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STRONG ESTIMATE FOR SQAURE FUNCTIONS IN HIGHER DIMENSIONS

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Abstract. In this paper, we give a generalization to multidimensional case of strong estimate for square functions obtained by Jones Rosenblatt and Ostrovskii [**3**].

We assume that the averages are taken over squares and the operators are commuting and contraction in L^2. For the non-commutative case we need a supplementary condition. Some weak type inequalities are proved.

Introduction

Let (Ω, β, μ) be a σ-finite measure space, and let $\tau\colon \Omega \to \Omega$ be an invertible β-measurable transformation preserving μ. For $Tf = fo\tau$, R. Jones, I. Ostrovskii and J. Rosenblatt [**3**] proved strong and weak L^1estimates for square functions and sqaure maximal functions. In this paper, we give a generalization to multidimensional case of some strong and weak L^1 estimates. In the first section, we prove strong estimates for linear contractions and for power bounded operators in L^2. In the second section, strong estimates for square maximal functions are obtained. In the third section, some weak estimates for linear contractions are proved. In [**3**] it was shown the following result:

Theorem 1. *Given the usual averages $A_nf = \frac{1}{n}\sum_{k=1}^{n} fo\tau^k$ in ergodic theory, let $n_1 \leq n_2 \leq \dots$ and $Sf = \left(\sum_{k=1}^{\infty} \left|A_{n_k}f - A_{n_{k-1}}f\right|^2\right)^{\frac{1}{2}}$*

1. *For all $f \in L^2$ we have $\|Sf\|_2 \leq 25\|f\|_2$.*
2. *For all $f \in L^1$we have $m\{Sf > \lambda\} \leq 7000\|f\|_1$.*

Theorem 2. *Let T be a contraction on a Hilbert space H. Let (n_k) be a sequence in Z^+ with $n_k \leq n_{k+1}$ for all $k \geq 1$. Let $A_nf = \frac{1}{n}\sum_{k=1}^{n} T^kf$ for all $f \in H$. Then $\left(\sum_{k=1}^{\infty} \|A_{n_k}f - A_{n_{k-1}}f\|_H^2\right)^{\frac{1}{2}} \leq 25\|f\|_H$ for all $f \in H$.*

For $H = L^2$, we extend Theorem 2 to multidimensional case, such that the averages are taken over squares and the operators are commuting contractions in L^2.

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The non commuting case for power-bounded operators needs a supplementary condition.

I. Square Functions

a) Commuting case for contractions

We study the two cases:

1. The operators are commuting contractions in L^2 and the averages are taken over squares.
2. The power-bounded operators (may be not commuting). In this case we need a supplementary condition.

We study the case where the averages are taken over squares. Let $T_1, \dots, T_d$ be linear contractions on L^2 and let

$$A_n(T_1, \dots, T_d)f = A_n(T_1) \dots A_n(T_d)f = \frac{1}{n^d} \sum_{k_1 < n} \cdots \sum_{k_d < n} T_1^{k_1} \dots T_d^{k_d} f$$

and

$$S_d f = \left(\sum_{k=1}^{\infty} \left| A_{n_{k+1}}(T_1, \dots, T_d)f - A_{n_k}(T_1, \dots, T_d)f \right|^2 \right)^{\frac{1}{2}}.$$

From Theorem 2, we can easily deduce the following:

Theorem 3. *Let T be a contraction L^2. Let (n_k) be a sequence in Z^+ with $n_k \le n_{k+1}$ for all $k \ge 1$. Let $A_n f = \frac{1}{n} \sum_{k=1}^{n} T^k f$for all $f \in L^2$. Then*

$$\|Sf\|_2 = \left\| \left(\sum_{k=1}^{\infty} \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 \right)^{\frac{1}{2}} \right\|_2 \le 25 \|f\|_2$$

for all $f \in L^2$.

Proof. It suffices to remark that

$$\|Sf\|_2 = \int \sum_{k=1}^{\infty} \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 d\mu \le \sum_{k=1}^{\infty} \int \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 d\mu$$

$$= \sum_{k=1}^{\infty} \|A_{n_k} f - A_{n_{k-1}} f\|_{L^2}^2 \le 25 \|f\|_2.$$

□

Theorem 4 is our main result in this section, that is an extension of Theorem 3 to higher dimensions.

Theorem 4. *Let $T_1, \dots, T_d$ be linear commuting contractions on L^2. For $f \in L^2$ we have the following inequality*

$$\|S_d f\|_2 \le (26^d - 1)\|f\|_2.$$

Proof. First, we study the case $d = 2$. Let $T_1 = T$ and $T_2 = S$ we can write

$$\begin{aligned}
&A_{n_{k+1}}(T)A_{n_{k+1}}(S)f - A_{n_k}(T)A_{n_k}(S)f \\
&\quad= \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right) f \\
&\qquad+ \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) A_{n_k}(S)f + A_{n_k}(T)\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) f
\end{aligned}$$

using the triangle inequality we see that

$$\begin{aligned}
S_2 f &= \left(\sum_{k=1}^{\infty} \left|A_{n_{k+1}}(T)A_{n_{k+1}}(S)f - A_{n_k}(T)A_{n_k}(S)f\right|^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{\infty} \left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right) f\right|^2\right)^{\frac{1}{2}} \\
&\quad+ \left(\sum_{k=1}^{\infty} \left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) A_{n_k}(S)f\right|^2\right)^{\frac{1}{2}} \\
&\quad+ \left(\sum_{k=1}^{\infty} \left|A_{n_{k+1}}(T)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right) f\right|^2\right)^{\frac{1}{2}} \\
&= Bf + Cf + Df.
\end{aligned}$$

For any $N \ge 1$, the partial sum $\sum_{k=1}^{N} \left|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right|^2$ is in L^1 if $f \in L^2$ and

$$\left\|\sum_{k=1}^{N} \left|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right|^2\right\|_1 = \sum_{k=1}^{N} \|A_{n_{k+1}}(T)f - A_{n_k}(T)f\|_2^2.$$

Now, let $f_{n_k} = A_{n_{k+1}}(S)f - A_{n_k}(S)f$. We first show that $\|Bf\|_2 \le 25^2\|f\|_2$. To see this we write

$$\begin{aligned}
\|B_N f\|_2^2 &= \int \sum_{k=1}^{N} \left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) f_{n_k}\right|^2 \, d\mu \\
&\le \int \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \left|\left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T)\right) f_{n_{k_2}}\right|^2 \, d\mu
\end{aligned}$$

$$= \sum_{k_2=1}^{N} \left\| \sum_{k_1=1}^{N} \left| \left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T)\right) f_{n_{k_2}} \right|^2 \right\|_1$$

$$= \sum_{k_2=1}^{N} \left\| \left(\sum_{k_1=1}^{N} \left| \left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T)\right) f_{n_{k_2}} \right|^2 \right)^{\frac{1}{2}} \right\|_2^2$$

$$= \sum_{k_2=1}^{N} \|S_2(f_{n_{k_2}})\|_2^2 \leq 25^2 \sum_{k_2=1}^{N} \|f_{n_{k_2}}\|_2^2 \qquad \text{(by Theorem 1 on } T\text{)}$$

$$= 25^2 \sum_{k_2=1}^{N} \|A_{n_{k+1}}(S)f - A_{n_k}(S)f\|_2^2 \leq 25^4 \|f\|_2^2 \qquad \text{(by Theorem 1 on } S\text{)}.$$

Let $N \to \infty$, the monotone convergence Theorem says $\|Bf\|_2 \leq 25^2 \|f\|_2$.

To find a majorization for C we shall use the commutation of T and S:

$$\|C_N f\|_2^2 = \int \sum_{k=1}^{N} \left| \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) A_{n_k}(S) f \right|^2 \, d\mu$$

$$= \int \sum_{k=1}^{N} \left| A_{n_k}(S) \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) f \right|^2 \, d\mu \qquad \text{(since } TS = ST\text{)}$$

$$\leq \int \sum_{k=1}^{N} \left| \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) f \right|^2 \, d\mu \qquad \text{(since } \|S\|_2 \leq 1\text{)}$$

$$\leq \sum_{k=1}^{N} \left\| \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right) f \right\|_2^2 \leq 25^2 \|f\|_2 \qquad \text{(by Theorem 1 on } S\text{)}$$

and then $\|Cf\|_2 \leq 25\|f\|_2$.

By the same argument as in C and without the commutation of T and S we can write $\|Df\|_2 \leq 25\|f\|_2$. Finally, we have

$$\|S_2 f\|_2 \leq (25^2 + 25 + 25)\|f\|_2 = 675\|f\|_2 = (26^2 - 1)\|f\|_2. \qquad \square$$

For the general case, when $d > 2$ we need the following technical lemma where the proof can be done by induction on d:

Lemma 5. *Let $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ be real numbers. Then we have the following equality*

$$a_1 \ldots a_d - b_1 \ldots b_d = \prod_{i=1}^{d} (a_i - b_i) + \sum_{s=1}^{d-1} \left[\sum_{1 = i_1 < \cdots < i_s} b_{i_1} b_{i_2} \ldots b_{i_s} \right] \prod_{\substack{j<d \\ j \notin \{i_1, \ldots, i_s\}}} (a_j - b_j).$$

By this lemma we can write

$$
\begin{aligned}
&A_{n_{k+1}}(T_1)A_{n_{k+1}}(T_2)f\dots A_{n_{k+1}}(T_d)f - A_{n_k}(T_1)A_{n_k}(T_2)f\dots A_{n_k}(T_d)f\\
&= \prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right) f\\
&\quad + \sum_{s=1}^{d-1}\left[\sum_{1=i_1<\dots<i_s} A_{n_k}(T_{i_1})\dots A_{n_k}(T_{i_s})\right] \prod_{\substack{j<d\\ j\notin\{i_1,\dots,i_s\}}} \left(A_{n_{k+1}}(T_j)f - A_{n_k}(T_j)\right) f.
\end{aligned}
$$

Using the triangle inequality we see that

$$
\begin{aligned}
Sf &= \left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\dots A_{n_{k+1}}(T_{i_d})f - A_{n_k}(T_{i_1})\dots A_{n_k}(T_{i_d})f\right|^2\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{k=1}^{\infty}\left|\left[\prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right)\right]\right|^2 f\right)^{\frac{1}{2}}\\
&\quad + \sum_{i_1=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\prod_{j\neq i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}\\
&\quad + \sum_{1=i_1<i_2}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})A_{n_{k+1}}(T_{i_2})\prod_{j\notin\{i_1,i_2\}}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}\\
&\quad + \dots + \sum_{1=i_1<\dots<i_{d-1}}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\dots A_{n_{k+1}}\right.\right.\\
&\quad \left.\left.\times (T_{i_{d-1}})\prod_{j\notin\{i_1,\dots,i_{d-1}\}}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Let

$$
Mf = \left(\sum_{k=1}^{\infty}\left|\left[\prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right)\right] f\right|^2\right)^{\frac{1}{2}},
$$

$$
M_{i_1}f = \sum_{i_1=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\prod_{j\neq i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}
$$

and let

$$M_{i_1,\dots,i_s}f = \Bigg(\sum_{k=1}^{\infty}\Big|A_{n_{k+1}}(T_{i_1})\dots A_{n_{k+1}}(T_{i_{d-1}})$$
$$\times \prod_{j<dj\notin\{i_1,\dots,i_{s-1}\}} \left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\Big|^2\Bigg)^{\frac{1}{2}}$$

with these notations, we have

$$\text{(1)} \qquad Sf \le Mf + \sum_{i_1=1}^{d} M_{i_1}f + \sum_{1=i_1<i_2}^{d} M_{i_1,i_2}f + \dots + \sum_{1=i_1<\dots<i_{d-1}}^{d} M_{i_1,\dots,i_d}f.$$

We shall majorize each Mf and $M_{i_1,\dots,i_s}f$ in L^2 for $s = 1,\dots,d-1$.

$$\begin{aligned}\|Mf\|_2^2 &= \sum_{k=1}^{\infty}\int\left|\left[\prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right)\right]\right|^2 f\,d\mu \\ &\le \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\int\Big|\left(A_{n_{k_1+1}}(T_1) - A_{n_{k_1}}(T_1)\right)\dots \\ &\quad \Big(A_{n_{d^{k+1}}}(T_d) - A_{n_{k_d}}(T_d)\Big) f\Big|^2 d\mu \\ &\le 25^2\sum_{k_1=1}^{\infty}\cdots\sum_{k_{d-1}=1}^{\infty}\int\Big|\left(A_{n_{k_1+1}}(T_1) - A_{n_{k_1}}(T_1)\right)\dots \\ &\quad \Big(A_{n_{k_{d-1}+1}}(T_{d-1}) - A_{n_{k_{d-1}}}(T_{d-1})\Big) f\Big|^2 d\mu \\ &\le \dots \le 25^{2d}\|f\|_2^2 \quad \text{(by applying successively Theorem 3 on } T_d,\dots,T_1).\end{aligned}$$

Then

$$\|Mf\|_2 \le 25^d\|f\|_2.$$

To control each $M_{i_1,\dots,i_s}f$ we write

$$\begin{aligned}\|M_{i_1}f\| &\le \sum_{k=1}^{\infty}\int\Big|A_{n_{k+1}}(T_{i_1})\prod_{j\neq i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\Big|^2 d\mu \\ &\le \sum_{k=1}^{\infty}\int\Big|\prod_{j\neq i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\Big|^2 d\mu \quad (\text{since } \|T_1\|_2 \le 1) \\ &\le 25^{d-1}\|f\|_2.\end{aligned}$$

By the same argument we obtain $\|M_{i_1,\dots,i_s}f\|_2 \le 25^{d-s}\|f\|_2$.

Applying norm both sides of (1) and use the triangle inequality

$$\begin{aligned}\|S_d f\|_2 &\le \|Mf\|_2 + \sum_{i_1=1}^{d} \|M_{i_1} f\|_2 \\ &\quad + \sum_{1=i_1<i_2}^{\infty} \|M_{i_1,i_2} f\|_2 + \cdots + \sum_{1=i_1<\cdots<i_{d-1}}^{\infty} \|M_{i_1,\dots,i_{d-1}} f\|_2 \\ &\le (25^d + C_d^1 25^{d-1} + \cdots + C_d^s 25^{d-s} + \cdots + C_d^{d-1} 25)\|f\|_2 \\ &= (26^d - 1)\|f\|_2.\end{aligned}$$

Remark that in the case $d = 2$ we have obtained the constant $675 = 26^2 - 1$.

b) Non-commuting case for power-bounded operators:

We now study the multidimensional averages over rectangles

$$A_{n_1,\dots,n_d}(T_1,\dots,T_d)f = A_{n_1}(T_1)\dots A_{n_d}(T_d)f.$$

Let (n_{k_j}), $j = 1,\dots,d$; be increasing sequences of integers. In this case, instead of using the strong estimate for square functions, we shall use the dominated ergodic theorem of Akcoglu for positive contraction 1975 [**5**, p. 186], (or positive power-bounded operators) in L^p.

Theorem 6. *Let $T_1,\dots,T_d$ be linear power-bounded operators in L^q, i.e. $\sup_j \|T_k^j\| \le M_k$, $k = 1,\dots,d$ with $1 < q < \infty$. Assume that $\sum_{k_j=1,j=1,\dots,d}^{\infty}\big(1 - \prod_{i=1}^{j} \frac{n_{k_i-1}}{n_{k_i}}\big) < \infty$. Then the q-variation operator*

$$S_d f = \left(\sum_{k_1=2}^{\infty} \cdots \sum_{k_d=2}^{\infty} \left|A_{n_{k_1},\dots,n_{k_d}} f - A_{n_{k_1-1},\dots,n_{k_d-1}} f\right|^q\right)^{\frac{1}{q}}$$

is finite a.e. for all bounded f. In fact, $S_d\, f$ verifies a strong estimate in L^q

$$\|S_d f\|_q \le C_{q,d}\|f\|_q.$$

Proof. First we study the case $d = 2$. The general case can be done by a similar argument

$$\begin{aligned}&A_{n_{k_1},n_{k_2}} f - A_{n_{k_1-1},n_{k_2-1}} f \\ &= \frac{1}{n_{k_1} n_{k_2}} \sum_{i=0}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f - \frac{1}{n_{k_1-1} n_{k_2-1}} \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \\ &= \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}}\right) \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \\ &\quad - \frac{1}{n_{k_1} n_{k_2}} \left[\sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f + \sum_{i=0}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f + \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f\right]\end{aligned}$$

Using the triangle inequality we see that

$$
\begin{aligned}
S_2 f \leq & \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right) \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=0}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
= & \, Af + Bf + Cf + Df.
\end{aligned}
$$

We first show that $\|Af\|_q \leq C_q \|f\|_q$. To see this we just write

$$
\begin{aligned}
\|Af\|_q^q \leq & \int \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right) \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q \right) d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right)^q (n_{k_1-1} n_{k_2-1})^q \\
& \times \int \left| \frac{1}{n_{k_1-1} n_{k_2-1}} \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(1 - \frac{n_{k_1-1} n_{k_2-1}}{n_{k_1} n_{k_2}} \right)^q \int \left| A_{n_{k_1-1}, n_{k_2-1}} f \right|^q d\mu \\
\leq & M_1 M_2 \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(1 - \frac{n_{k_1-1} n_{k_2-1}}{n_{k_1} n_{k_2}} \right)^q \int |f|^q \, d\mu \\
= & \, C_q^q \|f\|_q^q
\end{aligned}
$$

For B, we write

$$
\begin{aligned}
\|Bf\|_q^q \leq & \int \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(\frac{1}{n_{k_1} n_{k_2}} \right)^q \int \left| T_1^{n_{k_1}-1} T_2^{n_{k_2}-1} \sum_{i=0}^{n_{k_1}-n_{k_{1-1}}} \sum_{j=0}^{n_{k_2}-n_{k_{2-1}}} T_1^i T_2^i f \right|^q d\mu
\end{aligned}
$$

$$\leq \sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(\frac{1}{n_{k_1}n_{k_2}}\right)^q (n_{k_1}-n_{k_1-1})\,(n_{k_2}-n_{k_2-1})$$
$$\times \int \left|T_1^{n_{k_1}-1}T_2^{n_{k_2}-1}A_{n_{k_1}-n_{k_1-1},n_{k_2}-n_{k_2-1}}f\right|^q \, d\mu$$
$$\leq M_1M_2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(1-\frac{n_{k_1-1}n_{k_2-1}}{n_{k_1}n_{k_2}}\right)^q \int \left|A_{n_{k_1}-n_{k_1-1},n_{k_2}-n_{k_2-1}}f\right|^q \, d\mu$$
$$\leq M_1^2M_2^2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(1-\frac{n_{k_1-1}n_{k_2-1}}{n_{k_1}n_{k_2}}\right)^q \int |f|^q \, d\mu$$
$$= C_q'^q\|f\|f_q^q.$$

By the same argument we can prove that $\|Cf\|_q \leq C_q''\|f\|_q$, and $\|Df\|_q \leq C_q'''\|f\|_q$. Finally we obtain $\|S_{q,d}f\|_q \leq C_{q,d}\|f\|_q$.

II. Square Maximal Functions

a) One dimensional case for power bounded operators

We now study for $1 < q \leq \infty$, the following maximal q-variation operator

$$S_q^*f = \left(\sum_{k=1}^{\infty}\sup_{n_{k-1}\leq n\leq n_k}\left|A_nf - A_{n_k}f\right|^q\right)^{\frac{1}{q}}.$$

In [**3**] it was shown the following result:

Theorem 7. *Let (n_k) denote an increasing sequence of integers. If $n_k = p(k)$ for some polynomial p of degree $s > 0$, then there is a constant C such that*

$$\|S_q^*f\|_2 \leq C\|f\|_2.$$

We shall extend Theorem 7: first to power bounded operator with a class of sequences (n_k) increasing and satisfy the following hypothesis: for $1 < q < \infty$ $\sum_{k=2}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q < \infty$.

We notice that the sequences of the form $n_k = p(k)$ for some polynomial satisfy this condition. Since if $p(k)$ is a polynomial of degree s then $n_k - n_{k-1} = p(k) - p(k-1)$ is a polynomial of degree $s-1$ and then the series has the same nature as

$$\sum_{k=2}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q = \sum_{k=2}^{\infty}\left(\frac{p(k)-p(k-1)}{p(k)}\right)^q = C\sum_{k=2}^{\infty}\frac{1}{k^q} < \infty.$$

Theorem 8. *Let T be a linear power-bounded operator on $1 < q < \infty$. Assume that $\rho = \sum_{k=2}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q < \infty$. Then for $f \in L^q(\Omega, R)$*

$$\|S_q^* f\|_q \le 8\sqrt{\rho}\|f\|_q.$$

Remark 1. In [**1**] M. Akcoglu, R. Jones and P. Schwartz proved that for $q < 2$, there is a function $f \in L^\infty$ such that $S_q f = \left(\sum_{k=1}^{\infty} \left|A_{n_k} f - A_{n_{k-1}} f\right|^q\right)^{\frac{1}{q}} = \infty$. So that there is no strong estimate for $S_q f$ and hence for $S_q^* f$ $(S_q f \le S_q^* f)$.

In the proof of Theorem 1.5 we shall use the dominated ergodic theorem to obtain a strong estimate for mutidimensional square functions.

Proof. We write

$$\left|A_n f - A_{n_{k-1}} f\right| = \left|A_{n_{k-1}} f - A_n f\right| = \left|\left(\frac{1}{n_{k-1}} - \frac{1}{n}\right) \sum_{i=0}^{n_{k-1}} T^i f - \frac{1}{n} \sum_{i=n_{k-1}+1}^{n} T^i f\right|$$

then

$$\begin{aligned}
\sup_{n_{k-1} \le n \le n_k} |A_n f - A_{n_k} f| &\le \sup_{n_{k-1} \le n \le n_k} \left|\left(\frac{1}{n_{k-1}} - \frac{1}{n}\right) \sum_{i=0}^{n_{k-1}} T^i f - \frac{1}{n} \sum_{i=n_{k-1}+1}^{n} T^i f\right| \\
&\le \sup_{n_{k-1} \le n \le n_k} \left|\left(\frac{1}{n_{k-1}} - \frac{1}{n}\right) \sum_{i=0}^{n_{k-1}} T^i f\right| + \sup_{n_{k-1} \le n \le n_k} \left|\frac{1}{n} \sum_{i=n_{k-1}+1}^{n} T^i f\right| \\
&= \sup_{n_{k-1} \le n \le n_k} \left|\left(1 - \frac{n_{k-1}}{n}\right) A_{n_{k-1}} f\right| \\
&\quad + \sup_{n_{k-1} \le n \le n_k} \left|\left(1 - \frac{n_{k-1}}{n}\right) T^{n_{k-1}+1} A_{n-n_{k-1}} f\right| \\
&\le \sup_{n_{k-1} \le n \le n_k} \left|\left(1 - \frac{n_{k-1}}{n}\right) \sup_{n_{k-1} \le n \le n_k} \left|A_{n_{k-1}} f\right|\right| \\
&\quad + \sup_{n_{k-1} \le n \le n_k} \left|\left(1 - \frac{n_{k-1}}{n}\right)\right| \sup_{n_{k-1} \le n \le n_k} \left|T^{n_{k-1}+1} A_{n-n_{k-1}} f\right| \\
&\le \left(1 - \frac{n_{k-1}}{n_k}\right) \left|A_{n_{k-1}} f\right| \\
&\quad + \left(1 - \frac{n_{k-1}}{n_k}\right) \sup_{n_{k-1} \le n \le n_k} \left|T^{n_{k-1}+1} A_{n-n_{k-1}} f\right|.
\end{aligned}$$

Using the triangle inequality we see that

$$\begin{aligned}
\left|S_q^* f\right| &\le \left(\sum_{k=1}^{\infty} \left(\left(1 - \frac{n_{k-1}}{n_k}\right) \left|A_{n_{k-1}} f\right|\right)^q\right) \\
&\quad + \left(\sum_{k=1}^{\infty} \left(\left(1 - \frac{n_{k-1}}{n_k}\right) \sup_{n_{k-1} \le n \le n_k} \left|T^{n_{k-1}+1} A_{n-n_{k-1}} f\right|\right)^q\right)^{\frac{1}{q}} \\
&= Af + Bf
\end{aligned}$$

by integration we have

$$\begin{aligned}
\|Af\|_q &\leq \int \sum_{k=1}^{\infty}\Big(\Big(1-\frac{n_{k-1}}{n_k}\Big)\,\big|A_{n_{k-1}}f\big|\Big)^q\, d\mu \\
&\leq \sum_{k=1}^{\infty}\Big(1-\frac{n_{k-1}}{n_k}\Big)^q \int \big|A_{n_{k-1}}f\big|^q\, d\mu \\
&\leq M\sum_{k=1}^{\infty}\Big(1-\frac{n_{k-1}}{n_k}\Big)^q \int |f|^q\, d\mu \\
&= C_q^q\|f\|_q^q \qquad (T \text{ is power-bounded in } L^q).
\end{aligned}$$

For Bf we have a similar argument.

$$\begin{aligned}
\|Bf\|_q &\leq \int \sum_{k=1}^{\infty}\left((1-\frac{n_{k-1}}{n_k})\sup_{n_{k-1}\leq n\leq n_k}\big|T^{n_{k-1}+1}A_{n-n_{k-1}}f\big|\right)^q d\mu \\
&= \int \sum_{k=1}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q\left(\sup_{n_{k-1}\leq n\leq n_k}\big|T^{n_{k-1}+1}A_{n-n_{k-1}}f\big|\right)^q d\mu \\
&\leq \int \sum_{k=1}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q\left(T^{n_{k-1}+1}\sup_{n_{k-1}\leq n\leq n_k}\big|A_{n-n_{k-1}}f\big|\right)^q d\mu \quad (T\geq 0) \\
&\leq M\int \sum_{k=1}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q\left(\sup_{n_{k-1}\leq n\leq n_k}\big|A_{n-n_{k-1}}f\big|\right)^q d\mu \\
&\quad (\sup_j \|T\|_q\leq M) \\
&\leq M\int \sum_{k=1}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q\left(\sup_m |A_m f|\right)^q d\mu.
\end{aligned}$$

(By Brunel' Theorem [**2**] or the dominated ergodic theorem for power-bounded operators on T)

$$\begin{aligned}
&\leq Kk_q\sum_{k=1}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q\int |f|^q\, d\mu \\
&= C_q'\|f\|_q^q.
\end{aligned}$$

To obtain Theorem 7 it suffices to take $q=2$ and $Tf = fo\tau$ where τ is a measure preserving transformations on Ω.

b) Multidimensional case

We now study by a similar argument the multidimensional version of Theorem 7.

Let for (n_{k_j}), $j = 1, \dots, d$, be increasing sequences of integers. Let

$$S^*_{q,d} f =$$
$$\left(\sum_{k_1=1}^{\infty} \dots \sum_{k_d=1}^{\infty} \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \dots \sup_{n_{k_d} \le m_2 \le n_{k_d+1}} \left| A_{m_1,\dots,m_d} f - A_{n_{k_1},\dots,n_{k_d}} f \right|^q \right)^{\frac{1}{q}}.$$

We shall prove the following result:

Theorem 9. *Let $T_1, \dots, T_d$ be linear positive power-bounded operators in L^q, $\sup_j \|T_k^j\|_q \le M_k$, $k = 1, \dots, d$, $1 < q < \infty$. Assume that $\sum_{k=2, j=1,\dots,d}^{\infty} \big(1 - \prod_{i=0}^{j} \frac{n_{k_i-1}}{n_{k_i}}\big)^q < \infty$. Then the q-variation operator satisfies the strong estimate: for all $f \in L^q(\Omega, R)$*

$$\|S^*_{q,d} f\|_q \le C_{q,d} \|f\|_q.$$

Proof. It suffices to prove the case where $d = 2$: we can write as above

$$\begin{aligned}
A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f &= \frac{1}{m_1 m_2} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} T_1^i T_2^i f - \frac{1}{n_{k_1} n_{k_2}} \sum_{i=0}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f \\
&= \left(\frac{1}{m_1 m_2} - \frac{1}{n_{k_1} n_{k_2}} \right) \sum_{i=n_{k_1}}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f \\
&\quad - \frac{1}{n_{k_1} n_{k_2}} \left[\sum_{i=n_{k_1}}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f + \sum_{i=0}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f + \sum_{i=n_{k_1}}^{m_1} \sum_{j=0}^{m_2} T_1^i T_2^i f \right] \\
&= \left(1 - \frac{m_1 m_2}{n_{k_1} n_{k_2}} \right) \{ A_{n_{k_1},n_{k_2}} f + T_1^{n_{k_1}} T_2^{n_{k_2}} A_{m_1-n_{k_1},m_2-n_{k_2}} f \\
&\quad + T_2^{n_{k_2}} A_{m_1,m_2-n_{k_2}} f + T_1^{n_{k_1}} A_{m_1-n_{k_1},m_2} f \}.
\end{aligned}$$

Applying sup on both sides we obtain

$$\begin{aligned}
&\sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right| \\
&\qquad \le \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}} \right) \Big\{ \left| A_{n_{k_1},n_{k_2}} f \right| \\
&\qquad\quad + \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_1^{n_{k_1}} T_2^{n_{k_2}} A_{m_1-n_{k_1},m_2-n_{k_2}} f \right| \\
&\qquad\quad + \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_2^{n_{k_2}} A_{m_1,m_2-n_{k_2}} f \right| \\
&\qquad\quad + \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_1^{n_{k_1}} A_{m_1-n_{k_1},m_2} f \right| \Big\}.
\end{aligned}$$

Using the triangle inequality we see that

$$
\begin{aligned}
Sf \le & \left(\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \left|A_{n_{k_1}, n_{k_2}} f\right|^q \right)^{\frac{1}{q}} \\
& + \Bigg[\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \\
& \quad \times \left| T_1^{n_{k_1}} T_2^{n_{k_2}} \left(\sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1 - n_{k_1}, m_2 - n_{k_2}} f \right| \right) \right|^q \Bigg]^{\frac{1}{q}} \\
& + \Bigg[\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \\
& \quad \times \left| T_2^{n_{k_2}} \left(\sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1, m_2 - n_{k_2}} f \right| \right) \right|^q \Bigg]^{\frac{1}{q}} \\
& + \Bigg[\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \\
& \quad \times \left| T_1^{n_{k_1}} \left(\sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1 - n_{k_1} ++, m_2} f \right| \right) \right|^q \Bigg] \\
= & \; Af + Bf + Cf + Df.
\end{aligned}
$$

We first show that $\|Af\|_q \le C_q \|f\|_q$. We can write

$$
\begin{aligned}
\|Af\|_q^q &\le \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \int \left|A_{n_{k_1}, n_{k_2}} f\right|^q \, d\mu \\
&\le M_1 M_2 \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \int |f|^q \, d\mu \\
&= C_{q,1}^q \|f\|_q^q.
\end{aligned}
$$

For Bf, Cf, and Df we shall use the dominated ergodic theorem of Brunel [**2**]. Cesaro bounded operators (or the dominated ergodic theorem power-bounded positive operator) which was extended by Olsen to higher dimension.

We see that

$$
\begin{aligned}
\|Bf\|_q^q \le & \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \\
& \times \int \left| T_1^{n_{k_1}} T_2^{n_{k_2}} \left(\sup_{m_1} \sup_{m_2} \left|A_{m_1, m_2} f\right| \right) \right|^q \, d\mu
\end{aligned}
$$

$$\leq M_1 M_2 \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q$$
$$\times \int \left| \left(\sup_{m_1} \sup_{m_2} |A_{m_1,m_2} f| \right) \right|^q d\mu$$
$$\leq M_1 M_2 k_q^q \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \int |f|^q \, d\mu = C_{q,2}^q \|f\|_q^q.$$

By the same argument we can majorize Cf and Df in L^q. □

Remark 2. Since $S_d \leq S_{q,d}^*$ then the result of Theorem 6 can be obtained from Theorem 9. But we have another multiple constant.

III. Weak Estimates

From Theorem 5 we can deduce the following result:

Theorem 10. *Let $T_1, \ldots, T_d$ be linear positive power-bounded operators in L^q, $\sup_j \|T_k^j\|_q \leq M_k$, $k = 1, \ldots, d$, $1 < q < \infty$. Assume that $\sum_{k=2,j=1,\ldots,d}^{\infty} \big(1 - \prod_{i=0}^{j} \frac{n_{k_i-1}}{n_{k_i}}\big)^q < \infty$. Then the q-variation operator satisfies the strong estimate: for all $f \in L^q(\Omega, R)$*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} m \left\{ \sup_{n_{k_1} \leq m_1 \leq n_{k_1+1}} \cdots \sup_{n_{k_d} \leq m_d \leq n_{k_d+1}} \left| A_{m_1,\ldots,m_d} f - A_{n_{k_1},\ldots,n_{k_d}} f \right| > \lambda \right\}$$
$$\leq \frac{C_q^q}{\lambda^q} \|f\|_q^q.$$

Proof. Study the case $d = 2$. The general case can be done similarly.

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} m \left\{ \sup_{n_{k_1} \leq m_1 \leq n_{k_1+1}} \sup_{n_{k_2} \leq m_2 \leq n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right| > \lambda \right\}$$
$$= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} m \left\{ \sup_{n_{k_1} \leq m_1 \leq n_{k_1+1}} \sup_{n_{k_2} \leq m_2 \leq n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right|^q > \lambda^q \right\}$$
$$\leq \frac{1}{\lambda^q} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left\| \sup_{n_{k_1} \leq m_1 \leq n_{k_1+1}} \sup_{n_{k_2} \leq m_2 \leq n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right| \right\|_q^q$$
$$\leq \frac{C_q^q}{\lambda^q} \|f\|_q^q \qquad \text{(by Theorem 5).}$$
□

Corollary 11. *Under the same hypothesis of Theorem* 10 *we have: for all* $f \in L^q(\Omega, R)$

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} m\left\{\left|A_{n_{k_1}+1,\dots,n_{k_d}+1}f - A_{n_{k_1},\dots,n_{k_d}}f\right| > \lambda\right\} \le \frac{C_q^q}{\lambda^q}\|f\|_q^q.$$

From Theorem 2 and Theorem 6 we can deduce the following:

Corollary 12. *Let* $T_1, \dots, T_d$ *be linear contracting commuting operators on* L^2. *For* $f \in L^q$ *we have the following weak type inequality*

$$\sum_{k=1}^{\infty} m\left\{|A_{n_k+1}(T_1,\dots,T_d)f - A_{n_k}(T_1,\dots,T_d)f| > \lambda\right\} \le \frac{(26^d - 1)}{\lambda^2}\|f\|_2^2.$$

In [**4**] R. Jones proved that if $Tf = fo\theta$, then the square functions

$$Sf = \left(\sum |A_{n_k+1}f - A_{n_k+1}f|^2\right)^{\frac{1}{2}}$$

there is a weak estimate, $m\{Sf > l\} \le \frac{C}{\lambda}\|f\|_1$ valid for some constant$C < \infty$ and $f \in L^1$. We shall prove that for a linear positive contraction T on L^1 such that

$$m\left\{\sup_n |T^n f| > \lambda\right\} \le \frac{C}{\lambda}\|f\|_1$$

the result of Jones remains true.

Theorem 13. *Let* T *be a linear positive on* L^1*:*

(i) *If* T *is a self-adjoint positive contraction on* L^2, *then*

$$m\left\{\left(\sum_{k=1}^{\infty} |A_{n_k+1}(T)f - A_{n_k}(T)f|^2\right)^{\frac{1}{2}} > \lambda\right\} \le \frac{124}{\lambda^2}\|f\|_2^2.$$

(ii) *If* T *is contraction on* L^1 *and on* L^∞ *and satisfies that*

$$m\left\{\sup_n |T^n f| > \lambda\right\} \le \frac{C'}{\lambda}\|f\|_1. \qquad (*)$$

Then there is a constant $C < \infty$ *such that*

$$m\left\{\left(\sum_{k=1}^{\infty} |A_{n_k+1}(T)f - A_{n_k}(T)f|^2\right)^{\frac{1}{2}} > \lambda\right\} \le \frac{C'}{\lambda}\|f\|_1.$$

Proof. In [**5**, pp. 190] Stein proved that if T is a self-adjoint positive operator in L^2 then $\| \sup_n T^n |f| \|_2 \leq 6\|f\|_2$. We shall use this estimate to prove the inequality (i). we can write

$$A_k(T)f - A_{k+1}(T)f = \left(\frac{1}{k} - \frac{1}{k+1}\right) \sum_{j=0}^{k} T^j f - \frac{1}{k+1} T^{k+1} f$$

so

$$|A_k(T)f - A_{k+1}(T)f| \leq \left(\frac{1}{k} - \frac{1}{k+1}\right) \left|\sum_{j=0}^{k} T^j f\right| + \frac{1}{k+1} \left|T^{k+1} f\right|$$

using the triangle inequality we see

$$\begin{aligned}
Sf &\leq \left[\sum_{k=1}^{\infty} \left(1 - \frac{k}{k+1}\right)^2 \left|\frac{1}{k} \sum_{j=0}^{k} T^j f\right|^2\right]^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} \left|T^{k+1} f\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \sup_n |A_n(T)f|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sup_n \left|T^{n+1} f\right|^2\right)\right)^{\frac{1}{2}} \\
&\leq \left[\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sup_n |A_n(T)f|\right)^2\right]^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \left(\sup_n \left|T^{n+1} f\right|\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} \left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1} f\right|\right\} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} \left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1} f\right|\right\} \\
&\leq \sqrt{2} \left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1} f\right|\right\}.
\end{aligned}$$

Using the dominated ergodic theorem of Akcoglu for positive contraction and that of Stein we have

$$\begin{aligned}
\|Sf\|_2 &\leq \sqrt{2} \left\{\| \sup_n \left|A_n(T)f\right|\|_2 + \| \sup_n \left|T^{n+1} f\right|\|_2\right\} \\
&\leq \sqrt{2}(2+6)\|f\|_2 = 8\sqrt{2}\|f\|_2.
\end{aligned}$$

Clearly,

$$m\{Sf > \lambda\} \leq \frac{1}{\lambda^2}\|f\|_2^2 \leq \frac{124}{\lambda^2}\|f\|_2^2.$$

For (ii), the set

$$\{Sf > \lambda\} \subseteq \left\{\sup_n \left|A_n(T)f\right| > \frac{\lambda}{2\sqrt{2}}\right\} \bigcup \left\{\sup_n \left|T^{n+1} f\right| > \frac{\lambda}{2\sqrt{2}}\right\}.$$

But by the Dunford-schwartz theorem we have

$$m\left\{\sup_n\left|A_n(T)f\right| > \frac{\lambda}{2\sqrt{2}}\right\} \leq \frac{2\sqrt{2}}{\lambda}\|f\|_1$$

and by the condition on T

$$m\left\{Sf > \lambda\right\} \leq \frac{2\sqrt{2}+C}{\lambda}\|f\|_1.$$

Remark 3. The condition $(*)$ in Theorem 13 can be replaced by an operator T for which $T^n f$ already converges a.e. at least for $f \in L^2$.

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