |C(G)|$ holds and is sharp. We shall prove the upper bound.

Let $P = P(u, v)$ be a diametral path of G. Then the path P contains all central vertices of G and, moreover, the induced subgraph $\langle V(P)\rangle = P$, because otherwise P would not be a diametral path. Let $P \equiv (u = w_0, w_1, \ldots, w_{d-1}, w_d = v)$. Then $e(u) = e(v) = d$. The graph G is not self-centered because otherwise G would be an L_3-graph (see Remark 12b). Hence the vertices u and v do not belong to $C(G)$ and $|C(G)| \leq d-1$. Next we will prove by contradiction that $|C(G)| \neq d-1, d-2$.

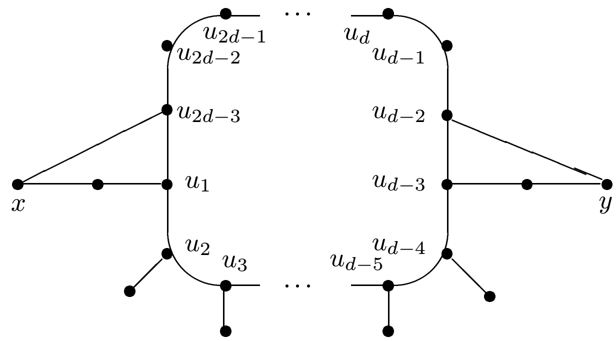
1. Suppose $|C(G)| = d - 1$. Then $w_i \in C(G)$ for $i = 1, 2, \ldots, d - 1$; $d(w_1, v) = d-1$ and then $e(w_i) = d-1$ for $i = 1, 2, \ldots, d-1$. Let s be any vertex of G different from w_2. If s belongs to $P(u, v)$, then $d(w_2, s) \leq d - 2$. If s does not belong to $P(u, v)$, then $s \notin C(G)$, there exists a vertex t such that $e(s) = d = d(s, t)$ and the diametral path $Q(s, t)$ contains all central vertices of G. But the vertex s must be adjacent to either w_1 or w_{d-1} because $d(s, t) = d$ and the induced subgraph $\langle V(P)\rangle = P$. In both cases $d(w_2, s) \leq d - 2$. Therefore we have $e(w_2) \leq d - 2$, which is a contradiction. Hence $|C(G)| \leq d - 2$.

2. Suppose $|C(G)| = d-2$. Then $d-2$ vertices from the set $\{w_1, w_2, \ldots, w_{d-1}\}$ belong to $C(G)$. We distinguish two cases:

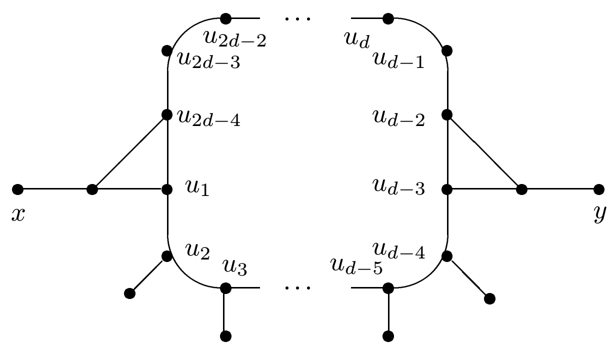
(a) $C(G)$ does not contain either w_1 or w_{d-1}, e.g. $w_1 \notin C(G)$. Then $d(w_0, w_{d-1}) = d - 1$; $e(w_{d-1}) = d - 1; e(w_i) = d - 1$ for $i = 2, 3, \ldots, d - 1$ and $e(x) = d$ for every $x \notin C(G)$. Similarly to part 1) we denote by s any vertex of G different from w_2. Using analogous arguments as in part 1) we obtain that $d(w_2, s) \leq d - 2$, i.e. $e(w_2) \leq d - 2$, which is a contradiction.

(b) $C(G)$ does not contain a vertex $w_j, 2 \leq j \leq d - 2$. Then $e(w_j) = d = d(w_j, t)$ for some $t \in V(G)$ and any diametral path $Q(w_j, t)$ contains all central vertices $\{w_1, w_2, \ldots, w_{j-1}, w_{j+1}, \ldots, w_{d-1}\}$. One can easily verify that the diametral path $Q(w_j, t)$ can be shortened in all cases by using either the edge (w_j, w_{j-1}) or (w_j, w_{j+1}), which is a contradiction.

Thus, we have $|C(G)| \leq d - 3$. This bound is attained at graphs that are depicted in Fig. 4a for $d = 2k$, $k > 2$ and in Fig. 4b for $d = 2k + 1$, $k > 2$. The base of these graphs is a circuit $C_{2d-3} : u_1, u_2, \ldots, u_{2d-3}$ for an even d and C_{2d-4}: $u_1, u_2, \ldots, u_{2d-4}$ for an odd d. The central vertices are $u_1, u_2, \ldots, u_{d-3}$. The diametral pair of vertices is x, y. This completes the proof. □



(a) $d = 2k,\ k > 2$



(b) $d = 2k + 1,\ k > 2$

Figure 4. L_1-graphs for which $|C(G)| = d - 3$.

Theorem 11. *Let G be an L_3-graph with $p \geq 3$ vertices. Then $2 \leq |C(G)| \leq p$.*

Proof. If G is an L-graph with one central vertex, then G is an L_1-graph. The lower bound 2 is attained at the graph in Fig. 3b.

The upper bound $|C(G)| \leq p$ follows if the graph G is a circuit C_p with $p \geq 3$ vertices. It is clear that then G is an L_3-graph and has p central vertices.

The theorem follows. $\square$

Next we will show that several well-known classes of graphs are L_3-graphs.

Remark 12. (a) A complete graph K_n, $n \geq 3$ is an L_3-graph with n central vertices. A complete bipartite graph $K_{m,n}$, $m \geq 2$, $n \geq 2$ is an L_3-graph of radius two with $m + n$ central vertices.

(b) A self-centered graph G such that $G \neq P_2$ is an L_3-graph. (If G is a self-centered graph and $G \neq P_2$ then G contains at least one circuit $C_i, i \geq 3$, and then any diametral path of G does not contain all vertices of C_i and hence all vertices of G.)

(c) Next we will describe a general construction of some L_3-graphs. Let $n \geq 3$ and $r \geq 2$ be given integers. Let K_n be a complete graph on n vertices $u_1, u_2, \dots, u_n$. Let $G_i, i = 1, 2, \dots, n$ be any graph that has one vertex v_i of eccentricity $r-1$. Let all graphs $K_n, G_1, G_2, \dots, G_n$ be mutually disjoint. Finally, let the graph H arise from the above graphs K_n and $G_1, G_2, \dots, G_n$ only by identification of the vertices u_i and v_i, for $i = 1, 2 \dots, n$. One can easily verify that H is an L_3-graph of radius r with n central vertices. An example of these graphs for $n = 3$ and $r = 3$ is in Fig. 5.

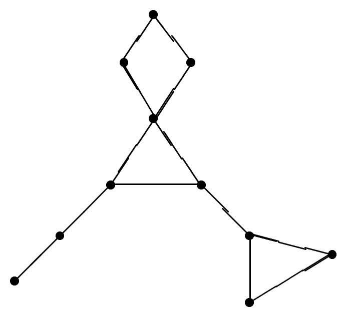


Figure 5. An L_3-graph for $r = 3$, $n = 3$.

Remark 13. (a) An L_2-graph G of diameter 1 does not exist.
(b) An L_2-graph G of diameter 2 does not exist.

Proof. Since $d(G) = 2$, then according to Remark 12(b) G cannot be self-centered and therefore $r(G) = 1$. L-graphs with diameter two and radius one are described in Theorem 3. One can easily verify that these graphs are not L_2-graphs. □

Next we will prove a bound for $|C(G)|$ of L_2-graphs of diameter $d \geq 3$.

Theorem 14. *Let G be an L_2-graph of diameter $d \geq 3$. Then $2 \leq |C(G)| \leq d-1$.*

Proof. If G is an L-graph and $|C(G)| = 1$, then G is an L_1-graph. So $|C(G)| \geq 2$ and this lower bound is attained at the graph in Fig. 6a. □

Since G is an L_2-graph, then there exists a diametral path $P(x, y)$ containing all central vertices of G. According to Remark 12(b) the graph G is not self-centered. Then there exists at least one vertex of G which does not belong to $C(G)$. But the maximum of eccentricities of all vertices of G is $d(G) = e(x) = e(y)$. Hence $x \notin C(G)$, $y \notin C(G)$ and then $|C(G)| \leq d-1$.

This upper bound is attained in graphs G depicted in Fig. 6a for $d = 3$, and in Fig. 6b for $d \geq 4$, where

— the levels of vertices A, B, C consist of the mutally disjoint paths P_{d-1};

— the levels of vertices X, Y consist of paths P_{d-3};

— the other edges of G are depicted in Fig. 6 and we did not describe them in details.

All central vertices of G are vertices of level A. One can easily verify that the graphs G are L_2-graphs of diameter $d \geq 3$. This completes the proof. □

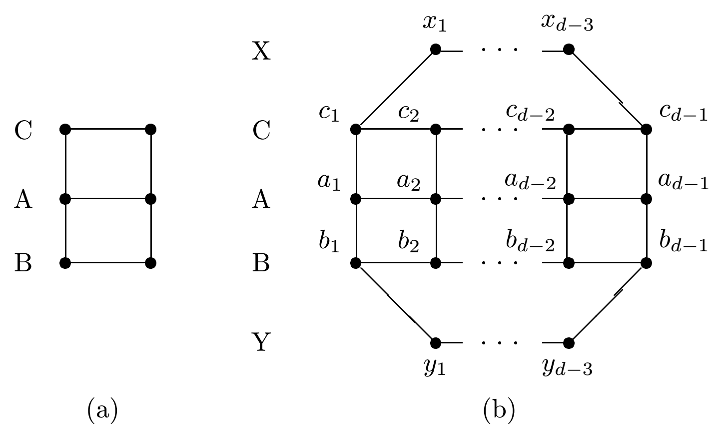


Figure 6. L_2−graphs for which $|C(G)| = d - 1$.

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