Acta Math. Univ. Comenianae
Vol. LXVII, 2(1998), pp. 217-230

ORTHOGONAL DECOMPOSITIONS IN HILBERT C*–MODULES AND STATIONARY PROCESSES

D. POPOVICI

Abstract. It is obtained a Wold-type decomposition for an adjointable isometry on a Hilbert $\mathbf{C}^*$-module which is sequentially complete with respect to some locally convex topology, denoted by s. Particularly self-dual Hilbert $\mathbf{C}^*$-modules satisfy this condition. Finally, as an application we shall give a new proof of the Wold decomposition theorem for discrete stationary processes in complete correlated actions.

1. Introduction

Hilbert modules over a $\mathbf{C}^*$-algebra were introduced by I. Kaplansky in [**2**], the variety of applications, emphasized later by the papers of W. L. Paschke [**6**] and M. A. Rieffel [**9**], inciting the interest on these objects.

Today Hilbert $\mathbf{C}^*$-modules represent an important instrument of study in a general K-theory introduced by Kasparov (to see for example [**3**]) and called KK-theory, in the $\mathbf{C}^*$-algebraic approach to quantum group theory (to see [**15**]), but also in the study of various prediction problems in correlated actions.

In the development of the prediction theory, at the same time with factorization theorems by analytic functions (for example, using the Szegö [**12**] factorization of a positive scalar function by an analytic scalar function, Kolmogorov [**4**] give an elegant solution for the univaried prediction problem), another result, of geometric type, namely the Wold decomposition theorem, in its various variants, played a very important role. This decomposition theorem for discrete stationary processes in complete correlated actions allow us to obtain (in some supplementary Harnack-type condition) the predictible part of a process, and also the prediction error operator. We cannot omit here the contribution of H. Wold in [**14**], and also of I. Suciu and I. Valuşescu for example in [**10**], [**11**] or [**13**].

In the following we shall approach this last subject in connection with Hilbert modules. The decomposition theorem for discrete stationary processes in complete correlated actions mentioned above can be obtained as an application of the main

Received July 6, 1997; revised February 26, 1998.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 46L10; Secondary 60G15.
Key words and phrases. Hilbert $\mathbf{C}^*$-modules, Wold decomposition, stationary processes in complete correlated actions.

result of this paper, a Wold-type decomposition for adjointable isometries on certain Hilbert $\mathbf{C}^*$-modules (in a class which contains in particular self-dual Hilbert modules).

2. Notations and Preliminaries

Let A be a $\mathbf{C}^*$-algebra. A **pre-Hilbert A-module** is a right A-module E (having a compatible vector space structure) equipped with a map $\langle\cdot,\cdot\rangle\colon E\times E\to A$ linear in the second variable and having the properties:

(i) $\langle x,x\rangle\geq 0,\ x\in E;\quad \langle x,x\rangle=0$ if and only if $x=0$;
(ii) $\langle x,y\rangle^*=\langle y,x\rangle,\quad x,y\in E$;
(iii) $\langle x,ya\rangle=\langle x,y\rangle a,\quad x\in E,a\in A$.

E is said to be a **Hilbert A-module** if verifies in addition

(iv) E is complete with respect to the norm

$$\|x\|:=\|\langle x,x\rangle\|^{1/2},\quad x\in E.$$

A pre-Hilbert A-module E is said to be **self-dual** if every continuous A-module map $\tau\colon E\to A$ has the form

$$\tau(x)=\langle y,x\rangle,\quad x\in E,$$

$y\in E$ being fixed.

If A is a von Neumann algebra, the **s-topology** on E is the locally convex topology on E generated by the family of seminorms $(p_\phi)_{\phi\in A_*^+}$,

$$p_\phi(x)=\Big(\phi(\langle x,x\rangle)\Big)^{1/2},\quad x\in E$$

(we denoted by A_*^+ the set of all positive linear functionals in the predual of A).

If E is self-dual then E is the dual of a Banach space E_* ([**6**]), so its closed unit ball is σ ($\sigma(E,E_*)$)-compact. Furthermore s contains σ and if $x_\alpha\xrightarrow{\sigma}x$ then

$$\phi(\langle x_\alpha,y\rangle)\to\phi(\langle x,y\rangle),\quad \phi\in A_*^+,y\in E. \tag{1}$$

We can prove now that (E,s) is quasi-complete and therefore sequentially complete. Indeed, if $(x_\alpha)_\alpha$ is a bounded s-Cauchy net in E, it is also σ-Cauchy and therefore σ-convergent to an $x\in E$. For each $\phi\in A_*^+$, $(x_\alpha)_\alpha$ being Cauchy, converges to an x_ϕ in a Hilbert space E_ϕ, the completion of E with respect to the sesquilinear form $\phi\circ\langle\cdot,\cdot\rangle$. Due to (1) $x_\phi=x$, $\phi\in A_*^+$ and so $(x_\alpha)_\alpha$ is s-convergent to x, remark which complete the proof.

For a pre-Hilbert module collection $(E_\alpha)_\alpha$ over a von Neumann algebra A its **ultraweak direct sum** (according to [**6**]) is the pre-Hilbert A-module

$$\boxplus_{\alpha\in I} E_\alpha = \Big\{ x = (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} E_\alpha \;\Big|\; \sup_F \Big\| \sum_{\alpha\in F} \langle x_\alpha, x_\alpha\rangle \Big\| < \infty \Big\},$$

F belonging to the set $\mathcal{F}$ of all finite parts of I.

A submodule E_0 of a Hilbert module E over a $\mathbf{C}^*$-algebra A is called **complementable** if there exits a submodule E_1 such that $E = E_0 + E_1$ and $\langle E_0, E_1\rangle = 0$. We use the notation $E = E_0 \oplus E_1$.

If E and F are Hilbert A-modules a map $T\colon E \to F$ is called **adjointable** if there exists $T^*\colon F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in E, \quad y \in F.$$

Denote by $\mathcal{L}_A(E,F)$ ($\mathcal{L}_A(E)$ if $E = F$) the set of all these maps. For T in $\mathcal{L}_A(E,F)$ we shall use the notation $[E,F,T]$ (respectively $[E,T]$ if $E = F$). Every adjointable operator is a bounded A-module map. In some additional conditions on E the converse is also true:

Proposition 2.1. *If E is self-dual then every bounded A-module map is adjointable.*

An A-module map $U\colon E \to F$ is called **unitary** if it is isometric and surjective. Using the terminology from [**7**] an isometry $[E,V]$ is called a **shift** if

$$\begin{aligned} E &= \bigoplus_{n=0}^{\infty} V^n L \\ &:= \Big\{ x = \sum_{n=0}^{\infty} V^n l_n \;\Big|\; l_n \in L \text{ and } \sum_n \langle l_n, l_n\rangle \text{ converges in norm in } A \Big\}, \end{aligned}$$

where $L = \ker V^*$.

The following result gives a necessary and sufficient condition on a Hilbert module adjointable isometry in order to admit a Wold-type decomposition.

Theorem 2.2 ([**7**]). *An isometry $[E,V]$ admits a unique decomposition of the form $E = E_0 \oplus E_1$ where:*

- *E_0, E_1 reduces V;*
- *$V|E_0$ is a unitary operator;*
- *$V|E_1$ is a shift*

if and only if

$$(\langle V^{*n}x, V^{*n}x\rangle)_n \text{ converges in norm in } A \text{ for all } x \in E.$$

3. s-Shifts and Unitary Operators

For the sake of completeness we shall give now other proofs for the results obtained by E. C. Lance in [**5**] regarding the characterizations of unitary operators and adjointable isometries on Hilbert $\mathbf{C}^*$-modules.

Lemma 3.1. *Let A be a $\mathbf{C}^*$-algebra, E, F be two Hilbert A-modules and $V\colon E \to F$ be an isometric A-module map. Then*

$$\langle Vx, Vy\rangle_F = \langle x, y\rangle_E, \quad x, y \in E.$$

Proof. Firstly let us observe that $VE \subset F$ is a closed submodule. The operator

$$V_0\colon E \to VE, \quad V_0 x := Vx, \quad x \in E$$

is bijective and, in addition, V_0^{-1} is a bounded A-module map. Using the characterization in [**6**] of bounded A-module maps $T\colon E \to F$ by

$$\langle Tx, Tx\rangle_F \le \|T\|^2 \langle x, x\rangle_E, \quad x \in E,$$

we can write

$$\langle x, x\rangle_E = \langle V_0^{-1}Vx, V_0^{-1}Vx\rangle_E \le \|V_0^{-1}\|^2 \langle Vx, Vx\rangle_{VE} = \langle Vx, Vx\rangle_F.$$

Applying the same inequality for $T = V$ we obtain

$$\langle Vx, Vx\rangle_F \le \langle x, x\rangle_E$$

which leads, using the polarization identity, to the conclusion. $\square$

Starting with this lemma, result which represents, in fact, the main part in the proofs given by E. C. Lance we can enunciate:

Proposition 3.2. *Let A, E, F be as in the previous lemma and $U\colon E \to F$ be an A-module map. Then U is a unitary operator (that is isometric and surjective) if and only if $U \in \mathcal{L}_A(E, F)$, $U^*U = I_E$ and $UU^* = I_F$.*

Proposition 3.3. *With A, E, F as above we consider $V\colon E \to F$ a linear map. Then are equivalent:*

(*i*) *V is an isometric A-module map with complentable range;*
(*ii*) *$V \in \mathcal{L}_A(E, F)$ and $V^*V = I_E$.*

In the following we suppose that A is a von Neumann algebra if it is not otherwise specified.

Definition 3.4. Let E be a Hilbert A-module. A map $S\colon E \to E$ is called an **s-shift** if there exists a Hilbert A-module F such that S and S_F are unitary equivalent (we denoted by S_F the operator $S_F\colon \boxplus_{n=0}^{\infty} F \to \boxplus_{n=0}^{\infty} F$, $S_F(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$).

We deduce some remarks from this definition.

Remark 3.5.

• Since S and S_F are unitary equivalent (that is $S = U^* S_F U$, $[E, \boxplus_{n=0}^{\infty} F, U]$ being unitary), S is an adjointable isometry on E.

• Identifying F with $\{(x, 0, \dots) \mid x \in F\}$ we can consider $L = U^* F$, relation which permits to observe that $S^n L = U^* S_F^n F$, $n \in \mathbf{N}$. Because

$$\langle S^n L, L\rangle = \langle U^* S_F^n F, U^* F\rangle = \langle S_F^n F, F\rangle = 0, \quad n > 0$$

L is wandering for S (that is $\langle S^n L, S^m L\rangle = 0$, $m, n \in \mathbf{N}$, $m \neq n$).

Lemma 3.6. *Let E be a pre-Hilbert A-module, $(E_\alpha)_{\alpha\in I}$ a parwise orthogonal family of submodules in E and $(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} E_\alpha$. If $(x_\alpha)_{\alpha\in I}$ is s-summable then $\sup_{F\in\mathcal{F}} \|\sum_{\alpha\in F} \langle x_\alpha, x_\alpha\rangle\| < \infty$.*

Proof. Denote by $x = s\text{-}\lim_{F\in\mathcal{F}} \sum_{\alpha\in F} x_\alpha$. Because for each $\phi \in A_*^+$, $\phi(\sum_{\alpha\in F}\langle x_\alpha, x_\alpha\rangle) \xrightarrow{F\in\mathcal{F}} \phi(\langle x, x\rangle)$ the uniform boundedness principle applyied to the family $(T_F)_{F\in\mathcal{F}}$,

$$T_F\colon A_* \to \mathbf{C}, \quad T_F(\phi) = \sum_{\alpha\in F} \phi(\langle x_\alpha, x_\alpha\rangle), \quad \phi \in A_*$$

shows that the net $(\|T_F\|)_{F\in\mathcal{F}}$ is bounded. The conclusion follows because $\|T_F\|^2 = \|\sum_{\alpha\in F}\langle x_\alpha, x_\alpha\rangle\|$, $F \in \mathcal{F}$. □

Definition 3.7. Let E be a pre-Hilbert A-module and $(E_\alpha)_{\alpha\in I}$ a family of parwise orthogonal submodules of E. We call the **direct s-sum** of the submodules E_α, $\alpha \in I$ the set

$$\boxplus_{\alpha\in I} E_\alpha := \Big\{ x = s\text{-}\lim_{F\in\mathcal{F}} \sum_{\alpha\in F} x_\alpha \;\Big|\; x_\alpha \in E_\alpha \ (\alpha \in I) \text{ and } (x_\alpha)_{\alpha\in I} \text{ is } s\text{-summable} \Big\}.$$

Remark 3.8.

• Because the map $(E, s) \ni x \mapsto xa \in (E, s)$, $a \in A$ fixed, is continuous it can be easily proved that $\boxplus_{\alpha\in I} E_\alpha$ is an A-submodule of E. Furthermore, if $x \in \boxplus_{\alpha\in I} E_\alpha$

and $\phi \in A_*^+$ then

$$\begin{aligned}\left|\Big[\phi(\sum_{\alpha\in F}\langle x_\alpha, x_\alpha\rangle)\Big]^{1/2} - [\phi(\langle x,x\rangle)]^{1/2}\right| &= \Big|p_\phi(\sum_{\alpha\in F} x_\alpha) - p_\phi(x)\Big| \\ &\le p_\phi(\sum_{\alpha\in F} x_\alpha - x) \overset{F\in\mathcal{F}}{\longrightarrow} 0\end{aligned}$$

and consequently $\Big(\sum_{\alpha\in F}\langle x_\alpha, x_\alpha\rangle\Big)_{F\in\mathcal{F}}$ ultraweakly converges to $\langle x,x\rangle$. Using the polarization identity we can assert the same about the convergence of the net $\Big(\sum_{\alpha\in F}\langle x_\alpha, y_\alpha\rangle\Big)_{F\in\mathcal{F}}$ to $\langle x,y\rangle$ for all $x,y \in \boxplus_{\alpha\in I} E_\alpha$.

- $(x_n)_n \in \boxplus_{m=0}^{\infty} F$ if and only if $(x_n)_n = s\text{-}\lim_{m\to\infty}\sum_{k=0}^{m}(0,\dots,x_k,0,\dots)$.

Indeed, for all $\phi \in A_*^+$ and $k \in \mathbf{N}$,

$$\begin{aligned}&\phi(\langle(x_0,\dots,x_k,0,\dots) - (x_n)_n, (x_0,\dots,x_k,0,\dots) - (x_n)_n\rangle) \\ &\qquad = \phi\Big(\sum_{p=0}^{k}\langle x_p, x_p\rangle - \langle(x_n)_n,(x_n)_n\rangle\Big) \overset{k\to\infty}{\longrightarrow} 0 \quad ([6]).\end{aligned}$$

We used here that

$$\langle(x_0,\dots,x_k,0,\dots),(x_n)_n\rangle = \sum_{p=0}^{k}\langle x_p, x_p\rangle,$$

which is a true relation because $(\langle x_0,x_0\rangle + \cdots + \langle x_k,x_k\rangle + \langle 0,x_{k+1}\rangle + \cdots + \langle 0,x_m\rangle)_m$ ultraweakly converges to $(\langle(x_0,\dots,x_k,0,\dots),(x_n)_n\rangle$ and is a constant equal to $\sum_{p=0}^{k}\langle x_p,x_p\rangle$.

The converse is an immediate consequence of Lemma 3.6.

Proposition 3.9. *Let E be a Hilbert A-module and $S\colon E\to E$ an isometric A-module map. If S is an s-shift then there exists a closed A-submodule L of E wandering for S such that*

$$E = \boxplus_{n=0}^{\infty} S^n L.$$

Furthermore, if (E,s) is sequentially complete the converse also holds.

Proof. For the first part it is sufficient to prove that $E = \boxplus_{n=0}^{\infty} S^n L$, where $L = U^*F$ (we used the notations and results in the Remark 3.5). Indeed

$$\begin{aligned}
E &= U^* \boxplus_{n=0}^{\infty} F \\
&= \Big\{ U^*(x_n)_n \mid (x_n)_n = s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} (0,\dots,x_k,0,\dots) \Big\} \\
&= \Big\{ x \in E \mid x = s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} U^*(0,\dots,x_k,0,\dots) \Big\} \\
&= \Big\{ x \in E \mid x = s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} S^k l_k \Big\} \\
&= \boxplus_{n=0}^{\infty} S^n L,
\end{aligned}$$

the last equality, more precisely the inclusion from left to right (the other one being obvious), being obtained due to the relations

$$\begin{aligned}
&\phi(\langle \sum_{k\in F} x_k - x, \sum_{k\in F} x_k - x\rangle) \\
&\quad = \phi(\langle \sum_{k=0}^{n} x_k - x, \sum_{k=0}^{n} x_k - x\rangle) - \phi(\langle \sum_{k\in F\setminus\{0,\cdots n\}} x_k, \sum_{k\in F\setminus\{0,\cdots n\}} x_k\rangle)
\end{aligned}$$

and

$$\phi(\langle \sum_{k\in F\setminus\{0,\cdots n\}} x_k, \sum_{k\in F\setminus\{0,\cdots n\}} x_k\rangle) \le \phi(\langle \sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} x_k\rangle),$$

with $x \in E$, $x_k \in S^k L$, $k = \overline{0,n}$, $n \in \mathbf{N}$, $F \in \mathcal{F} : \{0,\dots,n\} \subset F$, $m = \max F$.

For the converse let L be a closed A-submodule, wandering for S with the property $E = \boxplus_{n=0}^{\infty} S^n L$. We build $U\colon E \to \boxplus_{n=0}^{\infty} L$, $U(x) = U(s\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} S^k l_k) = (l_n)_n$, $x \in E$ and $l_n \in L$, $n \in \mathbf{N}$. We used here that if $x \in \boxplus_{n=0}^{\infty} S^n L$, x is defined by an s-summable family $(S^n l_n)_{n\in\mathbf{N}}$ namely $x = s\text{-}\lim_{F\in\mathcal{F}} \sum_{k\in F} S^k l_k$. So we can deduce that $x = s\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} S^k l_k$ also. The facts that the definition is correct and U is isometric can be obtained from Lemma 3.6 and the following relations

$$\langle Ux, Ux\rangle = \langle (l_n)_n, (l_n)_n\rangle = \lim_{n\to\infty} \sum_{k=0}^{n} \langle l_k, l_k\rangle = \lim_{n\to\infty} \Big\langle \sum_{k=0}^{n} S^k l_k, \sum_{k=0}^{n} S^k l_k \Big\rangle = \langle x, x\rangle$$

(the symbol lim above indicates the ultraweak convergence, the last equality being a consequence of the first part in Remarks 3.8). Using the sequential completeness of (E, s) we can easily prove the surjectivity of U. Therefore, with the results in Proposition 3.2, $S = U^* S_F U$, that is S is an s-shift. □

4. The Wold-Type Decomposition

As we stated before A will denote a von Neumann algebra unless otherwise specified.

Let E be a Hilbert A-module and $[E, V]$ an isometry. As we detailedly proved in [**7**], for each $n \in \mathbf{N}$,

$$E = L \oplus VL \oplus \cdots \oplus V^n L \oplus V^{n+1} E, \quad \text{where } L = \ker V^*.$$

So each $x \in E$ can be written as

$$x = \sum_{k=0}^{n} V^k l_k + V^{n+1} z_{n+1}, \tag{2}$$

$\{l_k\}_{k=0}^n \subset L$, $z_{n+1} \in E$ being given by the formulas

$$l_k = (I_E - VV^*) V^{*k} x, \quad z_{n+1} = V^{n+1} V^{*n+1} x, \quad n \in \mathbf{N}.$$

Before we enunciate the main theorem of this section let us give a definition.

Definition 4.1. $[E, V]$ admits a **Wold-type decomposition** if there exist two submodules E_0, $E_1 \subset E$ such that

(i) $E = E_0 \oplus E_1$;
(ii) E_0 (and consequently E_1) reduces V;
(iii) $V|E_0$ is a unitary operator and $V|E_1$ is an s-shift.

Theorem 4.2. *Suppose that (E, s) is sequentially complete. Then the isometry $[E, V]$ admits a Wold-type decomposition. This decomposition is unique.*

Proof. For $x \in E$ and $x_n = \sum_{k=0}^n V^k (I_E - VV^*) V^{*k} x$ we obtain, due to (2), the equality

$$x = x_n + V^{n+1} V^{*(n+1)} x, \quad n \in \mathbf{N}. \tag{3}$$

Let us observe that $(V^n V^{*n} x)_n$ is an s-Cauchy sequence. Indeed, because $(\langle V^{*n} x, V^{*n} x\rangle)_n$ is a decreasing sequence of positive elements there exists an element $a \in A$ such that

$$\phi(\langle V^{*n} x, V^{*n} x\rangle) \overset{n\to\infty}{\longrightarrow} \phi(a), \quad \text{for all } \phi \in A_*^+$$

(every functional $\phi \in A_*^+$ being normal). Consequently

$$\phi(\langle V^{*n}x, V^{*n}x\rangle - \langle V^{*m}x, V^{*m}x\rangle) \xrightarrow{m,n} 0\,. \tag{4}$$

Furthermore, for $m, n \in \mathbf{N}$, $m < n$,

$$\langle V^nV^{*n}x - V^mV^{*m}x, V^nV^{*n}x - V^mV^{*m}x\rangle = \langle V^{*m}x, V^{*m}x\rangle - \langle V^{*n}x, V^{*n}x\rangle$$

which, due to (4), permits us to conclude that $(V^nV^{*n}x)_n$ is s-Cauchy. Since (E, s) is sequentially complete there exists $x_u = s\text{-}\lim\limits_{n\to\infty} V^nV^{*n}x$. By passing to limit in (3) it obtains the decomposition

$$x = x_u + x_s, \quad x_u = s\text{-}\lim_{n\to\infty} V^nV^{*n}x, \quad x_s = s\text{-}\lim_{n\to\infty}\sum_{k=0}^{n} V^kl_k \in \boxplus_{n=0}^{\infty} V^nL.$$

Since, for $n \in \mathbf{N}$ fixed, V^nE is s-closed and, for $m \geq n$, $V^mV^{*m}x \in V^nE$ we obtain that $x_u \in \bigcap\limits_{n\geq 0} V^nE$. Furthermore, if $x \in \boxplus_{n=0}^{\infty} V^nL$ and $y \in \bigcap\limits_{n\geq 0} V^nE$ then

$$\langle x, y\rangle = \big\langle s\text{-}\lim_{n\to\infty}\sum_{k=0}^{n} V^kl_k, y\big\rangle = 0$$

because for each $n \in \mathbf{N}$, $\langle V^nl_n, y\rangle = 0$. Using the notations $E_0 = \bigcap\limits_{n\geq 0} V^nE$ and $E_1 = \boxplus_{n=0}^{\infty} V^nL$ it obtains the decomposition we are looking for, that is $E = E_0 \oplus E_1$. It is immediate that E_0, E_1 reduce V, $V|E_0$ is a unitary operator, and $V|E_1$ is an s-shift.

For the uniqueness let us suppose that these exist other two submodules E_0', $E_1' \subset E$ with the properties (i)-(iii) in Definition 4.1. Since $V|E_0'$ is unitary, each $x \in E_0'$ has the form $x = V^nV^{*n}x$, $n \in \mathbf{N}$ and so $x \in \bigcap\limits_{n\geq 0} V^nE = E_0$, that is $E_0' \subset E_0$. Now if $x \in E$ then the decomposition $E = E_0' \oplus E_1'$ allows to assert that $x = x_u' + x_s'$, with $x_u' \in E_0' \subset E_0 \subset VE$, and $x_s' = s\text{-}\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n} V^kl_k' \in L' + VE$, where $L' = \ker(V^*|E_1') \subset L$. Hence $E = L' \oplus VE$, that is $L = L'$, $E_1 = E_1'$ and $E_0 = E_0'$. $\square$

Because, for a self-dual Hilbert module E, (E, s) is sequentially complete as we have already stated in the second section of this paper we can formulate

Corollary 4.3. *Let E be a self-dual Hilbert A-module. Every isometry $[E, V]$ admits a unique Wold-type decomposition.*

Remark 4.4.

• As we saw in [**7**], $E_0 = \{x \in E \mid \langle x, x\rangle = \langle V^{*n}x, V^{*n}x\rangle,\ n \in \mathbf{N}\}$. Furthermore $E_1 = \{x \in E \mid V^{*n}x \xrightarrow{s} 0\}$. Indeed, let $x \in E$. Then $x = x_n + V^{n+1}V^{*(n+1)}x \in E_1$ if and only if $V^{n+1}V^{*(n+1)}x \xrightarrow{s} 0$.

• If, in addition, $(\langle V^{*n}x, V^{*n}x\rangle)_n$ converges in norm in A for all $x \in E$ then the Wold-type decomposition obtained above coincides with the decomposition in Theorem 2.2. In this case

$$\bigoplus_{n=0}^{\infty} V^n L = \boxplus_{n=0}^{\infty} V^n L, \quad \text{where } L = \ker V^*.$$

• An isometry $[E, V]$ is an s-shift if and only if $V^{*n}x \xrightarrow{s} 0$, for all $x \in E$. Indeed, if $[E, V]$ is an s-shift and $x \in E$ then

$$\begin{aligned}\langle V^{*n}x, V^{*n}x\rangle &= \Big\langle V^{*n}\big(s\text{-}\lim_{m\to\infty}\sum_{k=0}^{m} V^k l_k\big), V^{*n}\big(s\text{-}\lim_{m\to\infty}\sum_{k=0}^{m} V^k l_k\big)\Big\rangle \\ &= \Big\langle s\text{-}\lim_{m\to\infty}\sum_{k=0}^{m} V^k l_{n+k}, s\text{-}\lim_{m\to\infty}\sum_{k=0}^{m} V^k l_{n+k}\Big\rangle\end{aligned}$$

which, by the first part of the Remarks 3.8, is the ultraweak limit of the sequence $\big(\sum_{k=0}^{m}\langle l_{n+k}, l_{n+k}\rangle\big)_m$. If $a \in A$ is the least upper bound of the sequence $\big(\sum_{k=0}^{n}\langle l_k, l_k\rangle\big)_n$, then $\phi\big(\sum_{k=0}^{n}\langle l_k, l_k\rangle\big) \xrightarrow{n\to\infty} \phi(a)$ for every $\phi \in A_*^+$, and the conclusion follows.

Conversely, if $V^{*n}x \xrightarrow{s} 0$, for every $x \in E$ relation (3) shows that $E = \boxplus_{n=0}^{\infty} V^n L$. The sequential completeness of (E, s) and Proposition 3.9 prove that V is s-shift.

Definition 4.5. Let E be a Hilbert A-module. An isometry $[E, V]$ is called **completely non-unitary** (c.n.u.) if the restriction to every submodule F reducing for V is not a unitary operator (excepting the case $F = \{0\}$).

Corollary 4.6. *Let E be a Hilbert A-module and $[E, V]$ an isometry. If V is an s-shift then V is c.n.u. Conversely, if (E, s) is sequentially complete and V is c.n.u. then V is an s-shift.*

Proof. Without any difficulty we obtain that V is c.n.u. if and only if $\bigcap_{n\geq 0} V^n E = \{0\}$. If V is an s-shift then $V^{*n}x \xrightarrow{s} 0$, for all $x \in E$. If there exists an A-submodule $E' \subset E$ reducing for V such that $V|E'$ is a unitary operator then for each $x \in E'$, $\langle x, x\rangle = \langle V^{*n}x,\ V^{*n}x\rangle$, $n \in \mathbf{N}$ that is $x = 0$.

Conversely, we write the Wold-type decomposition corresponding to V. The unitary part being null it obtains that V is an s-shift. $\square$

5. Application to Stationary Processes

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the space of all linear and bounded operators on $\mathcal{H}$, E a right $\mathcal{L}(\mathcal{H})$-module and $\langle\cdot,\cdot\rangle\colon E\times E\to\mathcal{L}(\mathcal{H})$ an application.

Definition 5.1. $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ is called the **correlated action of** $\mathcal{L}(\mathcal{H})$ **on** E if $(E,\langle\cdot,\cdot\rangle)$ is a pre-Hilbert $\mathcal{L}(\mathcal{H})$-module.

Example 5.2. Let $\mathcal{K}$ be a Hilbert space. If we define

$$\langle\cdot,\cdot\rangle\colon \mathcal{L}(\mathcal{H},\mathcal{K})\times\mathcal{L}(\mathcal{H},\mathcal{K})\to\mathcal{L}(\mathcal{H}),\ \langle S,T\rangle := S^*T,\quad S,T\in\mathcal{L}(\mathcal{H},\mathcal{K})$$

we obtain a correlated action $\{\mathcal{H},\mathcal{L}(\mathcal{H},\mathcal{K}),\langle\cdot,\cdot\rangle\}$ called the **operator model**.

Every correlated action can be embedded in a correlated action of the type presented in the previous example as it results from the following proposition. Its complete proof can be found for example in [**13**].

Proposition 5.3. *Let $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ be a correlated action. Then there exist a Hilbert space $\mathcal{K}$ and an algebraic embeding*

$$E\ni x\overset{\varphi}{\longmapsto}\varphi(x)\in\mathcal{L}(\mathcal{H},\mathcal{K})\ \textit{with the properties:}$$

(i) $\langle x,y\rangle=\varphi(x)^*\varphi(y),\quad x,y\in E$;

(ii) $\mathcal{K}=\overline{\{\varphi(x)h\mid x\in E,\ h\in\mathcal{H}\}}$.

This decomposition is unique (up to a unitary equivalence).

If the map φ from Proposition 5.3 is surjective $\{\mathcal{H},E,\langle\cdot,\cdot\rangle\}$ is called a **complete correlated action**.

In the following we shall use only complete correlated actions.

A **discrete stationary process** is a sequence $\{f_n\}_n$ of elements of E with the property that $\langle f_n,f_m\rangle$ depends only on the difference $m-n$ and not on m or n separately.

The stationary process $\{g_n\}_{n\in\mathbf{Z}}$ is called **white noise** if $\langle g_n,g_m\rangle=0$ for $m\neq n$. The stationary process $\{f_n\}_n$ **contains** the white noise $\{g_n\}_n$ if:

(i) $\langle f_n,g_m\rangle$ depends only on the difference $m-n$ and is equal to 0 for $m>n$;

(ii) $\varphi(g_0)\mathcal{H}\subset\bigvee\limits_{k=-\infty}^{0}\varphi(f_k)\mathcal{H}$;

(iii) $Re\langle f_n-g_n,g_n\rangle\geq 0$.

The stationary process $\{f_n\}_n$ is called **deterministic** if it contains no non-null white noise. $\{f_n\}_n$ is called a **moving average** for the white noise $\{g_n\}_n$ if

(i) $\{f_n\}_n$ contains $\{g_n\}_n$;

(ii) $\bigvee\limits_{n=-\infty}^{\infty}\varphi(f_n)\mathcal{H}=\bigvee\limits_{n=-\infty}^{\infty}\varphi(g_n)\mathcal{H}$.

Remark 5.4. As we detailedly presented in [**8**], for every complete correlated action $\{\mathcal{H}, E, \langle\cdot,\cdot\rangle\}$ E is a self-dual Hilbert $\mathcal{L}(\mathcal{H})$-module. Because an s-closed submodule in a self-dual Hilbert $\mathbf{C}^*$-module is also self-dual, it will be complementable (the complete proof can be found in [**1**]). Consequently, if E_1 is an $\mathcal{L}(\mathcal{H})$-submodule in E we can consider the orthogonal projection associated to $\overline{E_1}^s$ and denoted by $P_{\overline{E_1}^s}$.

Proposition 5.5. *Let $\{\mathcal{H}, E, \langle\cdot,\cdot\rangle\}$ be a complete correlated action and E_1 a $\mathcal{L}(\mathcal{H})$-submodule of E. Then*

$$\varphi(P_{\overline{E_1}^s}x) = P_{\mathcal{K}_1}\varphi(x),\ x \in E,$$

where $\mathcal{K}_1 = \bigvee_{x\in E_1} \varphi(x)\mathcal{H}$, and $P_{\mathcal{K}_1}$ is the orthogonal projection in $\mathcal{K}$ associated to its closed subspace $\mathcal{K}_1$.

Proof. Let $x \in E$. Then $x \perp E_1$ if and only if $\langle x, y\rangle = 0$, that is $\varphi(x)^*\varphi(y) = 0$, for every $y \in E_1$. Because $\{\varphi(z)h \mid z \in E_1,\ h \in \mathcal{H}\}$ is dense in $\mathcal{K}_1$ we obtain that $\varphi(x)^*|\mathcal{K}_1 = 0$, which is the same with $\varphi(x)^*P_{\mathcal{K}_1} = [P_{\mathcal{K}_1}\varphi(x)]^* = 0$. The conclusion follows using that $\varphi^{-1}P_{\mathcal{K}_1}\varphi$ is also an orthogonal projection as it is $P_{\overline{E_1}^s}$. □

The next theorem is called the Wold decomposition theorem for discrete stationary processes.

Theorem 5.6. *Let $\{f_n\}_{n\in\mathbf{Z}}$ be a discrete stationary process in the complete correlated action $\{\mathcal{H}, E, \langle\cdot,\cdot\rangle\}$. There exists a unique decomposition of the form*

$$f_n = u_n + v_n, \quad n \in \mathbf{Z}$$

where

(a) $\{u_n\}_n$ is a moving average of the maximal white noise contained in $\{f_n\}$;
(b) $\{v_n\}_n$ is a deterministic process;
(c) $\langle u_n, v_m\rangle = 0$, for all $m, n \in \mathbf{Z}$.

Proof. Consider E^f_∞ and respectively E^f_n ($n \in \mathbf{Z}$), the s-closed $\mathcal{L}(\mathcal{H})$-submodules generated by $\{f_m\}_{m\in\mathbf{Z}}$ and respectively $\{f_m\}_{m\le n}$. We build

$$U_f\colon E^f_\infty \to E^f_\infty,\ U_f(f_n) = f_{n+1},\ n \in \mathbf{Z}.$$

The relation

$$\begin{aligned}\Big\langle U_f\Big(\sum_n{}' f_nT_n\Big), U_f\Big(\sum_m{}' f_mS_m\Big)\Big\rangle &= \sum_{n,m}{}' T_n^*\langle f_{n+1}, f_{m+1}\rangle S_m \\ &= \sum_{n,m}{}' T_n^*\langle f_n, f_m\rangle S_m = \Big\langle \sum_n{}' f_nT_n, \sum_m{}' f_mS_m\Big\rangle, \quad T_n, S_m \in \mathcal{L}(\mathcal{H})\end{aligned}$$

(the notation $\sum'$ represents finite sums) shows that U_f is well-defined and isometric. The surjectivity proves that U_f is a unitary operator.

Because E_0^f is invariant to U_f^* we can define the isometry $V = U_f^*|E_0^f$ which, by Proposition 2.1, is adjointable. Using Corollary 4.3 V admits a Wold-type decomposition

$$E_0^f = \boxplus_{n=0}^{\infty} V^n L \oplus \bigcap_{n\geq 0} V^n E_0^f, \text{ where } L \oplus VE_0^f = E_0^f. \tag{5}$$

Write the decomposition (5) in an operator form

$$I_{E_0^f} = P + Q, \quad P, Q \text{ projections.}$$

Define, for every $n \in \mathbf{Z}$,

$$u_n = U_f^n P f_0 \quad \text{and} \quad v_n = U_f^n Q f_0.$$

We shall prove in the following that $\{u_n\}_n$ and $\{v_n\}_n$ verify the theorem conclusion.

It is obvious that $f_n = u_n + v_n$, $n \in \mathbf{Z}$ and because

$$\langle u_n, u_m\rangle = \langle U_f^n P f_0, U_f^m P f_0\rangle = \langle U_f^{n-m} P f_0, P f_0\rangle$$

$\{u_n\}_n$ is a discrete stationary process. Analogously it proves that $\{v_n\}_n$ is also a stationary process.

Furthermore, for $m, n \in \mathbf{Z}$,

$$\langle u_n, v_m\rangle = \langle U_f^n P f_0, U_f^m Q f_0\rangle = \langle P f_0, U_f^{m-n} Q f_0\rangle = 0$$

because $(U_f^*|E_0^f)|\bigcap_{n\geq 0} V^n E_0^f$ is unitary and so $U_f^*\big(\bigcap_{n\geq 0} V^n E_0^f\big) = \bigcap_{n\geq 0} V^n E_0^f$, that is $U_f^{m-n} Q f_0 \in \bigcap_{n\geq 0} V^n E_0^f$. This proves (c).

Let us denote by $g_n = U_f^n P_L f_0$, $n \in \mathbf{Z}$. We shall prove that $\{g_n\}_n$ is the maximal white noise contained in $\{f_n\}_n$.

Because for example for $m > n$,

$$\langle g_n, g_m\rangle = \langle P_L f_0, U_f^{m-n} P_L f_0\rangle = \langle V^{m-n} P_L f_0, P_L f_0\rangle = 0,$$

$\{g_n\}_n$ is a white noise. Furthermore $\{g_n\}_n$ is contained in $\{f_n\}_n$. Indeed (i) it obtains by

$$\langle f_n, g_m\rangle = \langle U_f^m f_0, U_f^n P_L f_0\rangle = \langle V^{n-m} f_0, P_L f_0\rangle = 0,$$

for $n > m$. For (ii) it is sufficient to observe that, for $h \in \mathcal{H}$,

$$\begin{aligned}\varphi(g_0)h &= \varphi(P_L f_0)h = \varphi(f_0)h - \varphi(P_{VE_0^f} f_0)h \\ &= \varphi(f_0)h - \varphi(P_{E_{-1}^f} f_0)h = \varphi(f_0)h - P_{\mathcal{K}_{-1}^f}\varphi(f_0)h \in \mathcal{K}_0^f.\end{aligned}$$

We used here Proposition 5.5 and the notations $\mathcal{K}_n^f = \bigvee_{k\le n} \varphi(f_k)\mathcal{H}$, $n \in \mathbf{Z}$. Since

$$\begin{aligned} Re\langle f_n - g_n, g_n\rangle &= Re\langle U_f^n f_0 - U_f^n P_L f_0, U_f^n P_L f_0\rangle \\ &= Re\langle f_0 - P_L f_0, P_L f_0\rangle = 0, \end{aligned}$$

(iii) is also proved.

The maximality, the facts that $\{u_n\}_n$ is a moving average for $\{g_n\}_n$, $\{v_n\}_n$ is a deterministic process and the uniqueness of the Wold decomposition can be proved in a way similar to the classic case. □

References

1. Havet J. F., *Calcul fonctionnel continu dans les modules hilbertiens autoduaux*, Preprint, Orléans.
2. Kaplansky I., *Modules over operator algebras*, Amer. J. Math. **75** (1953), 839–858.
3. Kasparov G. G., *Operator $K-$theory and extentions of* $\mathbf{C}^*$*-algebras*, Izv. Acad. Nauk. SSSR, Ser. Mat. **44** (1980), 571–636.
4. Kolmogorov A. N., *Sur l'interpolation et extrapolation des scietes stationaires*, Comt. Rend. Acad. Sci. Paris **208** (1939), 2043–2045.
5. Lance E. C., *Unitary operators on Hilbert* $\mathbf{C}^*$*-modules*, Bull. London Math. Soc. **26** (1994), 363–366.
6. Paschke W. L., *Inner product modules over* $\mathbf{B}^*$*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.
7. Popovici D., *Orthogonal decompositions of isometries in Hilbert* $\mathbf{C}^*$*-modules*, J. Operator Theory **39** (1998), 99–112.
8. ______, Annales Univ. Sci. Budapest 40 (1997), 113–125, *Self-dual Hilbert* $\mathbf{C}^*$*-modules and stationary processes.*
9. Rieffel M. A., *Induced representations of* $\mathbf{C}^*$*-algebras*, Advances in Math. **13** (1974), 176–257.
10. Suciu I. and Valuşescu I., *Factorization theorems and prediction theory*, Rev. Roum. Math. Pures et Appl. **XXIII (9)** (1978), 1393–1423.
11. ______, *Linear predictor for stationary processes in complete correlated actions*, Proc. of Banach Center Theory Seminar, 1977.
12. Szegö G., *Uber die Randwerte analysticher Functionen*, Math. An. **84** (1921), 232–244.
13. Valuşescu I., *Operator methods in prediction theory*, Mathematical Reports **33** (1981), 343–401.
14. Wold H., *A study in the Analysis of Stationary Time Series*, Uppsala, Stockolm, 1938.
15. Woronowicz S. L., *Unbounded elements affiliated with* $\mathbf{C}^*$*-algebras and non-compact quantum groups*, Comm. Math. Phys. **136** (1991), 399–432.

D. Popovici, Department of Mathematics, West University of Timişoara, Bd. V. Pârvan no. 4, 1900 Timişoara, Romania, *e-mail:* danp@gaspar2.uvt.ro, danp@tim1.uvt.ro