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STRICT REFINEMENT FOR DIRECT SUMS AND GRAPHS

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Abstract. We study direct sums of structures with a one element subuniverse. We give a characterization of direct sums reminescent to that of inner products of groups. The strict refinement property is adapted to direct sums and to restricted Cartesian products of graphs. If a structure has the strict refinement property (for direct products), it has the strict refinement property for direct sums. Connected graphs satisfy the strict refinement property for their restricted Cartesian products.

Chang, Jónsson and Tarski introduce in [**6**] the strict refinement property for relational structures. Some of the ideas also appear in Fell and Tarski [**9**]. They show that for algebras with the strict refinement property, such as lattices, rings with zero annihilators and perfect groups, if an algebra A is a direct product of directly indecomposable algebras, then not only the directly indecomposable factors are unique up to isomorphism, but also the resulting factor congruence set on A is unique. In [**23**], Sabidussi defines relations on the edges of graphs that give a representation of certain connected graphs as Cartesian products of finitely many Cartesian indecomposable graphs and again these Cartesian indecomposable factors are unique up to isomorphism and the defined relation itself is unique. Cartesian products of infinite sets of connected nontrivial graphs are not connected. The strict refinement property is not (easily) applicable to Cartesian decompositions of graphs. In the present paper, we study the possibility of strict refinement for direct sums of structures and follow this study with an adaptation of the strict refinement property to graphs.

For any set A we denote the identity or diagonal relation $\{(x,x) : x \in A\}$ on A by $\Delta(A)$, and sometimes simply by Δ. If α is an equivalence relation on a set A and $a \in A$, a/α is the α-equivalence class of a; i.e., $a/\alpha = \{x \in A : a\alpha x\}$. If α, β are equivalence relations on a set A, then $\alpha \circ \beta$ is the relational composition of α and β; i.e., $x(\alpha \circ \beta)y$ iff there is $z \in A$ such that $x\alpha z$ and $z\beta y$. A set of congruence relations $\{\alpha_i : i \in I\}$ on an algebra A is called a **direct factor set**

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(DFS) on A if $\bigcap\{\alpha_i : i \in I\} = \Delta$ and for any $a_i \in A$ $(i \in I)$ there is $a \in A$ such that $a\alpha_i a_i$ $(i \in I)$. The direct factor sets on an algebra A are the congruence relations $\alpha_i = \{(x, y) \in A \times A : x_i = y_i\}$ where A is identified with $\prod\{A_i : i \in I\}$ and conversely A_i can be identified with the quotient algebra A/α_i. If α_i $(i = 1, 2)$ is a direct factor set, then α_1, α_2 is called a **direct factor pair**. This is the case iff $\alpha_1 \cap \alpha_2 = \Delta$ and $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1 = A \times A$. A decomposition operation on an algebra (or a structure in general) is a homomorphism $f\colon A \times A \longrightarrow A$ satisfying the equations $f(x, x) \approx x$ and $f(f(x, y), z) \approx f(x, z) \approx f(x, f(y, z))$. If $v \in A$, $f_v(x) = f(x, v)$ and $f^v(y) = f(v, y)$, then $\ker f_v$, $\ker f^v$ is a direct factor pair. Conversely, if $A = B \times C$, then $f((b, c), (b', c')) = (b, c')$ is a decomposition operation. If $\alpha_i (i \in I)$ are equivalence relations on a set A, then $\bigvee\{\alpha_i : i \in I\}$ is the smallest equivalence relation on A that contains α_i $(i \in I)$. We shall present a similar concept for direct sums of algebras, or structures in general, with a one element subuniverse.

Unless otherwise stated we shall use the terminology of McKenzie, McNulty and Taylor [**20**]. For the general theory of universal algebras the reader may consult Burris and Sankappanavar [**5**], Cohn [**7**], Grätzer [**11**], Maltsev [**19**] and McKenzie, McNulty and Taylor [**20**]. For refinement properties of direct products of finite structures the reader may consult Jónsson and Tarski [**16**]. For the general theory of graphs the reader may consult Berge [**1**], Biggs [**2**], Bollobás [**3**], Bondy and Murty [**4**] and Harary [**13**].

Definition 1. Let A_i $(i \in I)$ be algebras of a given similarity type such that for every $i \in I$, there is a one element subuniverse a_i of A_i. The subset of all $x \in \prod\{A_i : i \in I\}$ such that $\{i \in I : x_i \neq a_i\}$ is finite is a subalgebra of the Cartesian product $\prod\{A_i : i \in I\}$. This subalgebra will be denoted by $\sum\{(A_i, a_i) : i \in I\}$ and will be called the direct sum of (A_i, a_i) $(i \in I)$.

Definition 2. Let A be an algebra with a one element subuniverse 0. A set of congruences $\{\alpha_i : i \in I\}$ is called a direct sum set (DSS) modulo a congruence α and α is the direct sum of α_i $(i \in I)$ and we write $\alpha = \sum\{\alpha_i : i \in I\}$ if

(i) $\alpha = \bigcap\{\alpha_i : i \in I\}$.
(ii) For every $x \in A$, the set $\{i \in I : (x, 0) \notin \alpha_i\}$ is finite.
(iii) For any family x_i $(i \in I)$ of elements of A such that $\{i \in I : (x_i, 0) \notin \alpha_i\}$ is finite, there is $x \in A$ such that $(x, x_i) \in \alpha_i$ for every $i \in I$.

If the congruences α_i $(i \in I)$ form a direct sum set modulo $\Delta(A)$, then α_i $(i \in I)$ will be called a direct sum set.

If I is a finite set, then α_i $(i \in I)$ is a DSS iff it is a DFS. If α_1, α_2 is a direct factor pair modulo α, we write $\alpha = \alpha_1 \oplus \alpha_2$.

The notation $\alpha = \sum\{\alpha_i : i \in I\}$ is used here similar to the parallel notation for the case when the congruences $\{\alpha_i : i \in I\}$ is a direct factor set modulo α (as for instance in [**20**]).

Let A be an algebra and let α_i $(i \in I)$ be congruences on A and $\alpha = \bigcap\{\alpha_i : i \in I\}$. The epimorphism $x \longrightarrow x/\alpha_i$ of A onto A/α_i will be denoted by p_i. The resulting homomorphism of A into $\prod\{A/\alpha_i : i \in I\}$ will be denoted by p; i.e., $p(x) = (p_i(x) : i \in I)$.

Theorem 1. *Suppose A is an algebra with a one element subuniverse 0 and α_i $(i \in I)$ are congruences on A. Then $p(A)$ is a direct sum of $\{(p_i(A), p_i(0)) : i \in I\}$ iff α_i $(i \in I)$ is a direct sum set modulo $\alpha = \ker(p)$.*

Proof. Let α_i $(i \in I)$ be a DSS modulo $\alpha = \bigcap\{\alpha_i : i \in I\} = \ker(p)$. Let $a \in \sum\{p_i(A) : i \in I\}$. Then $F = \{i \in I : a_i \neq p_i(0)\}$ is finite. Let $x_i = 0$ if $i \in I\backslash F$ and $x_i/\alpha_i = a_i$ if $i \in F$. Thus x_i $(i \in I)$ satisfies $F = \{i \in I : (x_i, 0) \notin \alpha_i\}$ is finite. There is $x \in A$ such that $(x, x_i) \in \alpha_i$ for every $i \in I$. Thus $p_i(x) = p_i(x_i) = a_i$ if $i \in F$ and $p_i(x) = p_i(0)$ if $i \in I\backslash F$. Thus $p(x) = a$. Let $y \in A$. Then $\{i \in I : (y, 0) \notin \alpha_i\}$ is finite. Hence $\{i \in I : p_i(y) \neq p_i(0)\}$ is finite and so $p(y) \in \sum\{(p_i(A), p_i(0)) : i \in I\}$.

Conversely, let $p(A) = \sum\{(p_i(A), p_i(0)) : i \in I\}$. If $x \in A$, then $p(x) \in p(A)$ and the set $\{i \in I : p_i(x) \neq p_i(0)\}$ is finite. Hence $\{i \in I : (x, 0) \notin \alpha_i\}$ is finite. Let x_i $(i \in I)$ be elements of A satisfying $\{i \in I : (x_i, 0) \notin \alpha_i\}$ is finite. Then $\{i : p_i(x_i) \neq p_i(0)\}$ is finite. Hence there is $a \in \sum\{(p_i(A), p_i(0)) : i \in I\}$ such that $a_i = p_i(x_i), i \in I$. So, there is $x \in A$ such that $p(x) = a$; i.e., $p_i(x) = a_i = p_i(x_i)$, $i \in I$. Thus $(x, x_i) \in \alpha_i$, $i \in I$. Also $\ker(p) = \bigcap\{\alpha_i : i \in I\}$. This shows that α_i $(i \in I)$ is a DSS modulo $\ker(p)$. □

Definition 3. Suppose A is an algebra and φ_i $(i \in I)$ are congruences on A. The family φ_i $(i \in I)$ is called a dual direct sum set (DDSS) modulo a congruence α if

(i) $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$, for all $i, j \in I$,
(ii) $\varphi_i \cap (\bigvee\{\varphi_j : j \in I\backslash\{i\}\}) = \alpha$ for all $i \in I$,
(iii) $\bigvee\{\varphi_i : i \in I\} = A \times A$.

The motivation behind this definition will be clear from the following theorem:

Theorem 2. *Let A be an algebra with a one element subuniverse 0. Then*

(i) If α_i $(i \in I)$ is a direct sum set modulo α and $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$ for all $i \in I$, then φ_i $(i \in I)$ is a dual direct sum set modulo α. Furthermore, $\alpha_i = \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$ for all $i \in I$ and φ_i, α_i is a direct factor pair modulo α.
(ii) If φ_i $(i \in I)$ is a dual direct sum set modulo α and $\alpha_i = \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$ for all $i \in I$, then α_i $(i \in I)$ is a direct sum set modulo α. Furthermore, $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$ for all $i \in I$ and φ_i, α_i is a direct factor pair modulo α.

Proof. Let α_i $(i \in I)$ be a DSS modulo α and let $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$, $i \in I$. We need to show that φ_i $(i \in I)$ is a DDSS modulo α. Denote by A_i

the quotient algebra A/α_i and identify A/α with $\sum\{(A_i, 0/\alpha_i) : i \in I\}$. Then $\alpha_i = \{(x,y) : p_i(x) = p_i(y)\}$ and $\varphi_i = \{(x,y) : p_j(x) = p_j(y), j \in I\backslash\{i\}\}$. Thus $(x,y) \in \varphi_i \circ \varphi_j$ iff there is $z \in A$ such that $(x,z) \in \varphi_i$ and $(z,y) \in \varphi_j$. Thus $\varphi_i \circ \varphi_j = \{(x,y) : p_r(x) = p_r(y), r \in I\backslash\{i,j\}\} = \varphi_j \circ \varphi_i$. Let $(x,y) \in A \times A$. Then $F = \{i \in I : (x,y) \notin \alpha_i\}$ is finite. Thus $p_j(x) = p_j(y)$ for all $j \notin F$. Hence $(x,y) \in \bigvee\{\varphi_j : j \in F\} \subseteq \bigvee\{\varphi_j : j \in I\}$. If $(x,y) \in \bigvee\{\varphi_j : j \in I, j \neq i\}$, there is a finite set $G \subseteq I\backslash\{i\}$ such that $(x,y) \in \bigvee\{\varphi_j : j \in G\}$. Then $p_r(x) = p_r(y)$ for all $r \in I\backslash G$. Thus $p_i(x) = p_i(y)$; i.e., $(x,y) \in \alpha_i$ and $\bigvee\{\varphi_j : j \in I\backslash\{i\}\} \subseteq \alpha_i \subseteq \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$. If $(x,y) \in \varphi_i \cap \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$, then $(x,y) \in \varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$ and $(x,y) \in \alpha_i$; i.e., $(x,y) \in \bigcap\{\alpha_j : j \in I\} = \alpha$. Since φ_i, α_i permute, $\varphi_i \cap \alpha_i = \alpha$ and $\varphi_i \vee \alpha_i = A \times A$, $(A/\varphi_i) \times (A/\alpha_i) \cong A/\alpha$.

We need to establish the statement (ii). For any $X \subseteq I$, let $\widetilde{X} = \bigvee\{\varphi_j : j \in X\}$. As $\varphi_i \cap (\bigvee\{\varphi_j : j \in I\backslash\{i\}\}) = \alpha$, $\alpha \subseteq \widetilde{X}$ for every non-void $X \subseteq I$. Then

Claim 1. *For any subsets S, T of I, $\widetilde{S} \circ \widetilde{T} = \widetilde{T} \circ \widetilde{S}$.*

Indeed, since the relations φ_i $(i \in I)$ are mutually permutable, $\varphi_i \circ \varphi_j = \varphi_i \vee \varphi_j$. The assertion follows easily.

Claim 2. *If $S, T \subseteq I$ and $S \cap T = \emptyset$, then $\widetilde{S} \cap \widetilde{T} = \alpha$.*

Let $(x,y) \in \widetilde{S} \cap \widetilde{T}$. Then there are finite subsets $F \subseteq S$ and $G \subseteq T$ such that $(x,y) \in \widetilde{F} \cap \widetilde{G}$. We show that $\widetilde{F} \cap \widetilde{G} = \alpha$ by induction on $|F| \geq 1$. It is true for $|F| = 1$. Let $|F| > 1$ and $\ell \in F$, $\ell \notin V$ and $F = V \cup \{\ell\}$. By induction, $\widetilde{V} \cap \widetilde{G} = \alpha$. Let $a(\widetilde{F} \cap \widetilde{G})b$. As $\widetilde{V}, \varphi_\ell$ are permutable, there is $c \in A$ such that $a\widetilde{V}c\varphi_\ell b$ and $a\widetilde{G}b$. Then $c(\varphi_\ell \circ \widetilde{G})a$; i.e., $a(\varphi_\ell \vee \widetilde{G})c$ and $a\widetilde{V}c$. As V and $G \cup \{\ell\}$ are disjoint, $\widetilde{V} \cap (\varphi_\ell \vee \widetilde{G}) = \alpha$ by induction. So $a\alpha c$. Thus $a\widetilde{G}b$ and $a\varphi_\ell b$. Hence $a\alpha b$ as $\varphi_\ell \cap \widetilde{G} = \alpha$.

Claim 3. *If $S, T \subseteq I$, then $\widetilde{S} \cap \widetilde{T} = (\widetilde{S \cap T})$.*

We need to show only the case $S \cap T \neq \emptyset$, $S \nsubseteq T$ and $T \nsubseteq S$. Let $\lambda = (\widetilde{S \cap T})$, $\mu = (\widetilde{S\backslash T})$ and $\nu = (\widetilde{T\backslash S})$. Then $\lambda\cap\mu = \lambda\cap\nu = \mu\cap\nu = \alpha$ and $\widetilde{S} = \lambda\vee\mu$, $\widetilde{T} = \lambda\vee\nu$. Let $a(\widetilde{S} \cap \widetilde{T})b$. As λ, μ, ν are permutable, there are $c, d \in A$ such that $a\lambda c\mu b$ and $a\lambda d\nu b$. Hence $c\lambda d$. Also, $c(\mu \vee \nu)d$ since $c\mu b$ and $d\nu b$. But $(\mu \vee \nu) \cap \lambda = \alpha$. Thus $c\alpha d$. But then $c(\mu \cap \nu)b$. Hence $c\alpha b$ and so, $a\lambda b$. Thus $\widetilde{S} \cap \widetilde{T} \subseteq \lambda = (\widetilde{S \cap T})$. The reverse inclusion is obvious.

From Claims 1, 2, 3, $\alpha_i = \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$ $(i \in I)$ are mutually permutable. Let $(x,y) \in \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$. Then $(x,y) \in \alpha_j$ for every $j \in I\backslash\{i\}$. Thus there are finite subsets $F_j \subseteq I\backslash\{j\}$ such that $(x,y) \in \widetilde{F_j}$. Fix $r \in I\backslash\{i\}$. For every $s \in F_r$, $s \neq i$, $s \notin F_s$. Thus $F_r \cap (\bigcap\{F_s : s \in F_r, s \neq i\}) \subseteq \{i\}$. Thus $(x,y) \in \varphi_i$; i.e. $\bigcap\{\alpha_j : j \in I, j \neq i\} \subseteq \varphi_i$. As $\varphi_i \subseteq \alpha_j$ for every $j \neq i$. the reverse inclusion also holds. Thus for every $i \in I$, $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$. We need to show that $\bigcap\{\alpha_i : i \in I\} = \alpha$. $\bigcap\{\alpha_i : i \in I\} = \alpha_i \cap (\bigcap\{\alpha_j : j \in I\backslash\{i\}\}) = \alpha_i \cap \varphi_i = \alpha$.

As φ_i, α_i are permutable and $\varphi_i \vee \alpha_i = A \times A$, φ_i, α_i are a direct factor pair modulo α. Let $x \in A$. We need to show that $\{i \in I : (x, 0) \notin \alpha_i\}$ is finite. Since $A \times A = \bigvee\{\varphi_i : i \in I\}$, there is a finite set $F \subseteq I$ such that $(x, 0) \in \widetilde{F} \subseteq \alpha_i$ for every $i \in I \backslash F$. Thus $\{i \in I : (x, 0) \notin \alpha_i\} \subseteq F$. Finally, suppose $x_i \in A$ $(i \in I)$ satisfy $\{i \in I : (x_i, 0) \notin \alpha_i\} = G$ is finite. We need to find $x \in A$ such that $(x, x_i) \in \alpha_i$ for every $i \in I$. This is possible by induction on $|G|$. If $|G| = 0$, then $x = 0$ will do. Let $|G| > 0$ and $G = H \cup \{\ell\}$, $\ell \notin H$. Then $|H| < |G|$. Put $y_i = x_i$ if $i \neq \ell$ and $y_\ell = 0$. Then $\{i \in I : (y_i, 0) \notin \alpha_i\} = H$. By induction there is $y \in A$ such that $(y, y_i) \in \alpha_i$ for every $i \in I$. Now $(y, x_\ell) \in A \times A = \varphi_\ell \vee \alpha_\ell = \varphi_\ell \circ \alpha_\ell$. Thus there is $x \in A$ such that $(y, x) \in \varphi_\ell$ and $(x, x_\ell) \in \alpha_\ell$. Hence $(y, x) \in \alpha_i$ for all $i \in I \backslash \{\ell\}$. Hence $x \alpha_i y \alpha_i x_i$ for all $i \in I$, $i \neq \ell$ and $(x, x_i) \in \alpha_i$ for all $i \in I$. □

Remark 1. The characterization of dual direct sums in the case when I is finite also works for algebras without one element subuniverses and is similar to the case of internal direct sums in groups. In fact condition (ii) of Definition 3 can be replaced by the following condition:

(ii′) $\varphi_i \cap \bigvee\{\varphi_j : 1 \leq j < i\} = \alpha$ for $1 < i \leq n$.

We shall show that condition (ii′) implies condition (ii) of Definition 3 in the presence of conditions (i) and (iii) of Definition 3 in the case $I = \{1, 2, \ldots, n\}$. Assume that conditions (i), (iii) of Definition 3 and condition (ii)′ hold. We first show by induction on $n - k$ that $(\bigvee\{\varphi_j : 1 \leq j \leq k\}) \cap (\bigvee\{\varphi_j : k < j \leq n\}) = \alpha$, for all $1 \leq k < n$. Since this is true for $k = n - 1$, assume it is true for all $k > m$ where $m < n$. Let $\lambda = \bigvee\{\varphi_j : 1 \leq j \leq m\}$ and $\mu = \bigvee\{\varphi_j : m + 1 < j \leq n\}$. We need to show that $\lambda \cap (\varphi_{m+1} \vee \mu) = \alpha$. Let $a, b \in A$ and $a(\lambda \cap (\varphi_{m+1} \vee \mu))b$. As the φ_{m+1}, μ are permutable, there is $c \in A$ such that $a \varphi_{m+1} c \mu b$. Thus $c(\varphi_{m+1} \vee \lambda)b$. As $\mu \cap (\varphi_{m+1} \vee \lambda) = \alpha$, $c \alpha b$. Hence $a(\varphi_{m+1} \cap \lambda)b$ and by condition (ii′) $a \alpha b$. Since conditions (ii) and (ii′) are identical for $i = n$, we shall prove that (ii′) implies (ii) by induction on $n - i$. Let condition (ii) be true for all $i > m$ for some $m < n$. Let $\gamma = \bigvee\{\varphi_j : 1 \leq j < m\}$ and $\delta = \bigvee\{\varphi_j : m < j \leq n\}$. We need to show that $\varphi_m \cap (\gamma \vee \delta) = \alpha$. Let $a(\varphi_m \cap (\gamma \vee \delta))b$. Then there is $c \in A$ such that $a \gamma c \delta b$. Hence $c(\gamma \vee \varphi_m)b$. But $(\gamma \vee \varphi_m) \cap \delta = \alpha$, so $c \alpha b$. Thus $a(\varphi_m \cap \gamma)b$. Then $a \alpha b$.

A DDSS is essentially an internal characterization of a direct sum.

Theorem 3. *Let A be an algebra and let φ_i $(i \in I)$ be a dual direct sum set on A. Suppose 0 is a one element subuniverse. Then there is an isomorphism of A onto $\sum\{(0/\varphi_i, 0) : i \in I\}$.*

Proof. Let φ_i $(i \in I)$ be a DDSS and let 0 be a one element subuniverse. By Theorem 2, $\alpha_i = \bigvee\{\varphi_j : j \in I \backslash \{i\}\}$ $(i \in I)$ is a DSS and for every $i \in I$, φ_i, α_i is a direct factor pair. Thus for every $x \in A$, there is $x_i \in A$ such that $0 \varphi_i x_i \alpha_i x$. As $\varphi_i \cap \alpha_i = \Delta$, x_i is unique. By Theorem 1, $p(x) = (x/\alpha_i : i \in I)$ gives an isomorphism of A onto $\sum\{A/\alpha_i : i \in I\}$. Also $A \cong A/\alpha_i \times A/\varphi_i$. The mapping

$x/\alpha_i \longrightarrow x_i$ is an isomorphism of A/α_i onto the subalgebra $0/\varphi_i$, $i \in I$. Thus $x \longrightarrow (x_i : i \in I)$ is an isomorphism of A onto $\sum\{(0/\varphi_i, 0) : i \in I\}$. $\square$

The following lemmas prepare for characterizations of the strict refinement property for direct sums:

Lemma 1. *Let A be an algebra with a one element subuniverse 0. Suppose α_i $(i \in I)$ is a direct sum set modulo α and β_i $(i \in I)$ are congruences on A such that for every $i \in I$, $\alpha_i \subseteq \beta_i$ and $\beta = \bigcap\{\beta_i : i \in I\}$. Then β_i $(i \in I)$ is a direct sum set modulo β.*

Proof. Let $x_i \in A$ $(i \in I)$ and let $\{i \in I : (x_i, 0) \notin \beta_i\} = F$ be finite. Let $y_i = x_i$ if $i \in F$ and $y_i = 0$ if $i \in I \backslash F$. Then $\{i \in I : (y_i, 0) \notin \alpha_i\} = F$ is finite and so there is $x \in A$ such that $(x, y_i) \in \alpha_i \subseteq \beta_i$ for every $i \in I$. As $y_i = 0$ if $i \in I \backslash F$ and $(x_i, 0) \in \beta_i$ if $i \in I \backslash F$, $(x, x_i) \in \beta_i$ for every $i \in I$. If $x \in A$, then $\{i \in I : (x, 0) \notin \beta_i\} \subseteq \{i \in I : (x, 0) \notin \alpha_i\}$ is finite. Thus the family β_i $(i \in I)$ is a DSS modulo β. $\square$

Lemma 2. *Let A be an algebra with a one element subuniverse 0. Let α_i $(i \in I)$ be a direct sum set modulo α and let $\alpha = \bigcap\{\beta_i : i \in I\}$ where β_i $(i \in I)$ are congruences on A such that $\alpha_i \subseteq \beta_i$ for every $i \in I$. Then $\alpha_i = \beta_i$ for every $i \in I$.*

Proof. Let $k \in I$. Put $\gamma_i = \alpha_i$ if $i \in I \backslash \{k\}$ and $\gamma_k = \beta_k$. By Lemma 1, γ_i $(i \in I)$ is a DSS modulo $\bigcap\{\gamma_i : i \in I\} \subseteq \bigcap\{\beta_i : i \in I\} = \alpha$. Now φ_k, α_k and φ_k, γ_k are direct factor pairs modulo α, where $\varphi_k = \bigcap\{\alpha_i : i \in I \backslash \{k\}\} = \bigcap\{\gamma_i : i \in I \backslash \{k\}\}$. Let $a\gamma_k b$. Then there is $c \in A$ such that $a\varphi_k c \alpha_k b$. Thus $c\gamma_k b$ and $a\gamma_k b$. Hence $a\gamma_k c$. As $\varphi_k \cap \gamma_k = \alpha$, $a\alpha c$ and so, $a\alpha_k b$. Thus $\alpha_k = \gamma_k = \beta_k$. $\square$

Lemma 3. *Let A be an algebra with a one element subuniverse 0 and let α_i $(i \in I)$ and β_j $(j \in J)$ be direct sum sets modulo α. Then the following conditions are equivalent:*

- *(i) There are congruences γ_{ij} $((i,j) \in I \times J)$ such that $\alpha_i = \bigcap\{\gamma_{ij} : j \in J\}$ and $\beta_j = \bigcap\{\gamma_{ij} : i \in I\}$ for every $i \in I$ and $j \in J$.*
- *(ii) $\alpha_i \vee \beta_j$ $((i,j) \in I \times J)$ is a direct sum set modulo α and $\alpha_i = \bigcap\{\alpha_i \vee \beta_j : j \in J\}$ and $\beta_j = \bigcap\{\alpha_i \vee \beta_j : i \in I\}$ for every $i \in I$ and $j \in J$.*
- *(iii) $\alpha_i \vee \beta_j$ $((i,j) \in I \times J)$ is a direct sum set modulo α.*

Proof. It is clear that (ii) $\Rightarrow$ (i), (iii). We need to show that (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). As $\alpha_i \subseteq \alpha_i \vee \beta_j$ for all $j \in J$, $\alpha_i \subseteq \bigcap\{\alpha_i \vee \beta_j : j \in J\} = \gamma_i$. By Lemma 1, γ_i $(i \in I)$ is a DSS modulo $\bigcap\{\gamma_i : i \in I\} = \bigcap\{\alpha_i \vee \beta_j : (i,j) \in I \times J\} = \alpha$. By Lemma 2, $\alpha_i = \gamma_i = \bigcap\{\alpha_i \vee \beta_j : j \in J\}$, $i \in I$. The equalities $\beta_j = \bigcap\{\alpha_i \vee \beta_j : i \in I\}$ $(j \in J)$ are similar. It remains to show that (i) $\Rightarrow$ (iii). For every $(i,j) \in I \times J$, $\alpha_i \vee \beta_j \subseteq \gamma_{ij}$. $\bigcap\{\alpha_i \vee \beta_j : j \in J\} \subseteq \bigcap\{\gamma_{ij} : j \in J\} = \alpha_i$. Hence $\bigcap\{\alpha_i \vee \beta_j : (i,j) \in I \times J\} = \bigcap\{\alpha_i : i \in I\} = \alpha$. Let $x \in A$. As α_i

$(i \in I)$ and β_j $(j \in J)$ are DSSs modulo α, the sets $F = \{i \in I : (x, 0) \notin \alpha_i\}$ and $G = \{j \in J : (x, 0) \notin \beta_j\}$ are finite. Hence $\{(i, j) \in I \times J : (x, 0) \notin \alpha_i \vee \beta_j\} \subseteq F \times G$ is finite. Suppose $x_{ij} \in A$ $((i, j) \in I \times J)$ satisfy $\{(i, j) : (x_{ij}, 0) \notin \alpha_i \vee \beta_j\}$ is finite. Fix $k \in I$. Then $\{j \in J : (x_{kj}, 0) \notin \alpha_k \vee \beta_j\}$ is finite. As $\alpha_k \vee \beta_j$ $(j \in J)$ is a DSS modulo α_k, by Lemma 1, there is $x_k \in A$ such that $(x_k, x_{kj}) \in \alpha_k \vee \beta_j$ for all $j \in J$. Let $G = \{k \in I : (x_k, 0) \notin \alpha_k\}$. As $\alpha_k = \bigcap\{\alpha_k \vee \beta_j : j \in J\}$, for every $k \in G$, there is $j(k) \in J$ such that $(x_k, 0) \notin \alpha_k \vee \beta_{j(k)}$. But $(x_k, x_{kj}) \in \alpha_k \vee \beta_j$ for all $k \in I$ and $j \in J$. Hence $(x_{kj(k)}, 0) \notin \alpha_k \vee \beta_{j(k)}$ for every $k \in G$. Thus G is finite and there is $x \in A$ such that $(x, x_k) \in \alpha_k$ for all $k \in I$. Hence $(x, x_{kj}) \in \alpha_k \vee (\alpha_k \vee \beta_j) = \alpha_k \vee \beta_j$ for all $k \in I$ and $j \in J$. This shows that the family $\alpha_i \vee \beta_j$ $((i, j) \in I \times J)$ is a DSS modulo α. $\square$

Lemma 4. *Let A be an algebra and let φ_i $(i \in I)$ and ψ_j $(j \in J)$ be dual direct sum sets on A. If $\varphi_i \cap \psi_j$ $((i, j) \in I \times J)$ is a dual direct sum set, then for every $i \in I$, $\varphi_i = \bigvee\{\varphi_i \cap \psi_j : j \in J\}$ and for every $j \in J$, $\psi_j = \bigvee\{\varphi_i \cap \psi_j : i \in I\}$.*

Proof. Let $a\varphi_k b$. Since $A \times A = \bigvee\{\varphi_i \cap \psi_j : (i, j) \in I \times J\}$, there is a finite set $F \subseteq I \times J$ such that $(a, b) \in \bigvee\{\varphi_i \cap \psi_j : (i, j) \in F\}$. As $\varphi_i \cap \psi_j$ are mutually permutable, there is $c \in A$ such that $a\gamma c$ and $c\delta b$, where $\gamma = \bigvee\{\varphi_k \cap \psi_j : (k, j) \in F\} \subseteq \varphi_k$ and $\delta = \bigvee\{\varphi_i \cap \psi_j : (i, j) \in F, i \neq k\}$. Thus $b\varphi_k c$ and $(b, c) \in \bigvee\{\varphi_i : i \in I \backslash \{k\}\}$. Hence $b = c$ and $\varphi_k \subseteq \bigvee\{\varphi_k \cap \psi_j : j \in J\} \subseteq \varphi_k$. $\square$

The strict refinement property can be defined for direct sums. Later we shall see that an algebra or a structure satisfies the strict refinement property for direct sums iff it satisfies the strict refinement property (for direct products.)

Definition 4. Let A be an algebra with a one element subuniverse 0. Then A satisfies the strict refinement property for direct sums (SRPS) if for any direct sum sets α_i $(i \in I)$ and β_j $(j \in J)$, there is a direct sum set γ_{ij} $((i, j) \in I \times J)$ such that $\alpha_i = \bigcap\{\gamma_{ij} : j \in J\}$ for every $i \in I$ and $\beta_j = \bigcap\{\gamma_{ij} : i \in I\}$ for every $j \in J$.

For direct sums, the strict refinement property implies the refinement property. In other words, if A is an algebra with a one element subuniverse 0 and A satisfies the SRPS and $A \cong \sum\{B_i : i \in I\} \cong \sum\{C_j : j \in J\}$, then there are algebras D_{ij} $((i, j) \in I \times J)$ such that for every $i \in I$, $B_i \cong \sum\{D_{ij} : j \in J\}$ and for every $j \in J$, $C_j \cong \sum\{D_{ij} : i \in I\}$. Furthermore, if A has SRPS and $A \cong \sum\{B_i : i \in I\} \cong \sum\{C_j : j \in J\}$ where all B_i, $i \in I$ and C_j, $j \in J$ are directly indecomposable algebras, then there is a DDSS φ_i $(i \in I)$ on A and a bijective mapping $g \colon I \longrightarrow J$ such that $0/\varphi_i \cong B_i \cong C_{g(i)}$ for every $i \in I$. Algebras with SRPS and with a one element subuniverse that are direct sums of directly indecomposable algebras have a unique DDSS φ_i $(i \in I)$ such that the substructures $0/\varphi_i$ are directly indecomposable.

We are now ready to show that for structures with a one element subuniverse, the strict refinement property is equivalent to SRPS.

Theorem 4. *For an algebra A with a one element subuniverse 0, the following conditions are equivalent:*

(i) A has the strict refinement property for direct sums.

(ii) A has the strict refinement property for direct sums for finite index sets I and J.

(iii) The set of factor congruences of A forms a Boolean lattice (i.e., it is a sublattice of the congruence lattice of A and the distributive laws hold on this sublattice).

(iv) If $\Delta = \alpha \oplus \alpha' = \beta \oplus \beta'$ for $\alpha, \alpha', \beta, \beta'$ congruences on A, then $(\alpha \vee \beta) \wedge \alpha' \leq \beta$.

(v) $f_v g_v = g_v f_v$ for all decomposition operations f, g, and all $v \in A$.

(vi) There is $v \in A$ such that $f_v g_v = g_v f_v$ for all decomposition operations f, g.

(vii) A has the strict refinement property for direct sums for index sets I and J such that $|I| = |J| = 2$.

(viii) For any dual direct sum sets φ_i $(i \in I)$ and ψ_j $(j \in J)$, $\varphi_i \cap \psi_j$ $((i,j) \in I \times J)$ is a dual direct sum set, and $\varphi_i = \bigvee\{\varphi_i \cap \psi_j : j \in J\}$ for every $i \in I$, and $\psi_j = \bigvee\{\varphi_i \cap \psi_j : i \in I\}$ for every $j \in J$.

(ix) For any dual direct sum sets φ_1, φ_2 and ψ_1, ψ_2, $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a dual direct sum set.

Proof. Conditions (i), (ii), (iii), (iv), (v), (vi) are the formulation for direct sums of the corresponding conditions for direct products in ([**20**, Theorem 5.17, p. 303]). Since for any finite set I, α_i $(i \in I)$ is a DFS iff it is a DSS, Lemma 2 of ([**20**, p. 302]) holds for direct sums as its proof uses only finite direct factor sets. The only place that needs a change in the proof of ([**20**, Theorem 5.17, p. 303]) is that, after obvious notational changes, following $\alpha_i = \bigcap\{\alpha_i \vee \beta_j : j \in J\}$ and $\beta_j = \bigcap\{\alpha_i \vee \beta_j : i \in I\}$, by Lemma 3, the family $\alpha_i \vee \beta_j$ $((i,j) \in I \times J)$ forms a DSS and $\alpha_i = \sum\{\alpha_i \vee \beta_j : j \in J\}$ and $\beta_j = \sum\{\alpha_i \vee \beta_j : i \in I\}$. It is clear that condition (ii) implies condition (vii). We can show that condition (vii) implies condition (iii) in the same way as (ii) implies (iii) in ([**20**, Theorem 5.17, p. 303]). That conditions (viii) and (i) are equivalent and condition (vii) is equivalent to condition (ix) follows from Theorem 2, Lemma 3, and Lemma 4. □

From Theorem 4, the strict refinement property for direct sums holds iff it is true for finite index sets I and J. As for finite index sets DSS coincides with DFS, the following is valid.

Corollary 1. *Let A be an algebra with a one element subuniverse 0. Then A has the strict refinement property iff A satisfies the strict refinement property for direct sums.*

Let A be any algebra with a one element subuniverse 0 that has the strict refinement property such as congruence distributive algebras, perfect or centerless algebras in congruence permutable varieties such as rings with zero annihilator, etc. If A is the direct sum of directly indecomposable algebras A_i ($i \in I$), then there is precisely one DDSS φ_i ($i \in I$) such that $0/\varphi_i \cong A_i$, $i \in I$. If $A \cong \sum\{B_j : j \in J\}$ where every B_j is directly indecomposable, and ψ_j ($j \in J$) is the DDSS such that $B_j \cong 0/\psi_j$, $j \in J$, then $|I| = |J|$ and $\{\varphi_i : i \in I\} = \{\psi_j : j \in J\}$. Examples: If a lattice L is a direct sum of directly indecomposable sublattices containing an element $a \in L$, then this set of "direct summands" is unique. If a ring is a direct sum of its directly indecomposable ideals and every such ideal is a ring with zero annihilator, then this set of ideals is unique. A direct sum of a set of finite centerless groups is a direct sum of directly indecomposable groups and the set of resulting normal subgroups is unique.

Now we study the applicability of refinement properties to graphs. All the results of this paper can be carried over to directed graphs without loops and for which from any given vertex to another there can be no more than one directed edge. By a graph Γ we mean a pair of not necessarily finite sets $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is the set of vertices of Γ and $E(\Gamma)$ (the set of edges of Γ) is a set of unordered pairs of distinct elements of $V(\Gamma)$. Thus, in this article, graphs have neither loops nor multiple edges. A graph may be viewed as a set with a symmetric irreflexive binary relation. We write $a \in V(\Gamma)$ to mean that a is a vertex of Γ and if a pair $\{a, b\}$ is an edge of Γ, we write $ab \in E(\Gamma)$. A **path** in Γ connecting the vertices $a, b \in V(\Gamma)$ is a sequence of vertices $c_0, c_1, \ldots, c_n \in V(\Gamma)$ such that for $1 \le i \le n$, $c_{i-1}c_i \in E(\Gamma)$ and $a = c_0$, $b = c_n$. If $n \ge 3$, a graph with n distinct vertices $c_0, c_1, \ldots, c_{n-1}$ and n edges $c_0c_1, c_1c_2, \ldots, c_{n-2}c_{n-1}, c_{n-1}c_0$ is called a **cycle** of length n and denoted by C_n. A graph Γ is connected if for any distinct vertices $a, b \in V(\Gamma)$, there is a path in Γ connecting a, b. A **homomorphism** of a graph Γ_1 into a graph Γ_2 is a mapping f from $V(\Gamma_1)$ into $V(\Gamma_2)$ such that for any $a, b \in V(\Gamma_1)$, if $f(a) \neq f(b)$ and $ab \in E(\Gamma_1)$, then $f(a)f(b) \in E(\Gamma_2)$. Two graphs Γ_1, Γ_2 are **isomorphic** and we write $\Gamma_1 \cong \Gamma_2$ if there is a bijective mapping f from $V(\Gamma_1)$ onto $V(\Gamma_2)$ such that f and f^{-1} are homomorphisms. For any non-void set A of vertices of a graph Γ, by $\Gamma[A]$ we denote the **subgraph** of Γ whose vertex set is A and for any $a, b \in A$, ab is an edge of $\Gamma[A]$ iff $ab \in E(\Gamma)$. The **Cartesian product** of graphs Γ_i, $i \in I$ is the graph Γ such that $V(\Gamma)$ is the direct product of the $V(\Gamma_i)$, $i \in I$ (i.e., the set of all $x = (\ldots, x_i, \ldots)$ where $x_i \in V(\Gamma_i)$, $i \in I$. For $x, y \in V(\Gamma)$, $xy \in E(\Gamma)$ iff there is precisely one $i \in I$ such that $x_iy_i \in E(\Gamma_i)$ and for every $j \in I\backslash\{i\}$, $x_j = y_j$. We denote the Cartesian product of graphs Γ_1, Γ_2 by $\Gamma_1 \oplus \Gamma_2$. This construction appeared in Harary [**12**], Miller [**21**], Sabidussi [**22**], [**23**] and, in Shapiro [**24**]. The **restricted Cartesian product** of graphs (Γ_i, v_i), $i \in I$, where $v_i \in V(\Gamma_i)$ is given for every $i \in I$, is the graph Γ such that $V(\Gamma)$ is the direct sum of the pointed sets $(V(\Gamma_i), v_i)$, $i \in I$ (i.e., the set of all $x = (\dots, x_i, \dots)$ where $x_i \in V(\Gamma_i)$, $i \in I$ such that $\{i \in I : x_i \neq v_i\}$ is finite). For $x, y \in V(\Gamma)$, $xy \in E(\Gamma)$ iff there is precisely one $i \in I$ such that $x_i y_i \in E(\Gamma_i)$ and for every $j \in I \backslash \{i\}$, $x_j = y_j$. If Γ is the restricted Cartesian product of (Γ_i, v_i), $i \in I$, we shall write $\Gamma = \sum\{(\Gamma_i, v_i) : i \in I\}$. A graph Γ is called **Cartesian indecomposable** if it is non-trivial, i.e., contains more than one vertex, and Γ is not isomorphic to a Cartesian product of any two non-trivial graphs. The general theory of strict refinement of relational structures introduced in Chang, Jónsson, and Tarski [**6**], is applicable to the direct product of graphs; i.e., the direct product $\Gamma_1 \otimes \Gamma_2$ where $V(\Gamma_1 \otimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$ and $(u_1, u_2)(v_1, v_2) \in E(\Gamma_1 \otimes \Gamma_2)$ iff $u_i v_i \in E(\Gamma_i)$ for $i = 1, 2$. A graph Γ satisfies the **refinement property** for restricted Cartesian products if whenever $\Gamma \cong \sum\{(\Gamma_i, v_i) : i \in I\} \cong \sum\{(\Xi_j, u_j) : j \in J\}$, there are graphs Ψ_{ij}, and $w_{ij} \in V(\Psi_{ij})$, $i \in I$, $j \in J$ such that $\Gamma_i \cong \sum\{(\Psi_{ij}, w_{ij}) : j \in J\}$ and $\Xi_j \cong \sum\{(\Psi_{ij}, w_{ij}) : i \in I\}$ for every $i \in I$, $j \in J$. Some of the graphs Ψ_{ij} may be composed of one vertex only. A similar definition can be given for the refinement property relative to direct product decompositions. If G is any finite bipartite graph and $2C_3$ is the disjoint union of two cycles of length 3, then $G \otimes C_6 \cong G \otimes 2C_3$. (cf. Lovász [**17**], [**18**] and McKenzie, McNulty and Tayler [**20**, p. 331].) The cycle C_4 is directly indecomposable, i.e., not isomorphic to the direct product of any two nontrivial graphs. The same is true of $2C_3$, but $C_6 \cong K_2 \otimes C_3$ where K_2 is a graph with two vertices and one edge. Since C_4 is bipartite, $C_4 \otimes K_2 \otimes C_3 \cong C_4 \otimes 2C_3$ and so the direct product does not satisfy the refinement property even for finite graphs. A directly indecomposable graph may not be Cartesian indecomposable and vice-versa. $K_2 \otimes K_2 \cong 2K_2$ and $K_2 \oplus K_2 \cong C_4$. C_6 is Cartesian indecomposable. However, the restricted Cartesian product of a set of connected graphs is connected and every connected graph is, up to an isomorphism, uniquely the restricted Cartesian product of Cartesian indecomposable graphs. (cf. Sabidussi [**23**], Imrich [**14**])

Sabidussi gives in [**23**], an internal characterization of Cartesian decomposition of connected graphs by means of an equivalence relation on the set of edges. A similar method was given by Vizing [**25**]. Similar equivalence relations on the edges of a graph are used in Feder [**8**], Graham and Winkler [**10**] and Imrich and Zerovnik [**15**] to give efficient algorithms for the Cartesian decompositions of finite connected graphs. Imrich shows that every connected graph is, up to isomorphism, uniquely the restricted Cartesian product of Cartesian indecomposable graphs ([**14**, Szatz 4 and Szatz 5]). We shall give another characterization using equivalence relations on the set of vertices in a fashion reminiscent of the inner product of groups. We shall adapt the definition of the strict refinement property so that we can apply it to the restricted Cartesian product of graphs.

The following definition and lemma provide a connection between dual direct sum sets and restricted Cartesian decompositions of graphs:

Definition 5. Let φ, ψ be equivalence relations on the set of vertices of a graph Γ. The relation φ satisfies the edge condition relative to the relation ψ if for any a, b, c of $V(\Gamma)$ such that $a\varphi b$, $a\psi c$ and $ab \in E(\Gamma)$, there is $d \in V(\Gamma)$ such that $c\varphi d$, $b\psi d$ and $cd \in E(\Gamma)$. A family of equivalence relations φ_i $(i \in I)$ on $V(\Gamma)$ satisfies the edge condition if φ_i satisfies the edge condition relative to φ_j for any ordered pair $(i, j) \in I \times I$, $i \neq j$.

Lemma 5. *Let φ, ψ_i $(i \in I)$ be equivalence relations on the set of vertices of a graph Γ and $\psi = \bigvee\{\psi_i : i \in I\}$. If for every $i \in I$, φ satisfies the edge condition relative to ψ_i, then φ satisfies the edge condition relative to ψ.*

Proof. This is routine from the definition. □

Definition 6. Let Γ be a graph and let φ_i $(i \in I)$ be equivalence relations on $V(\Gamma)$. The family φ_i $(i \in I)$ is called a graph dual direct sum set (GDDSS) if

(i) The family φ_i $(i \in I)$ is a dual direct sum set on $V(\Gamma)$.
(ii) The family φ_i $(i \in I)$ satisfies the edge condition.
(iii) If $ab \in E(\Gamma)$, then $a\varphi_i b$ for some $i \in I$.

Now we give an internal characterization for Cartesian decompositions of graphs.

Theorem 5. *Let Γ, Γ_i $(i \in I)$ be graphs and $v_i \in V(\Gamma_i)$ $(i \in I)$. There is an isomorphism of Γ onto $\sum\{(\Gamma_i, v_i) : i \in I\}$ iff there is a graph dual direct sum set φ_i $(i \in I)$ on $V(\Gamma)$ and $v \in V(\Gamma)$ such that for every $i \in I$, $(\Gamma_i, v_i) \cong (\Gamma[v/\varphi_i], v)$.*

Proof. Let $(\Gamma, v) = \sum\{(\Gamma_i, v_i) : i \in I\}$. Define φ_i on $V(\Gamma)$ by $a\varphi_i b$ iff $a_j = b_j$ for all $j \in I\backslash\{i\}$. Checking that the family φ_i $(i \in I)$ is a GDDSS routinely follows from the definitions.

We need to show the converse. Suppose φ_i $(i \in I)$ is a GDDSS on $V(\Gamma)$. Let $v \in V(\Gamma)$ and let $\Gamma_i = \Gamma[v/\varphi_i]$, $v_i = v$, $i \in I$. We shall show that $(\Gamma, v) \cong \sum\{(\Gamma_i, v_i) : i \in I\}$. By Theorem 3, $(V(\Gamma), v) \cong \sum\{(V(\Gamma_i), v_i) : i \in I\}$ as pointed sets. For every $i \in I$, we define a mapping $\pi_i \colon \Gamma \to \Gamma_i$. Let $x \in V(\Gamma)$. As φ_i and $\alpha_i = \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$ is a factor pair, there is a unique $t \in V(\Gamma)$ such that $v\varphi_i t\alpha_i x$. Define $\pi_i(x) = t$. It is clear that π_i is surjective. Actually π_i is a graph homomorphism. Indeed, let $bc \in E(\Gamma)$ and let $\pi_i(b) \neq \pi_i(c)$. As $v\varphi_i\pi_i(b)\alpha_i b$ and $v\varphi_i\pi_i(c)\alpha_i c$, then $\pi_i(b)\varphi_i\pi_i(c)$. Also there is a unique $j \in I$ such that $b\varphi_j c$, since $bc \in E(\Gamma)$. If $j \neq i$, then $\varphi_j \subseteq \alpha_i$ and so $b\alpha_i c$, which in turn implies $\pi_i(b)\alpha_i\pi_i(c)$ and consequently, as $\alpha_i \cap \varphi_i = \Delta$, $\pi_i(b) = \pi_i(c)$. Thus $j = i$ and $b\varphi_i c$. As φ_i, $i \in I$ satisfy the edge condition and $\alpha_i = \bigvee\{\varphi_j : 1 \leq j \leq n, j \neq i\}$, by Lemma 5, the pair φ_i, α_i satisfies the edge condition. This and $\alpha_i \cap \varphi_i = \Delta$ implies $\pi_i(b)\pi_i(c) \in E(\Gamma[v/\varphi_i])$. We need to show that $cd \in E(\Gamma)$ iff $\pi(c)\pi(d)$ is an edge in $\sum\{(\Gamma_i, v_i) : i \in I\}$, where $\pi(x) = (\ldots, \pi_i(x), \ldots)$. Let $cd \in E(\Gamma)$. As φ_i $(i \in I)$ is a GDDSS, there is a unique $i \in I$, such that $c\varphi_i d$. As $\varphi_i \cap \alpha_i = \Delta$,

$\pi_i(c) \neq \pi_i(d)$. As π_i is a graph homomrphism, $\pi_i(c)\pi_i(d) \in E(\Gamma_i)$. If $j \in I \backslash \{i\}$, then $(c,d) \in \varphi_i \subseteq \alpha_j$ and so $\pi_j(c) = \pi_j(d)$. So $\pi_k(c)\pi_k(d) \in E(\Gamma_k)$ holds only for $k = i$. Hence $\pi(c)\pi(d)$ is an edge of the restricted Cartesian product. On the other hand, if $\pi(c)\pi(d)$ is an edge of the restricted Cartesian product, then $cd \in E(\Gamma)$ follows from the fact that there is precisely one $i \in I$ with $\pi_i(c)\pi_i(d) \in E(\Gamma_i)$ and for $j \in I\backslash\{i\}$, $\pi_j(c) = \pi_j(d)$ and φ_i, α_i satisfy the edge condition; i.e., the mapping π is a graph isomorphism of (Γ, v) onto $\sum\{(\Gamma_i, v_i) : i \in I\}$. □

Remark 2. Viewing the equivalence relations φ_i $(i \in I)$ as partitions, for any given $i \in I$, and any vertices a, b of Γ the graphs $\Gamma[a/\varphi_i]$ and $\Gamma[b/\varphi_i]$ are isomorphic subgraphs of Γ. The homomorphism π_i restricted to b/φ_i provides a graph isomorphism of $\Gamma[b/\varphi_i]$ onto $\Gamma[a/\varphi_i]$.

In order to adapt the definition of the strict refinement property to the case of graphs, we need to find what a DSS for graphs should be. This is achieved by the following definition.

Definition 7. Let Γ be a graph and $v \in V(\Gamma)$. A set α_i $(i \in I)$ of equivalence relations on $V(\Gamma)$ is called a graph direct sum set (GDSS) and every α_i is called a graph direct factor if

(i) α_i $(i \in I)$ is a direct sum set on the pointed set $(V(\Gamma), v)$.
(ii) If $ab \in E(\Gamma)$, then there is $i \in I$ such that $a\alpha_j b$ for every $j \in I\backslash\{i\}$.
(iii) For every $i \in I$, α_i, $\bigcap\{\alpha_j : j \in I\backslash\{i\}\}$ satisfy the edge condition.

Similar to Theorem 2 we have

Theorem 6. *Let Γ be a graph and $v \in V(\Gamma)$. Then*

(i) If α_i $(i \in I)$ is a graph direct sum set on Γ, then $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$ $(i \in I)$ is a graph dual direct sum set.
(ii) If φ_i $(i \in I)$ is a graph dual direct sum set on Γ, then $\alpha_i = \bigvee\{\varphi_j : j \in I\backslash\{i\}\}$ $(i \in I)$ is a graph direct sum set.

Proof. In view of Theorem 2, we need only show that in (i), φ_i $(i \in I)$ satisfy the edge condition and for every $ab \in E(\Gamma)$, there is $i \in I$ such that $a\varphi_i b$. The latter follows from (ii) of Definition 7. Let $i, j \in I$ and $i \neq j$, $a, b, c \in V(\Gamma)$, $a\varphi_i b$, $a\varphi_j c$ and $ab \in E(\Gamma)$. Since φ_i, α_i is a factor pair and $\varphi_j \subseteq \alpha_i$, there is a unique $d \in V(\Gamma)$ such that $c\varphi_i d$ and $b\alpha_i d$. As α_i, φ_i satisfy the edge condition ((iii) of Definition 7), $cd \in E(\Gamma)$. Now φ_i and φ_j are permutable. Hence there is $e \in V(\Gamma)$ such that $c\varphi_i e$ and $b\varphi_j e$. Again $\varphi_j \subseteq \alpha_i$. So $c\varphi_i e$ and $b\alpha_i e$. Then $e = d$ and $b\varphi_j d$.

To show (ii), let $ab \in E(\Gamma)$. Then there is a unique $i \in I$ such that $a\varphi_i b$. So $a\alpha_j b$ for every $j \in I\backslash\{i\}$. Since φ_i, φ_j satisfy the edge condition for $i \neq j$, by Lemma 5, φ_i satisfies the edge condition relative to $\bigvee\{\varphi_j : j \in I\backslash\{i\}\} = \alpha_i$. If $ab \in E(\Gamma)$, $a\alpha_i b$ and $a\varphi_i c$. There is $j \in I\backslash\{i\}$ such that $a\varphi_j b$. As $\varphi_j \subseteq \alpha_i$ and φ_j satisfies the edge condition relative to φ_i, α_i satisfies the edge condition relative to φ_i. Since $\varphi_i = \bigcap\{\alpha_j : j \in I\backslash\{i\}\}$, (iii) of Definition 7 is satisfied. □

Definition 8. Let Γ be a graph. A graph decomposition operation f on Γ is a graph homomorphism of the Cartesian product $\Gamma \oplus \Gamma$ onto Γ such that

(i) The equations $f(x, x) \approx x$ and $f(f(x, y), z) \approx f(x, f(y, z)) \approx f(x, z)$ hold in $V(\Gamma)$.

(ii) If $ab \in E(\Gamma)$, then $f(a, b) \in \{a, b\}$.

Theorem 7. *Let Γ be a graph. Then*

(i) If $\Gamma = \Gamma_1 \oplus \Gamma_2$ and $f((x_1, x_2), (y_1, y_2)) = (x_1, y_2)$, then f is a graph decomposition operation on Γ.

(ii) If f is a graph decomposition operation on Γ and $v \in V(\Gamma)$, then $\Gamma \cong \Gamma[v/\ker f_v] \oplus \Gamma[v/\ker f^v]$.

Proof. It is sufficient to show in (i) that $f(a, b) \in \{a, b\}$ if $ab \in E(\Gamma)$ and f is a graph homomorphism of $\Gamma \oplus \Gamma$ into Γ. If $x \in V(\Gamma)$, then $x = (x_1, x_2)$ where $x_i \in V(\Gamma_i)$, $i = 1, 2$. $f(a, b) = (a_1, b_2)$. As $ab \in E(\Gamma)$, either $a_1 = b_1$ in which case $f(a, b) = (a_1, b_2) = (b_1, b_2) = b$, or $b_1 = b_2$, in which case $f(a, b) = a$. If $(a, b)(c, d) \in E(\Gamma \oplus \Gamma)$, then either $a = c$ and $bd \in E(\Gamma)$ or $ac \in E(\Gamma)$ and $b = d$. $f(a, b) = (a_1, b_2)$, $f(c, d) = (c_1, d_2)$. If $a = c$, then $a_1 = c_1$. If $f(a, b) \neq f(c, d)$, then $b_2 \neq d_2$. As $bd \in E(\Gamma_1 \oplus \Gamma_2)$ and $b_2 \neq d_2$, $b_1 = d_1$ and $b_2 d_2 \in E(\Gamma_2)$. Hence $(a_1, b_2)(a_1, d_2) \in E(\Gamma_1 \oplus \Gamma_2)$. Thus $f(a, b)f(c, d) \in E(\Gamma)$. The other case is similar.

To show (ii), it suffices, in view of Theorem 5, to verify that the factor pair $\ker f_v$, $\ker f^v$ is a GDSS. Let $ab \in E(\Gamma)$. There is no loss in generality assuming $f(a, b) = a$. Thus $f(a, b) = a = f(a, a)$ and $(a, b) \in \ker f_a = \ker f_v$. We need to show that $\ker f_v$, $\ker f^v$ satisfy the edge condition. It suffices to show that $\ker f_v$ satisfies the edge condition relative to $\ker f^v$. Let $(a, b) \in \ker f_v$, $(a, c) \in \ker f^v$ and $ab \in E(\Gamma)$. There is $d \in V(\Gamma)$ such that $(c, d) \in \ker f_v$ and $(b, d) \in \ker f^v$. As $\ker f_v = \ker f_a$, $f(c, a) = f(d, a)$. Also $\ker f^v = \ker f^c$ and $f(c, a) = f(c, c) = c$. Similarly $f(d, d) = f(d, b) = d$ as $\ker f^v = \ker f^d$. So $f(d, a) = c$ and $f(d, b) = d$. If $c = d$, then $f(d, b) = d = c = f(c, a) = f(d, a)$. This implies that $(a, b) \in \ker f^v \cap \ker f_v = \Delta(V(\Gamma))$. So $a = b$ contradicting $ab \in E(\Gamma)$. Thus $c \neq d$. As f is a homomorphism of the Cartesian square of Γ onto Γ and $f(d, a) = c \neq d = f(d, b)$ and $ab \in E(\Gamma)$, $cd \in E(\Gamma)$. This shows that $\ker f_v$ satisfies the edge condition relative to $\ker f^v$. $\square$

Now we propose to define the strict refinement property for graphs.

Definition 9. A graph Γ has the strict refinement property for restricted Cartesian products (GSRP) if for any $v \in V(\Gamma)$ and graph direct sum sets α_i $(i \in I)$ and β_j $(j \in J)$ on Γ, there is a graph direct sum set γ_{ij} $((i, j) \in I \times J)$ such that $\alpha_i = \sum\{\gamma_{ij} : j \in J\}$ for every $i \in I$ and $\beta_j = \sum\{\gamma_{ij} : i \in I\}$ for every $j \in J$.

In view of Theorems 4, 5, 6 and 7, we have the following characterization of GSRP:

Theorem 8. *The following conditions for a graph Γ are equivalent:*

(i) *Γ has the strict refinement property for restricted Cartesian products.*
(ii) *Γ has the strict refinement property for restricted Cartesian products for finite index sets I and J.*
(iii) *The set of factor congruences of Γ forms a Boolean lattice (i.e., it is a sublattice of the lattice of equivalence relations on $V(\Gamma)$ and the distributive laws hold on this sublattice).*
(iv) *If $\Delta(V(\Gamma)) = \alpha \oplus \alpha' = \beta \oplus \beta'$ where α, α' and β, β' are graph direct sum sets on Γ, then $(\alpha \vee \beta) \wedge \alpha' \leq \beta$.*
(v) *$f_v g_v = g_v f_v$ for all graph decomposition operations f, g and all $v \in V(\Gamma)$.*
(vi) *There is $v \in V(\Gamma)$ such that $f_v g_v = g_v f_v$ for all graph decomposition operations f, g.*
(vii) *Γ has the strict refinement property for restricted Cartesian products for index sets I and J such that $|I| = |J| = 2$.*
(viii) *For any graph dual direct sum sets φ_i $(i \in I)$ and ψ_j $(j \in J)$, $\varphi_i \cap \psi_j$ $((i,j) \in I \times J)$ is a graph dual direct sum set, and $\varphi_i = \bigvee\{\varphi_i \cap \psi_j : j \in J\}$ for every $i \in I$ and $\psi_j = \bigvee\{\varphi_i \cap \psi_j : i \in I\}$ for every $j \in J$.*
(ix) *For any graph dual direct sum sets φ_1, φ_2 and ψ_1, ψ_2, the set $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a graph dual direct sum set.*

As in the general case GSRP implies the refinement property for restricted Cartesian products of graphs.

Theorem 9. *Every connected graph has the strict refinement property for restricted Cartesian products.*

A graph has the strict refinement property for restricted Cartesian products iff it satisfies condition (ix) of Theorem 8. First we prove the following lemma:

Lemma 6. *If φ_1, φ_2 and ψ_1, ψ_2 are graph dual direct sum sets on a graph Γ, then the family $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a graph dual direct sum set iff the equivalence relations $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ are mutually permutable and $\varphi_i = (\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$ and $\psi_i = (\psi_i \cap \varphi_1) \vee (\psi_i \cap \varphi_2)$, $i = 1, 2$.*

Proof. Let φ_1, φ_2 and ψ_1, ψ_2 be GDDSS on a graph Γ. Suppose the family $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a GDDSS. Then $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a DDSS on the set $V(\Gamma)$ and by Lemma 4, $\varphi_i = (\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$ and $\psi_i = (\psi_i \cap \varphi_1) \vee (\psi_i \cap \varphi_2)$, $i = 1, 2$.

Conversely, if $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ are mutually permutable and $\varphi_i = (\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$, $\psi_i = (\psi_i \cap \varphi_1) \vee (\psi_i \cap \varphi_2)$, $i = 1, 2$, then the family $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a DDSS. Indeed, they satisfy conditions (i) and (iii) of Definition 3. Let $a(\varphi_1 \cap \psi_1)b$ and $a((\varphi_1 \cap \psi_2) \vee (\varphi_2 \cap \psi_1) \vee (\varphi_2 \cap \psi_2))b$. As $(\varphi_1 \cap \psi_2) \vee (\varphi_2 \cap \psi_1) \vee (\varphi_2 \cap \psi_2) = (\varphi_1 \cap \psi_2) \vee \varphi_2$, there is $c \in V(\Gamma)$ such that $a(\varphi_1 \cap \psi_2)c\varphi_2 b$. Hence $c\varphi_1 a\varphi_1 b$ and $c(\varphi_1 \cap \varphi_2)b$. As $\varphi_1 \cap \varphi_2 = \Delta$, $c = b$. Then $a(\psi_1 \cap \psi_2)b$ and $a = b$. The other cases to verify condition (ii) of Definition 3 are similar. If $ab \in E(\Gamma)$, then $a\varphi_i b$ and $a\psi_j b$ for some $i, j = 1, 2$. It remains to verify the edge condition. Let $a(\varphi_1 \cap \psi_1)b$ and let $ab \in E(\Gamma)$. As ψ_1, ψ_2 satisfy the edge condition, if $a(\varphi_1 \cap \psi_2)c$, there is $d \in V(\Gamma)$ such that $c\psi_1 d$, $b\psi_2 d$ and $cd \in E(\Gamma)$. As $\varphi_1 \cap \psi_1$ and $\varphi_1 \cap \psi_2$ are permutable, there is a vertex e such that $b(\varphi_1 \cap \psi_2)e$ and $e(\varphi_1 \cap \psi_1)c$. Then $(e, d) \in \psi_1 \cap \psi_2 = \Delta$. Thus $e = d$ and so $\varphi_1 \cap \psi_1$ satisfies the edge condition relative to $\varphi_1 \cap \psi_2$. If $a(\varphi_2 \cap \psi_2)c$, again since ψ_1, ψ_2 satisfy the edge condition, there is $d \in V(\Gamma)$ such that $cd \in E(\Gamma)$, $c\psi_1 d$ and $b\psi_2 d$. As $\varphi_1 \cap \psi_1$ and $\varphi_2 \cap \psi_2$ are permutable and $c(\varphi_2 \cap \psi_2)a(\varphi_1 \cap \psi_1)b$, there is $e \in V(\Gamma)$ with $c(\varphi_1 \cap \psi_1)e$ and $e(\varphi_2 \cap \psi_2)b$. Thus $(e, d) \in \psi_1 \cap \psi_2 = \Delta$. So, $e = d$ and $\varphi_1 \cap \psi_1$ satisfies the edge condition relative to $\phi_2 \cap \psi_2$. The remaining cases are similar. Thus the family $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ is a GDDSS. □

The following definition is useful.

Definition 10. Let Γ be a graph and let φ, ψ be equivalence relations on $V(\Gamma)$. The relations φ, ψ are edge permutable if for any vertices $a, b, c \in V(\Gamma)$ such that $a\varphi b$, $a\psi c$, where ab, $ac \in E(\Gamma)$, there is $d \in V(\Gamma)$ such that $c\varphi d$, $b\psi d$ and cd, $bd \in E(\Gamma)$.

Lemma 7. *Let Γ be a graph and let φ, ψ be equivalence relations on $V(\Gamma)$. If φ, ψ are edge permutable and for every $v \in V(\Gamma)$, $\Gamma[v/\psi]$ is connected, then φ satisfies the edge condition relative to ψ.*

Proof. Let $a\varphi b$, $a\psi c$, $a \neq c$ and $ab \in E(\Gamma)$. Then there is a path $a = c_0, c_1, \ldots, c_n = c$ such that $c_i \psi c_{i+1}$ for all $0 \leq i < n$. As φ, ψ are edge permutable, there is $b_1 \in V(\Gamma)$ such that bb_1, $c_1 b_1 \in E(\Gamma)$, $c_1 \varphi b_1$ and $b\psi b_1$. By induction there is a path $b = b_0, b_1, \ldots, b_n$ such that $b_i \psi b_{i+1}$, $c_j \varphi b_j$, $c_j b_j \in E(\Gamma)$ for all $0 \leq i < n$ and $1 \leq j \leq n$. Thus $b\psi b_n$, $c\varphi b_n$ and $cb_n \in E(\Gamma)$. □

Proof of Theorem 9. Suppose Γ is a connected graph, $a \in V(\Gamma)$ and φ_1, φ_2 and ψ_1, ψ_2 are two GDDSSs on Γ. By Theorem 5, $\Gamma \cong \Gamma[a/\varphi_1] \times \Gamma[a/\varphi_2] \cong \Gamma[a/\psi_1] \times \Gamma[a/\psi_2]$. Since Cartesian factors of connected graphs are connected (cf. Sabidussi [**23**]), from Theorem 5, $\Gamma[a/\varphi_i]$, $\Gamma[a/\psi_i]$, $i = 1, 2$ are connected. We need to show that $\varphi_i \cap \psi_j$ $(i, j = 1, 2)$ is a GDDSS. First we show that φ_1, φ_2 and similarly ψ_1, ψ_2 are edge permutable. Indeed, suppose $a, b, c \in V(\Gamma)$, $a\varphi_1 b$, $a\varphi_2 c$ and ab, $ac \in E(\Gamma)$. As φ_1, φ_2 satisfy the edge condition, there is $d \in V(\Gamma)$ such that $cd \in E(\Gamma)$ and $c\varphi_1 d$, $b\varphi_2 d$. Reversing the roles of φ_1, φ_2, there is $e \in V(\Gamma)$ such that $be \in E(\Gamma)$ and $c\varphi_1 e$, $b\varphi_2 e$. Thus $d\varphi_1 c$, $d\varphi_2 b$, $e\varphi_1 c$, $e\varphi_2 b$. As $\varphi_1 \cap \varphi_2 = \Delta$, $d = e$ and φ_1, φ_2 are edge permutable. Next we show that any two of the four equivalence relations $\varphi_i \cap \psi_j$, $i, j = 1, 2$ are edge permutable.

This does not require that Γ be connected. Let ab, $ac \in E(\Gamma)$ and $a(\varphi_1 \cap \psi_1)b$. Suppose $a(\varphi_1 \cap \psi_2)c$. As ψ_1, ψ_2 are edge permutable, there is $d \in V(\Gamma)$ such that $c\psi_1 d$, $b\psi_2 d$ and bd, $cd \in E(\Gamma)$. Now $c\varphi_1 d$ or $c\varphi_2 d$ and $b\varphi_1 d$ or $b\varphi_2 d$. Also $c\varphi_1 d$ iff $b\varphi_1 d$. If $c\varphi_2 d$, then $b\varphi_2 d$ and $c(\varphi_1 \cap \varphi_2)b$. Thus $b = c$ and $a(\psi_1 \cap \psi_2)b$; i.e., $a = b$ contradicting $ab \in E(\Gamma)$. Hence $c(\varphi_1 \cap \psi_1)d$ and $b(\varphi_1 \cap \psi_2)d$. Thus $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$ are edge permutable. If $a(\varphi_2 \cap \psi_2)c$, there is $d \in V(\Gamma)$ such that $c\psi_1 d$ and $b\psi_2 d$ where bd, $cd \in E(\Gamma)$. Again $c\varphi_1 d$ or $c\varphi_2 d$ and $b\varphi_1 d$ or $b\varphi_2 d$. If $c\varphi_1 d$ and $b\varphi_1 d$, then $a(\varphi_1 \cap \varphi_2)c$ and $a = c$ contradicting $ac \in E(\Gamma)$. If $c\varphi_2 d$ and $b\varphi_1 d$, then $a(\varphi_1 \cap \varphi_2)d$ and $a = d$. Then $c(\psi_1 \cap \psi_2)d$ and $c = d$ contradicting $cd \in E(\Gamma)$. If $c\varphi_2 d$ and $b\varphi_2 d$, then $a(\varphi_1 \cap \varphi_2)b$ and $a = b$ contradicting $ab \in E(\Gamma)$. Thus the only possibility is $c\varphi_1 d$ and $b\varphi_2 d$; i.e., $c(\varphi_1 \cap \psi_1)d$, $b(\varphi_2 \cap \psi_2)d$. Thus $\varphi_1 \cap \psi_1$, $\varphi_2 \cap \psi_2$ are edge permutable. The treatment of the remaining pairs is similar. If Γ is connected, we shall show that $\Gamma[a/(\varphi_1 \cap \psi_1)]$ is connected. If $a, c \in V(\Gamma)$, $a \neq c$ and $a(\varphi_1 \cap \psi_1)c$, there is a path $a = c_0, c_1, \ldots, c_n = c$ in $\Gamma[a/\varphi_1]$. As $c_k c_{k+1} \in E(\Gamma)$ for $0 \leq k < n$, $c_k \psi_1 c_{k+1}$ or $c_k \psi_2 c_{k+1}$ for $0 \leq k < n$. Suppose for some $0 \leq s < n$, $(c_s, c_{s+1}) \notin \psi_1$. As $\varphi_1 \cap \psi_1$ and $\varphi_1 \cap \psi_2$ are edge permutable, if $(c_{k-1}, c_k) \in \psi_2$ and $(c_k, c_{k+1}) \in \psi_1$, there is c'_k such that $(c_{k-1}, c'_k) \in \varphi_1 \cap \psi_1$ and $(c'_k, c_{k+1}) \in \varphi_1 \cap \psi_2$. Thus we can assume that in the given path, for some $0 < r \leq n$, $(c_k, c_{k+1}) \in \psi_1$ for all $0 \leq k < r$ and $(c_k, c_{k+1}) \in \psi_2$ for all $r \leq k < n$. Thus $a(\varphi_1 \cap \psi_1)c_r(\varphi_1 \cap \psi_2)c$. Then $c_r(\psi_1 \cap \psi_2)c$ and $c = c_r$. Thus there is a path $a = c_0, c_1, \ldots, c_n = c$ in $\Gamma[a/(\varphi_1 \cap \psi_1)]$. This shows that for any $v \in V(\Gamma)$ and for any i, $j = 1, 2$, the subgraph $\Gamma[v/(\varphi_i \cap \psi_j)]$ is connected. Now we show that the family $\varphi_i \cap \psi_j$ $(i, j = 1, 2)$ satisfies the edge condition. This follows from Lemma 7. We need to show that any pair of $\varphi_i \cap \psi_j$ are permutable. Let $a(\varphi_i \cap \psi_j)b$, $a(\varphi_r \cap \psi_s)c$, where i, j, r, $s \in \{1, 2\}$ and $(i, j) \neq (r, s)$. As $\Gamma[a/(\varphi_i \cap \psi_j)]$ is connected, there is a path $a = b_0, b_1, \ldots, b_n = b$ such that $(b_k, b_{k+1}) \in \varphi_i \cap \psi_j$ for all $0 \leq k < n$. By the edge condition there is d_1 with $cd_1 \in E[\Gamma]$, $(b_1, d_1) \in \varphi_r \cap \psi_s$ and $(c, d_1) \in \varphi_i \cap \psi_j$. By induction there is a path $c = d_0, d_1, \ldots, d_n$ in $\Gamma[\varphi_i \cap \psi_j]$ such that $(b_k, d_k) \in \varphi_r \cap \psi_s$ for every $1 \leq k \leq n$. Thus there is $d(= d_n)$ such that $c(\varphi_i \cap \psi_j)d(\varphi_r \cap \psi_s)b$. This shows the permutability of $\varphi_i \cap \psi_j$ and $\varphi_r \cap \psi_s$. By Lemma 6, and by symmetry, it suffices to show that $\varphi_i = (\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$. Let $a\varphi_i b$. As $\Gamma[a/\varphi_i]$ is connected, there is a path $a = b_0, b_1, \ldots, b_n = b$ in $\Gamma[a/\varphi_i]$. Every $(b_k, b_{k+1}) \in \psi_r$ for some $r \in \{1, 2\}$ and thus belongs to $(\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$. Thus $(a, b) \in (\varphi_i \cap \psi_1) \vee (\varphi_i \cap \psi_2)$. This shows that $\varphi_1 \cap \psi_1$, $\varphi_1 \cap \psi_2$, $\varphi_2 \cap \psi_1$, $\varphi_2 \cap \psi_2$ form a GDDSS. $\square$

If a graph satisfies the strict refinement property for restricted Cartesian products, it satisfies the property for any two GDDSSs. However, the Cartesian product of an infinite family of nontrivial connected graphs is not connected as shown in [**23**]. For general structures, as indicated in [**6**], the strict refinement property carries over to infinite direct products. Since pointed sets do not satisfy the strict refinement property, there are (disconnected) graphs that do not satisfy GSRP. If φ_i ($i \in I$) and ψ_j ($j \in J$) are GDDSSs on a connected graph Γ, $a \in V(\Gamma)$ and all $\Gamma[a/\varphi_i]$ and $\Gamma[a/\psi_j]$ are Cartesian indecomposable, then $|I| = |J|$, $\{\varphi_i : i \in I\} = \{\psi_j : j \in J\}$ and so $\{\Gamma[a/\varphi_i] : i \in I\} = \{\Gamma[a/\psi_j] : j \in J\}$. The following theorem is due to Imrich ([**14**, Szatz 4]). We shall give a proof using methods from the present paper.

Theorem 10. *Every connected graph is a restricted Cartesian product of Cartesian indecomposable graphs.*

The proof will be based on the following lemmas:

Lemma 8. *Let Γ be a connected graph and let α be an equivalence relation on $V(\Gamma)$. Then α is a graph direct factor on Γ iff*

(i) If $a\alpha b$, $(a,c) \notin \alpha$ and ab, $ac \in E(\Gamma)$, then there is $d \in V(\Gamma)$ such that $c\alpha d$, $(b,d) \notin \alpha$ and bd, $cd \in E(\Gamma)$.

(ii) If $a_0 a_1 \dots a_n$ is a path and $(a_i, a_{i+1}) \notin \alpha$ for $0 \le i < n$ and $a_0 \ne a_n$, then $(a_0, a_n) \notin \alpha$.

Proof. If α, β is a GDSS, then (i) follows from the edge condition and ii follows from $(a_i, a_{i+1}) \in \beta$, for $0 \le i < n$ and so $(a_0, a_n) \in \beta$. As $\alpha \cap \beta = \Delta$, $(a_0, a_n) \notin \alpha$. Conversely, the set $\{(x,y) : xy \in E(\Gamma), (x,y) \notin \alpha\}$ generates an equivalence relation β on $V(\Gamma)$. We need to show that α, β is a GDSS. It is clear that for every $v \in V(\Gamma)$, $\Gamma[v/\beta]$ is connected. From (i), α, β are edge permutable. By Lemma 7, α satisfies the edge condition relative to β. We shall show that for every $v \in V(\Gamma)$, $\Gamma[v/\alpha]$ is connected. Indeed, let $a\alpha b$ and $a \ne b$. As Γ is connected, there is a path $a = c_0, c_1, \dots, c_n = b$. If $(c_i, c_{i+1}) \notin \alpha$, then $c_i \beta c_{i+1}$. In view of the edge permutabilty of α, β we can assume that $c_i \alpha c_{i+1}$ for all $0 \le i < r$ and $c_i \beta c_{i+1}$ for all $r \le i < n$. If $r = n$, we are through. Otherwise, $b\beta c_r$, $a\alpha c_r$ and $a\alpha b$. Thus $b\alpha c_r$. In view of (ii), $b = c_r$ and $\Gamma[a/\alpha]$ is connected. Again, applying Lemma 7, β satisfies the edge condition relative to α. Every edge belongs to either α or β. Thus we need only show that α, β is a DSS. If $a, b \in V(\Gamma)$ and $a \ne b$, then there is a path from a to b. As α, β are edge permutable, we can assume the existence of a path $a = b_0, b_1, \dots, b_n = b$ such that $b_i \alpha b_{i+1}$ and $b_j \beta b_{j+1}$ implies $i < j$. Thus $V(\Gamma) \times V(\Gamma) = \alpha \circ \beta$. From (ii), $\alpha \cap \beta = \Delta$. Thus α, β form a GDSS and α is a graph direct factor. $\square$

Lemma 9. *If φ_i is a graph direct factor on a connected graph Γ for every $i \in I$, then $\bigcap\{\varphi_i : i \in I\}$ is a graph direct factor on Γ.*

Proof. Let $\alpha = \bigcap\{\varphi_i : i \in I\}$. We need to show that α is a graph direct factor. Let $a\alpha b$, ab, $ac \in E(\Gamma)$ and $(a,c) \notin \alpha$. There is $k \in I$ such that $(a,c) \notin \varphi_k$. As φ_k is a graph direct factor, there is $d \in V(\Gamma)$ such that $c\varphi_k d$, $(b,d) \notin \varphi_k$,and bd, $cd \in E(\Gamma)$. If $c\varphi_k d'$, $bd' \in E(\Gamma)$ and $(b,d') \notin \varphi_k$, then $d = d'$, otherwise, dbd' is a path where db, $bd' \in E(\Gamma)$, $(d,b) \notin \varphi_k$, $(b,d') \notin \varphi_k$ and $d\varphi_k c \varphi_k d'$ contradicting

(ii) of Lemma 8. Let $j \in I$. Applying GSRP to the GDDSSs φ_k, φ'_k and φ_j, φ'_j, where φ_k, φ'_k and φ_j, φ'_j are (graph) factor pairs, $\varphi_k \cap \varphi_j$, $\varphi_k \cap \varphi'_j$, $\varphi'_k \cap \varphi_j$, $\varphi'_k \cap \varphi'_j$ is a GDDSS on Γ. Thus $\varphi_k \cap \varphi_j$ is a graph direct factor. As $a(\varphi_k \cap \varphi_j)b$, $(a, c) \notin \varphi_k \cap \varphi_j$ and ab, $ac \in E(\Gamma)$, there is $e \in V(\Gamma)$ such that $c(\varphi_k \cap \varphi_j)e$, $(b, e) \notin \varphi_k \cap \varphi_j$ and be, $ce \in E(\Gamma)$. If $(b, e) \in \varphi_k$, then $a\varphi_k e\varphi_k c$ contradicting $(a, c) \notin \varphi_k$. Thus $(b, e) \notin \varphi_k$. Hence $d = e$. Thus $c\varphi_i d$ for every $i \in I$; i.e., $c\alpha d$. As $(b, d) \notin \varphi_k$, $(b, d) \notin \alpha$ and so α satisfies (i) of Lemma 8. We need to show that α satisfy (ii) of Lemma 8. If $a_0a_1 \dots a_n$ is a path, $a_0 \neq a_n$ and $(a_i, a_{i+1}) \notin \alpha$ for $0 \leq i < n$, there is a finite set $F \subseteq I$ such that $\beta = \bigcap\{\varphi_r : r \in F\}$ and $(a_i, a_{i+1}) \notin \beta$ for $0 \leq i < n$. As F is finite, GSRP implies β is a graph direct factor and so $(a_0, a_n) \notin \beta$. Since $\alpha \subseteq \beta$, $(a_0, a_n) \notin \alpha$. Thus α is a graph direct factor. □

Proof of Theorem 10. Let Γ be a connected graph with at least two vertices. For every $ab \in E(\Gamma)$, let $\varphi_{ab} = \bigcap\{\varphi : a\varphi b \text{ and } \varphi \text{ is a graph direct factor on } \Gamma\}$. We shall show that $\{\varphi_{ab} : ab \in E(\Gamma)\}$ is a GDDSS on Γ and for any $v \in V(\Gamma)$, $\Gamma[v/\varphi_{ab}]$ is Cartesian indecomposable. Since $a \neq b$, $\varphi_{ab} \neq \Delta$. Every φ_{ab} is a graph direct factor on Γ by Lemma 9. If $\Gamma[v/\varphi_{ab}] \cong \Gamma_1 \oplus \Gamma_2$, then $\varphi_{ab} = \chi \vee \psi$, where χ, ψ are graph direct factors on Γ and $\Gamma[v/\varphi_{ab}] \cong \Gamma[v/\chi] \oplus \Gamma[v/\psi]$. Hence either $a\chi b$, or $a\psi b$. If $a\chi b$, then $\varphi_{ab} \subseteq \chi \subseteq \varphi_{ab}$. Thus $\Gamma[v/\varphi_{ab}]$ is Cartesian indecomposable. If $\varphi_{ab} \neq \varphi_{ef}$, then $\varphi_{ab} \cap \varphi_{ef} = \Delta$. Otherwise, $\varphi_{ab} \cap \varphi_{ef}$ is a nontrivial graph direct factor and $\varphi_{ab} \cap \varphi_{ef}$, $\varphi_{ab} \cap \varphi'_{ef}$, $\varphi'_{ab} \cap \varphi_{ef}$, $\varphi'_{ab} \cap \varphi'_{ef}$ is a GDDSS by GSRP and $\varphi_{ab} = (\varphi_{ab} \cap \varphi_{ef}) \vee (\varphi_{ab} \cap \varphi'_{ef})$. As $\Gamma[a/\varphi_{ab}]$ is Cartesian indecomposable, $\varphi_{ab} \subseteq \varphi_{ef}$ or $\varphi_{ab} \subseteq \varphi'_{ef}$. If $\varphi_{ab} \subseteq \varphi_{ef}$, then $\varphi_{ef} = (\varphi_{ab} \cap \varphi_{ef}) \vee (\varphi'_{ab} \cap \varphi_{ef})$, again by GSRP. As $\Gamma[v/\varphi_{ef}]$ is Cartesian indecomposable, $\varphi_{ef} = \varphi_{ab} \cap \varphi_{ef}$ or $\varphi_{ef} = \varphi'_{ab} \cap \varphi_{ef}$. The first option implies $\varphi_{ef} = \varphi_{ab}$ which is a contradiction. The other option $(\varphi_{ef} = \varphi'_{ab} \cap \varphi_{ef})$ contradicts $\varphi_{ab} \subseteq \varphi_{ef}$. Thus $\varphi_{ab} \subseteq \varphi'_{ef}$ and $\varphi_{ab} \cap \varphi_{ef} = \Delta$. Then $(e, f) \in \varphi'_{ab}$ for every $ef \in E(\Gamma)$, $\varphi_{ef} \neq \varphi_{ab}$ since $(e, f) \notin \varphi_{ab}$. Thus $\varphi_{ef} \subseteq \varphi'_{ab}$ for every $ef \in E(\Gamma)$ such that $\varphi_{ab} \neq \varphi_{ef}$. Hence $\bigvee\{\varphi_{ef} : ef \in E(\Gamma), \varphi_{ef} \neq \varphi_{ab}\} \subseteq \varphi'_{ab}$. Actually $\varphi'_{ab} = \bigvee\{\varphi_{ef} : ef \in E(\Gamma), \varphi_{ef} \neq \varphi_{ab}\}$, since for every $cd \in E(\Gamma)$, $(c, d) \notin \varphi_{ab}$ implies $\varphi_{cd} \neq \varphi_{ab}$. Thus $\varphi_{ab} \cap (\bigvee\{\varphi_{ef} : ef \in E(\Gamma), \varphi_{ef} \neq \varphi_{ab}\}) = \Delta$. If $\varphi_{ab} \neq \varphi_{cd}$, then φ_{ab}, φ'_{ab} and φ_{cd}, φ'_{cd} are GDDSSs. Hence by GSRP, φ_{ab}, φ_{cd}, $\varphi'_{ab} \cap \varphi'_{cd}$ is a GDDSS since $\varphi_{ab} \cap \varphi_{cd} = \Delta$, $\varphi'_{ab} \cap \varphi_{cd} = \varphi_{cd}$ and $\varphi_{ab} \cap \varphi'_{cd} = \varphi_{ab}$. Thus $\varphi_{ab} \circ \varphi_{cd} = \varphi_{cd} \circ \varphi_{ab}$. As $\bigvee\{\varphi_{xy} : xy \in E(\Gamma)\} = \varphi_{ab} \vee (\bigvee\{\varphi_{ef} : ef \in E(\Gamma), \varphi_{ef} \neq \varphi_{ab}\}) = \varphi_{ab} \vee \varphi'_{ab} = V(\Gamma) \times V(\Gamma)$ and if $uv \in E(\Gamma)$, then $u\varphi_{uv}v$. This shows that (iii) of Definition 6 holds. Thus φ_{xy} $(xy \in E(\Gamma))$ is a GDDSS on Γ. Since every $\Gamma[v/\varphi_{ab}]$ is Cartesian indecomposable for every $v \in V(\Gamma)$ and every $ab \in E(\Gamma)$, by Theorem 5, Γ is a restricted Cartesian product of Cartesian indecomposable graphs. □

The factorization in Theorem 10 is essentially unique. On a connected graph Γ, the relation $ab \sim cd$ iff $\varphi_{ab} = \varphi_{cd}$ is an equivalence relation on $E(\Gamma)$, where φ_{ab} is the smallest graph direct factor on Γ containing (a, b). Let $T(\Gamma)$ be a transversal of the equivalence relation $\sim$; i.e., $T(\Gamma) \subseteq E(\Gamma)$ such that if ab, $cd \in T(\Gamma)$ and $\{a,b\} \neq \{c,d\}$, then $\varphi_{ab} \neq \varphi_{cd}$ and if $xy \in E(\Gamma)$, there is $ef \in T(\Gamma)$ such that $\varphi_{xy} = \varphi_{ef}$. Then the uniqueness of factorization can be expressed as follows:

Theorem 11. *Let $\Gamma \cong \sum\{(\Gamma_i, v_i) : i \in I\}, v_i \in V(\Gamma_i)$, $i \in I$. Suppose Γ is connected and Γ_i is a Cartesian indecomposable graph for every $i \in I$. Then there is a bijective mapping $g\colon I \longrightarrow T(\Gamma)$ such that $\Gamma_i \cong \Gamma[v/\varphi_{g(i)}]$, $i \in I$ where $v \in V(\Gamma)$ corresponds to $(\dots, v_i, \dots) \in V(\sum\{(\Gamma_i, v_i) : i \in I\})$.*

This theorem states the uniqueness of decomposition of connected graphs as restricted Cartesian products of Cartesian indecomposable graphs. It is essentially Szatz 5 in Imrich [**14**].

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