Acta Math. Univ. Comenianae
Vol. LXVIII, 1(1999), pp. 27–35

AN OMEGA THEOREM ON DIFFERENCES OF TWO SQUARES, II

M. KÜHLEITNER

Abstract. Let $\rho(n)$ denote the number of pairs $(u,v)\in\mathbb{N}\times\mathbb{Z}$ with $u^2-v^2=n$. Due to a formula of Sierpinski, $\rho(n)$ is closely related to the classical divisor function $d(n)$. We establish a lower bound for the remainder term in the asymptotic expansion for the Dirichlet summatory function of $\rho(n)$.

1. Introduction

As in part I of this paper [**8**], let $\rho(n)$ denote the number of pairs $(u,v)\in\mathbb{N}\times\mathbb{Z}$ with $u^2-v^2=n$. For the more general case where the square is replaced by a "k"-th power $k\geq 2$ see Krätzel [**6**], [**7**] and the recent paper of Nowak [**9**]. Due to an elementary formula of Sierpinski, our function $\rho(n)$ is closely related to the classical divisor function $d(n)$ by

$$\rho(n)=d(n)-2\,d\Big(\frac{n}{2}\Big)+2\,d\Big(\frac{n}{4}\Big), \tag{1}$$

where $d(\cdot)=0$ for non-integers, due to Sierpinski.

For a large real variable x, we consider the remainder term $\theta(x)$ in the asymptotic formula

$$T(x)=\sum_{n\leq x}\rho(n)=\frac{x}{2}\log x+(2\gamma-1)\frac{x}{2}+\theta(x),$$

where γ denotes throughout this paper the Euler-Mascheroni constant.

Upper bounds for $\theta(x)$ can be readily established as a trivial generalization of the corresponding results for the Dirichlet divisor problem. It is known that

$$D(x)=x\log x+(2\gamma-1)x+\Delta(x)$$

with

$$\Delta(x)\ll x^{23/73}(\log x)^{461/146}.$$

Received August 4, 1997.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 11N37.
Key words and phrases. Divisor problem, Dirichlet summatory function, asymptotic expansion.

(See Huxley [**5**] for this upper bound and the textbook of Krätzel [**6**] for an enlightening survey of the theory of Dirichlet's divisor problem and the definition of the O- and the Ω- symbols.)

Concerning lower estimates, the author proved in [**8**], on the basis of [**1**] and Hafner's method [**3**], that

$$\theta(x) = \Omega_+\Big((x \log x)^{1/4} (\log\log x)^{(3+2\log 2)/4} \exp\big(-A\sqrt{\log\log\log x}\ \big)\Big).$$

The aim of the present article is an Ω_-- result for $\theta(x)$, corresponding to that of Corrádi and Kátai [**1**] for the divisor problem.

Theorem.
$$T(x) = \frac{x}{2}\log x + (2\gamma - 1)\frac{x}{2} + \theta(x),$$
with
$$\theta(x) = \Omega_-\Big(x^{1/4} \exp\big(c(\log\log x)^{1/4}(\log\log\log x)^{-3/4}\big)\Big),$$
where c *is a positive absolute constant.*

2. Notations and Lemmas

For large real x we define P_x as the set of all primes less than or equal to x, and Q_x the set of all square-free integers composed only of primes from P_x. We write $|P_x|$ for the cardinality of P_x and $M = 2^{|P_x|}$ for the cardinality of Q_x. We then have
$$|P_x| \asymp \frac{x}{\log x} \qquad \text{and} \qquad M \ll \exp\big(c_1 \frac{x}{\log x}\big),$$
for some positive constant c_1. The largest integer in Q_x is bounded by e^{2x}, since for $q \in Q_x$, we have
$$\log q \le \sum_{p \le x} \log p \le 2x.$$

Let S_x be the set of numbers defined by

$$S_x = \Big\{\mu = \sum_{q \in Q_x} r_q \sqrt{q} \quad \text{where } r_q \in \{0, \pm 1\} \text{ and at least two } r_q \ne 0\Big\}.$$

Finally let
$$\eta(x) = \inf\Big\{\big|\sqrt{n} + 2\mu\big| \ \text{ with } n \in \mathbb{N}_o \text{ and } \mu \in S_x\Big\},$$
and
$$q(x) = -\log\big(\eta(x)\big).$$

By a slight modification of the method used for the corresponding result in Gangadharan [**2**], one readily shows the following lemma.

Lemma 1. *For $x \to \infty$ we have*

$$x \ll q(x) \ll \exp\Big(c_2 \frac{x}{\log x}\Big),$$

for some positive constant c_2.

Lemma 2. *There exists a positive constant c_3 such that*

$$\sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} \gg \exp\Big(c_3 \frac{x^{1/4}}{\log x}\Big).$$

Proof. By the definition of Q_x, we have

$$\begin{aligned}\sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} &= \prod_{p \le x} (1 + 2\, p^{-3/4}) = \exp\Big(\sum_{p \le x} \log\,(1 + 2\, p^{-3/4})\Big)\\ &\ge \exp\Big(\sum_{p \le x} p^{-3/4} + O(1)\Big) \gg \exp\Big(c_3 \frac{x^{1/4}}{\log x}\Big).\end{aligned}$$

□

As in Gangadharan [**2**] define for real z,

$$V(z) = 2(\cos(\frac{z}{2}))^2 = 1 + \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2},$$

and

$$T_x(u) = \prod_{q \in Q_x} V\Big(u\sqrt{q} - \frac{5\pi}{4}\Big).$$

Lemma 3. *We have*

(1) $0 \le T_x(u) \le 2^M$, *for all u,*
(2) $T'_x(u) \ll M\, 2^M\, e^x$, *for all u,*
(3) $T_x(u) = T_0 + T_{1,x} + T_{2,x} + T_{3,x}$ *where,*

$$\begin{aligned}T_0 &= 1,\\ T_{1,x} &= \frac{e^{5\pi\mathrm{i}/4}}{2} \sum_{q \in Q_x} e^{-\mathrm{i}u\sqrt{q}}\\ T_{3,x} &= \sum_{\mu \in S_x} h_\mu e^{\mathrm{i}u\mu},\end{aligned}$$

$T_{2,x}$ is the complex conjugate of $T_{1,x}$ and $|h_\mu| \le 1/4$.

Proof. The proof of Lemma 3 is straightforward by the definition of $V(z)$ and $T_x(u)$.

3. Proof of the Theorem

We start with the well known Voronoi identity for

$$\Delta_1(x) \overset{\text{def}}{=} \int_0^x \Delta(t)\,dt = \frac{x}{4} + \frac{x^{3/4}}{2\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(1).$$

Inserting this in

$$\theta(x) = \Delta(x) - 2\,\Delta\left(\frac{x}{2}\right) + 2\,\Delta\left(\frac{x}{4}\right),$$

and substituting $T = 4\pi\sqrt{x}$, we get

$$\begin{aligned} E_1(T) &\overset{\text{def}}{=} \int_0^T E(t)\,t\,dt \\ &= T^{3/2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \Big(\sin\left(T\sqrt{n} - \pi/4\right) - 2^{5/4} \sin\left(T\sqrt{n/2} - \pi/4\right) \\ &\quad + 2^{3/2} \sin\left(T\sqrt{n/4} - \pi/4\right)\Big), \end{aligned}$$

with

$$E(t) = 2\pi\sqrt{2\pi}\Big(\,\theta(t^2/16\pi^2) - 1/4\Big).$$

Define

$$P(x) = \exp\left(a\frac{x}{\log x}\right)$$

such that

$$q(x) \le P(x) \qquad \text{and} \qquad M^2 \le P(x),$$

and let

$$\sigma_x = \exp\left(-2P(x)\right).$$

Next define for fixed x,

$$\gamma_x = \sup_{u>0} \frac{-2\pi\sqrt{2\pi}\,\theta(u^2/16\pi^2)}{u^{1/2+1/P(x)}}\,.$$

We may assume that $\gamma_x < \infty$, otherwise more than Theorem 1 would be true. Thus

$$\gamma_x u^{1/2+1/P(x)} + A + E(u) \ge 0, \tag{2}$$

for all u, where $A = 2\pi\sqrt{2\pi}/4$.

Let

$$J_x = \sigma_x^{5/2} \int_0^{\infty} \left(\gamma_x u^{1/2+1/P(x)} + A + E(u)\right) u \exp\left(-\sigma_x u\right) T_x(u)\,du.$$

The next lemma provides an asymptotic expansion for J_x.

Lemma 4. *For $x \to \infty$,*

$$J_x = e^2\,\Gamma\big(\frac{5}{2}\big)\gamma_x - \frac{1}{4}\,\Gamma\big(\frac{5}{2}\big)\sum_{q\in Q_x}\frac{d(q)}{q^{3/4}} + o(\gamma_x) + o(1).$$

Proof. Do deal with the first two terms of J_x, we observe that, for $r = 1$ or $r = \frac{3}{2} + \frac{1}{P(x)}$,

$$\int_0^\infty u^r \exp(-\sigma_x u) T_x(u)\,du = \Gamma(1+r)\,\sigma_x^{-(1+r)} + \sum_{i=1,2,3}\int_0^\infty u^r \exp(-\sigma_x u)\,T_{i,x}(u)\,du$$

where $1 \le r \le \frac{3}{2} + \frac{1}{P(x)}$.

The part of $T_{1,x}$ contributes exactly,

$$\frac{e^{5\pi i/4}}{2}\,\Gamma(1+r)\sum_{q\in Q_x}\frac{1}{(\sigma_x + i\sqrt{q})^{1+r}} \ll \sum_{q\in Q_x} q^{-(1+r)/2} \ll \sum_{q\in Q_x} 1 \ll M \ll \sqrt{P(x)} = o(\sigma_x^{-5/2}).$$

The contribution of $T_{2,x} = \overline{T_{1,x}}$ is obviously no more than this. Finally $T_{3,x}$ contributes

$$\sum_{\mu\in S_x}\frac{h_\mu}{(\sigma_x + i\mu)^{1+r}} \ll 3^M \eta(x)^{-(1+r)} \ll \exp\big(M\ln 3 + (1+r)(-\log\eta(x)\big) \ll \exp(3P(x)) = o(\sigma_x^{-5/2}).$$

Next we deal with the contribution of $E(u)$ to J_x. Our first step is to integrate by parts to introduce $E_1(u)$ in the integral. Thus,

$$I \stackrel{\text{def}}{=} \int_0^\infty E(u)u\exp(-\sigma_x u)\,T_x(u)\,du = -\int_0^\infty E_1(u)\frac{d}{du}\big(\exp(-\sigma_x u)\,T_x(u)\big)\,du,$$

since $E_1(u) \ll u^{3/2}$ for large u and $E_1(0) = 0$. Inserting the series representation for $E_1(u)$ and integrating term by term, noting that the series converges absolutely for every u and uniformly on compact sets, we get

$$I = -\sum_{n=1}^\infty \frac{d(n)}{n^{5/4}}\text{Im}\,(e^{-\pi i/4} I_n) + O\Big(\int_0^\infty \Big|\frac{d}{du}\big(\exp(-\sigma_x u) T_x(u)\big)\Big|\,du\Big) + O\Big(\int_0^\infty u^{1/2}\exp(-\sigma_x u)|T_x(u)|\,du\Big),$$

since

$$u^{3/2}\frac{d}{du}\big(\exp(-\sigma_x u)\,T_x(u)\big) = \frac{d}{du}\big(u^{3/2}\exp(-\sigma_x u)\,T_x(u)\big) - \frac{3}{2}u^{1/2}\exp(-\sigma_x u)\,T_x(u),$$

and

$$I_n \overset{\text{def}}{=} \int_0^\infty (\mathrm{e}^{iu\sqrt{n}} - 2^{5/4}\,\mathrm{e}^{iu\sqrt{n/2}} + 2^{3/2}\,\mathrm{e}^{iu\sqrt{n/4}})\frac{d}{du}\Big(u^{3/2}\exp(-\sigma_x u)\,T_x(u)\Big)\,du.$$

Estimating the contributions of the error terms, we see that

$$\begin{aligned}\int_0^\infty \left|\frac{d}{du}\big(\exp(-\sigma_x u)\,T_x(u)\big)\right|\,du &\le \int_0^\infty |T_x(u)' - \sigma_x\,T_x(u)|\exp(-\sigma_x u)\,du \\ &\le 4^M\sigma_x^{-1} + 2^M \\ &\ll \exp\big(c\sqrt{P(x)}\,\big)\big(1+\exp(2\,P(x))\big) = o(\sigma_x^{-5/2}),\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty u^{1/2}\exp(-\sigma_x u)\,|T_x(u)|\,du &\ll 2^M\int_0^\infty u^{1/2}\exp(-\sigma_x u)\,du \\ &\ll 2^M\sigma_x^{-3/2} \ll \exp\big(c\sqrt{P(x)} + 3\,P(x)\big) = o(\sigma_x^{-5/2}).\end{aligned}$$

We integrate I_n by parts once more and expand $T_x(u)$ as in (3) of Lemma 3, to get

$$\begin{aligned}I_n &= -i\sum_{k=0,\dots,3}\int_0^\infty \left(\sqrt{n}\mathrm{e}^{iu\sqrt{n}} - 2^{5/4}\sqrt{\frac{n}{2}}\mathrm{e}^{iu\sqrt{n/2}} + 2^{3/2}\sqrt{\frac{n}{4}}\mathrm{e}^{iu\sqrt{n/4}}\right) \\ &\quad\times u^{3/2}\exp(-\sigma_x u)\,T_{i,x}(u)\,du \\ &= I_0(n) + I_1(n) + I_2(n) + I_3(n),\end{aligned}$$

for short. We shall show that the main term of I_n comes from $I_1(n)$. In fact, the contribution of $I_0(n)$ is

$$\ll \sqrt{n}\,|\sigma_x - i\sqrt{n}|^{-5/2} \ll n^{-3/4},$$

that of $I_2(n)$ is

$$\ll \sqrt{n}\sum_{q\in Q_x}|\sigma_x - i(\sqrt{n}+\sqrt{q})|^{-5/2} \ll Mn^{-3/4}.$$

The contribution of $I_3(n)$ is bounded by

$$I_3(n) \ll \sqrt{n} \sum_{\mu \in S_x} |\sigma_x - i(\sqrt{n} - \mu)|^{-5/2}$$
$$\ll \begin{cases} \sqrt{n}\, 3^M (\eta(x))^{-5/2}, & \text{if } n \le 2\max\{|\mu| : \mu \in S_x\} \\ \\ n^{-3/4}\, 3^M\,, & \text{else.} \end{cases}$$

This $\max\{|\mu| : \mu \in S_x\}$ is bounded by $M\mathrm{e}^{cx}$ for some positive constant c. Hence the total contribution to I is bounded by

$$\begin{aligned}
&\ll \sum_{n \le 2M\mathrm{e}^{cx}} \frac{d(n)}{n^{5/4}} \sqrt{n}\, 3^M \exp\left(-5 \log \frac{\eta(x)}{2}\right) + O\left(3^M \sigma_x^{-5/4} \sum_{n > 2M\mathrm{e}^{cx}} \frac{d(n)}{n^2}\right) \\
&\ll 3^M \sigma_x^{-5/4} \sum_{n \le 2M\mathrm{e}^{cx}} n^{-3/4+\epsilon} + O(3^M \sigma_x^{-5/4}) \\
&\ll 3^M \sigma_x^{-5/4} (M\mathrm{e}^{cx})^{1/4+\epsilon} \\
&= o(\sigma_x^{-5/2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I = &-\frac{1}{2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \mathrm{Im}\Big(i \sum_{q \in Q_x} \int_0^\infty \Big(\sqrt{n}\, \mathrm{e}^{iu(\sqrt{n}-\sqrt{q})} - 2^{5/4} \sqrt{\frac{n}{2}} \mathrm{e}^{iu(\sqrt{n/2}-\sqrt{q})} \\
&+ 2^{3/2} \sqrt{\frac{n}{4}} \mathrm{e}^{iu(\sqrt{n/4}-\sqrt{q})}\Big) u^{3/2} \exp\left(-\sigma_x u\right) du\Big) + o(\sigma_x^{-5/2}) \\
= &-\frac{1}{2} \sum_{q \in Q_x} \Big(\frac{d(q)}{q^{5/4}} - 2^{5/4} \frac{d(2q)}{(2q)^{5/4}} + 2^{3/2} \frac{d(4q)}{(4q)^{5/4}}\Big) \int_0^\infty \sqrt{q}\, u^{3/2} \exp\left(-\sigma_x u\right) du \\
&\qquad + O\Big(\sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \sum_{\substack{q \in Q_x \\ n \ne q}} \Big| \int_0^\infty \sqrt{n}\, \mathrm{e}^{iu(\sqrt{n}-\sqrt{q})}\, u^{3/2} \exp\left(-\sigma_x u\right) \Big|\, du\Big).
\end{aligned}$$

For this last error term we get a bound exactly as above for $I_3(n)$ with M replacing the factor 3^M, since

$$\sqrt{n} - \sqrt{q} \gg (\sqrt{n} + \sqrt{q})^{-1} \gg \mathrm{e}^{-x} \gg \exp\left(-P(x)\right),$$

for $n \le 2\max\{q : q \in Q_x\} \gg 2\mathrm{e}^{2x}$ and $n \ne q$.

We get,

$$\begin{aligned}
I &= -\frac{1}{2} \Gamma\left(\frac{5}{2}\right) \sigma_x^{-5/2} \Big(\sum_{q \in Q_x} \left(d(q) - d(2q) + \frac{1}{2} d(4q)\right) q^{-3/4} + o(\sigma_x^{-5/2}) \\
&= -\frac{1}{4} \Gamma\left(\frac{5}{2}\right) \sigma_x^{-5/2} \sum_{q \in Q_x} d(q) q^{-3/4} + o(\sigma_x^{-5/2}),
\end{aligned}$$

since

$$d(q) - d(2q) + \frac{1}{2}d(4q) = \frac{1}{2}d(q).$$

This completes the proof of Lemma 4. □

Since $\sigma_x > 0$ and $J_x > 0$ by (2), we have

$$\exp\left(c\frac{x^{1/4}}{\log x}\right) \ll \sum_{q \in Q_x} d(q) q^{-3/4} \ll \gamma_x,$$

by Lemma 2 and the last assertion by Lemma 4.

Thus by the definition of γ_x there is a sequence u_x which tends to infinity with x, such that

$$-\theta(u_x^2) \gg u_x^{1/2} \exp\left(\frac{\log u_x}{P(x)} + c\frac{x^{1/4}}{\log x}\right),$$

since $\theta(u)$ is bounded for bounded u, which follows for small u from

$$\theta(u) = -\frac{u}{2}\log u - (2\gamma - 1)\frac{u}{2}\,,$$

and is obvious for the other values of u.

Consider first the values of u_x for which

$$\frac{\log u_x}{P(x)} \le c\frac{x^{1/4}}{\log x}\,. \tag{3}$$

Taking logarithms on both sides, we have

$$\log\log u_x \ll \frac{x}{\log x}\,.$$

Since $y^{1/4}(\log y)^{-3/4}$ is an increasing function of y for sufficiently large y, we have from (3)

$$\frac{(\log\log u_x)^{1/4}}{(\log\log\log u_x)^{3/4}} \ll \frac{x^{1/4}}{\log x}\,,$$

from which the desired estimate follows.

Consider now those values of x for which

$$c\frac{x^{1/4}}{\log x} \le \frac{\log u_x}{P(x)}\,. \tag{4}$$

We may assume that

$$\frac{(\log\log u_x)^{1/4}}{(\log\log\log u_x)^{3/4}} \gg \frac{\log u_x}{P(x)}\,,$$

otherwise the estimate holds obviously. Taking logarithms on both sides gives

$$\log\log u_x \ll \frac{x}{\log x}\,,$$

from which the estimate follows as above. This proves the theorem. □

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M. Kühleitner, Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel Straße 33, A-1180 Wien, Austria; *e-mail*: kleitner@mail.boku.ac.at