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SHOCK CAPTURING AND RELATED NUMERICAL METHODS IN COMPUTATIONAL FLUID DYNAMICS

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Abstract. Some developments and efforts in designing and analyzing shock capturing algorithms and related numerical methods in computational fluid dynamics are reviewed. The importance of numerical viscosity in shock capturing algorithms is analyzed; the convergence and stability of some shock capturing algorithms are presented; the role of shock capturing algorithms in a mathematical existence theory is exhibited, especially for the compressible Euler equations for gas dynamics in one dimension and in multi-dimensions with spherical symmetry. Applications of shock capturing ideas to the compressible Navier-Stokes equations are also discussed.

1. Introduction

Shock waves are one of the most fundamental nonlinear waves in nature and arise in supersonic or transonic flow, or from a very sudden release (explosion) of chemical, nuclear, electrical, radiation, or mechanical energy in a limited space (see, for example, Van Dyke [**98**], Glass [**39**], Courant-Friedrichs [**27**], and Whitham [**103**]). Tracking shocks, especially when and where new shocks arise and interact in the motion of fluids, is scientifically extremely important but numerically burdensome. The main motivation in developing numerical shock capturing algorithms is to treat the shock problem in fluids.

The basic equations governing the dynamics of shocks are the compressible Euler equations, consisting of conservation laws of mass, momentum, and energy. The compressible Euler equations in d-space dimensions are the system of $d+2$ conservation laws

$$\begin{cases} \partial_t\rho + \nabla\cdot\mathbf{m} = 0, \\ \partial_t\mathbf{m} + \nabla\cdot\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \nabla p = 0, \\ \partial_t E + \nabla\cdot\left(\frac{\mathbf{m}}{\rho}(E+p)\right) = 0, \end{cases} \tag{1.1}$$

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with initial data

$$(\rho, \mathbf{m}, E)|_{t=0} = (\rho_0, \mathbf{m}_0, E_0)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1.2}$$

where $(\rho_0, \mathbf{m}_0, E_0)(\mathbf{x})$ is a given vector function of $\mathbf{x} \in \mathbb{R}^d$. In (1.1), $\tau = 1/\rho$ is the deformation gradient (specific volume for fluids, strain for solids), $\mathbf{v} = (v_1, \dots, v_d)^\top$ is the fluid velocity with $\rho\mathbf{v} = \mathbf{m}$ the momentum vector, p is the scalar pressure, and $E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e(\tau, p)$ is the total energy with e the internal energy, a given function of (τ, p) or (ρ, p) defined through thermodynamical relations. The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of two vectors.

The other two thermodynamic variables are the temperature θ and the entropy S. If (ρ, S) are chosen as the independent variables, then we have the constitutive relations:

$$(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S)), \tag{1.3}$$

governed by

$$\theta dS = de + pd\tau = de - \frac{p}{\rho^2} d\rho. \tag{1.4}$$

For a polytropic gas,

$$p = R\rho\theta, \qquad e = c_v\theta, \qquad \gamma = 1 + \frac{R}{c_v}, \tag{1.5}$$

and

$$p = p(\rho, S) = \kappa\rho^\gamma e^{S/c_v}, \qquad e = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}e^{S/c_v} = \frac{\theta}{\gamma - 1}, \tag{1.6}$$

where R, c_v, κ are positive constants.

The system in (1.1) is complemented by the Clausius inequality:

$$\partial_t(\rho a(S)) + \nabla \cdot (\mathbf{m} a(S)) \geq 0 \tag{1.7}$$

in the sense of distributions for any $a(S) \in C^1, a'(S) \geq 0$, to identify physical shocks.

The compressible Euler equations for an isentropic fluid take the following simpler form:

$$\begin{cases} \partial_t\rho + \nabla \cdot \mathbf{m} = 0, \\ \partial_t\mathbf{m} + \nabla \cdot \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) + \nabla p = 0, \end{cases} \tag{1.8}$$

where the pressure is regarded as a function of the density, $p = p(\rho, S_0)$, with constant S_0. For an isentropic polytropic gas,

$$p(\rho) = \kappa_0\rho^\gamma, \qquad \gamma > 1, \tag{1.9}$$

where $\kappa_0 > 0$ is a constant.

Observe that solutions of the system in (1.8) are a genuine approximation to solutions of the system in (1.1) since the entropy increases along a shock to third-order in wave strength for solutions of (1.1), while in (1.8) the entropy is constant. Furthermore, the system in (1.8) is an excellent model for an isothermal gas with $\gamma = 1$ and a shallow water with $\gamma = 2$.

For the one-dimensional case, the system in (1.1) in Eulerian coordinates is

$$\begin{cases} \partial_t\rho + \partial_x m = 0, \\ \partial_t m + \partial_x\left(\frac{m^2}{\rho} + p\right) = 0, \\ \partial_t E + \partial_x\left(\frac{m}{\rho}(E+p)\right) = 0, \end{cases} \tag{1.10}$$

with $E = \frac{1}{2}\frac{m^2}{\rho} + \rho e$. The system above can be rewritten in Lagrangian coordinates in one-to-one correspondence when the fluid flow is away from the vacuum $\rho = 0$:

$$\begin{cases} \partial_t\tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t(e + \frac{v^2}{2}) + \partial_x(pv) = 0. \end{cases} \tag{1.11}$$

For the isentropic case, the system in (1.10) becomes

$$\begin{cases} \partial_t\rho + \partial_x m = 0, \\ \partial_t m + \partial_x\left(\frac{m^2}{\rho} + p\right) = 0, \end{cases} \tag{1.12}$$

where $p = p(\rho)$ is determined by (1.9) for a polytropic gas.

All these systems fit into the following general conservation form:

$$\partial_t\mathbf{u} + \nabla\cdot\mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{u}\in\mathbb{R}^m,\ \mathbf{x}\in\mathbb{R}^d, \tag{1.13}$$

where $\mathbf{f}\colon \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping. Besides (1.1)-(1.12), most of partial differential equations arising from physical or engineering science can be also formulated into form (1.13) or its variants, for example, with additional source terms or equations modeling the effects of dissipation, relaxation, memory, damping, dispersion, magnetization, etc. The hyperbolicity of the system in (1.13) requires that, for all $\xi\in S^{d-1}$, the matrix $(\xi\cdot\nabla\mathbf{f}(\mathbf{u}))_{m\times m}$ have m real eigenvalues $\lambda_j(\mathbf{u},\xi), j = 1, 2, \ldots, m$, and be diagonalizable.

The main difficulty to deal with (1.13) is that solutions of the Cauchy problem (even starting from smooth initial data) generally develop singularities in a finite time, because of the physical phenomena of focusing and breaking of waves and the development of shock waves. For this reason, attention focuses on solutions in the space of discontinuous functions. Therefore, one can not directly use the classical analytic techniques that predominate in the theory of partial differential equations of other types.

Another difficulty is nonstrict hyperbolicity or resonance, that is, there exist some $\xi_0\in S^{d-1}$ and $\mathbf{u}_0\in\mathbb{R}^d$ such that $\lambda_i(\mathbf{u}_0,\xi_0) = \lambda_j(\mathbf{u}_0,\xi_0)$ for some $i\neq j$. In particular, for the Euler equations, such a degeneracy occurs at the vacuum states or from the multiplicity of eigenvalues of the system.

The correspondence of (1.7) in the context of hyperbolic conservation laws is the Lax entropy inequality:

$$\partial_t\eta(\mathbf{u}) + \nabla\cdot\mathbf{q}(\mathbf{u}) \leq 0 \tag{1.14}$$

in the sense of distributions for any C^2 entropy-entropy flux pair $(\eta, \mathbf{q})\colon \mathbb{R}^m \to \mathbb{R}^{d+1}, \mathbf{q} = (q_1, \dots, q_d)$, satisfying

$$\nabla^2 \eta(\mathbf{u}) \geq 0, \qquad \nabla q_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), \quad k = 1, \dots, d.$$

In Section 2, we discuss some historic developments and recent efforts in designing numerical shock capturing algorithms. In Section 3, we analyze various ways to generate numerical viscosity, which is essential, for some successful shock capturing algorithms to understand the essence of the algorithms; we also present some convergence and stability results of the algorithms and some new phenomena of solutions discovered from the numerical results by a upwind scheme. Then, in Section 4, we exhibit some examples to show that effective numerical algorithms can yield a mathematical existence theory to construct rigorously global entropy solutions for the compressible Euler equations in one-dimension and in multi-dimensions with geometric structure. In Section 5, we present two examples to show that shock capturing ideas are also useful for establishing the global existence of solutions of the compressible Navier-Stokes equations with large discontinuous initial data.

2. Shock Capturing and von Neumann

The main difficulty in calculating fluid flows with shocks is that it is very hard to predict, even in the process of a flow calculation, when and where new shocks arise and interact; tracking the shocks, especially their interactions, is numerically burdensome (see Lax [**61**]). Modern numerical ideas of shock capturing for computational fluid dynamics can date back as early as 1944 when von Neumann first proposed a new numerical method, a centered difference scheme, to treat the hydrodynamical shock problem, for which numerical calculations showed oscillations on mesh scale (see Lax [**59**]). von Neumann's dream of capturing shocks was first realized when von Neumann and Richtmyer [**102**] in 1950 introduced the ingenious idea of adding to the hydrodynamic equations a numerical viscous term of the same size as the truncation error. Their numerical viscosity guarantees that the scheme is consistent with the Clausius inequality, the entropy inequality. The shock jump conditions, i.e. the Rankine-Hugoniot jump conditions, are satisfied provided that the equations of gas dynamics are discretized in conservation form. Then oscillations were eliminated by the judicious use of the artificial viscosity; solutions constructed by this method converge uniformly except in a neighborhood of shocks, where they remain bounded and are spread out over a few mesh intervals.

Related analytical idea of shock capturing, i.e. vanishing viscosity methods, is quite old. For example, there are some hints about the idea of regarding inviscid gases as viscous gases with vanishingly small viscosity in the seminal paper by Stokes [**92**] in 1848. Also see the important contributions of Rankine [**84**], Hugoniot [**53**], and Rayleigh [**84**]. See Dafermos [**29**] for the details.

The main challenge in designing shock capturing numerical algorithms is that weak solutions are not unique; and the numerical schemes should be consistent with the Clausius inequality, the entropy inequality. Excellent numerical schemes should be also numerically simple, robust, fast, and low cost, and have sharp oscillation-free resolutions and high accuracy in domains where the solution is smooth. It is also desirable that the schemes capture contact discontinuities and vortices, and are coordinate invariant, among others.

For the one-dimensional case, examples of success include the Lax-Friedrichs scheme (1954), the Glimm scheme (1965), the Godunov scheme (1959) and related high order schemes (e.g. van Leer's MUSCL (1981), Colella-Wooward's PPM (1984), Harten-Engquist-Osher-Chakravarthy's ENO (1987)), and the Lax-Wendroff scheme (1960) and its two-step version, the Richtmyer scheme (1967) and the MacCormick scheme (1969).

For the multi-dimensional case, one direct approach is to generalize directly the one-dimensional methods to solve multi-dimensional problems; such an approach has lead several useful numerical methods including semi-discrete methods and Strang's dimension-dimension splitting methods.

Observe that multi-dimensional effects do play a significant role in the behavior of the solution locally, and any approach that only solves one-dimensional Riemann problems in the coordinate directions is clearly not using all the multi-dimensional information. The development of fully multi-dimensional methods requires a good mathematical theory to understand the multi-dimensional behavior of entropy solutions; current efforts in this direction include to use more information about the multi-dimensional behavior of solutions, determine the direction of primary wave propagation and employ wave propagation in other directions, and use transport techniques, relaxation techniques, and kinetic techniques from the microscopic level. See Fey-Jeltsch [**37**], Godlewski-Raviart [**45**], LeVeque [**65**], Toro [**97**], and the references cited therein.

3. Shock Capturing and Numerical Viscosity

Numerical viscosity plays an essential role in shock capturing numerical algorithms to guarantee their stability and consistency with the Lax entropy inequality. In this section, we analyze various ways to generate numerical viscosity from some successful shock capturing schemes to shed light on the essence of the algorithms. For clarity, we focus on the scalar conservation laws:

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \ \mathbf{f}\colon \mathbb{R} \to \mathbb{R}^d. \tag{3.1}$$

3.1. Lax-Friderichs Scheme

The Lax-Friedrichs scheme for (3.1) in one dimension reads:

$$\frac{1}{\Delta t}\left(u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)\right) + \frac{1}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n)) = 0, \tag{3.2}$$

which can be regarded as a direct discretization of (3.1) for $d = 1$, where $u_j^n \equiv u(j\Delta x, n\Delta t)$, and $\Delta x, \Delta t > 0$ are the space mesh length and the time mesh length, respectively.

To see its numerical viscosity, we calculate its local truncation error in any smooth region of the solution $u(x,t)$. Using that $u(x,t)$ is a solution of (3.1), one has

$$\begin{aligned} &\frac{1}{\Delta t}\left(u(x,t+\Delta t)-\frac{1}{2}(u(x-\Delta x,t)+u(x+\Delta x,t))\right)\\ &\qquad+\frac{1}{2\Delta x}(f(u(x+\Delta x,t))-f(u(x-\Delta,t)))\\ &=\partial_x\left(\frac{1}{2}\lambda\Delta x(1-\lambda^2 f'(u)^2)\partial_x u\right)+O((\Delta x)^2), \end{aligned} \tag{3.3}$$

where $\lambda=\frac{\Delta t}{\Delta x}$. Therefore, if $\Delta x, \Delta t$ satisfy the Courant-Friedrichs-Lewy condition:

$$\lambda\sup\{|f'(u)|\}<1,$$

then the first term in the right-hand side of (3.3) presents the dissipative effect, and the quantity $\frac{1}{2}\lambda\Delta x(1-\lambda^2 f'(u)^2)>0$ presents the strength of artificial viscosity.

The Godunov scheme has similar features with more delicate numerical viscosity.

The convergence of the Lax-Friedrichs scheme and the Godunov scheme was proved in Oleinik [**81**], Conway-Smoller [**26**], Kuznetsov [**58**], and Crandall-Majda [**28**] for scalar conservation laws, and in Chen [**9**] and Ding-Chen-Luo [**31, 32**] (also see Chen-LeFloch [**14**]) for the isentropic Euler equations. Related entropy flux-splitting schemes were analyzed in Chen-LeFloch [**15**]. We also refer to Liu-Yu [**73**] for the existence and behavior of continuum shock profiles for finite difference schemes. Recent results on the (apriori or aposteriori) error estimates of these schemes and related methods for scalar conservation laws can be found in [**4, 22, 23, 24, 36, 55, 56, 80, 83, 95, 96**] and the references cited therein.

3.2. Lax-Wendroff Scheme

The Lax-Wendroff scheme for (3.1) in one dimension reads:

$$\begin{aligned} u_j^{n+1}&=u_j^n-\frac{\lambda}{2}(f(u_{j+1}^n)-f(u_{j-1}^n))+\frac{\lambda^2}{2}\Delta_-\left(f'\left(\frac{u_j^n+u_{j-1}^n}{2}\right)\Delta_- f(u_j^n)\right)\\ &\quad+\lambda\Delta_-(\beta_{j+1}^n|\Delta_+ f'(u_j^n)|\Delta_+ u_j^n), \end{aligned} \tag{3.4}$$

where $\Delta_+ u_j=u_{j+1}-u_j, \Delta_- u_j=u_j-u_{j-1}$, and $\beta_{j+\frac{1}{2}}^n$ is a smooth function of u_j^n and u_{j+1}^n satisfying $0<\beta_0\le\beta_{j+\frac{1}{2}}^n\le\beta_1<\infty$, which presents the strength of numerical viscosity.

To understand the role of the last term in (3.4), we calculate its local truncation error for the case $\beta_{j+\frac{1}{2}}^n=$ const. and $f(u)=au$ in the region where $u(x,t)$ is smooth. Similar argument as in §3.1 yields

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{f(u_{j+1}^n)-f(u_{j-1}^n)}{2\Delta x}=\beta(\Delta x)^2 u_{xx}+\frac{1}{6}a(\Delta x)^2(a\lambda^2-1)u_{xxx},$$

where the first term in the right-hand side presents dissipative effect, and the second term presents dispersive effect. To ensure the convergence and stability of this scheme, it is required that the dissipation dominate the dispersion.

Majda-Osher [**76, 77**] showed the L^2-stability for scalar conservation laws, the entropy consistency for the boundedly convergent approximate solutions for the semi-discrete cases as well as the complete-discrete scheme for the time-independent cases of general systems endowed with a convex entropy, the efficient choices of artificial viscosity such as the switching techniques, and the validity of the CFL number in their analysis. We also refer to [**91**] for the stability of local discrete shock profiles for the Lax-Wendroff scheme.

In Chen-Liu [**16**], the convergence of Lax-Wendroff type schemes with high resolution to weak entropy solutions for hyperbolic conservation laws was analyzed. These schemes include the original Lax-Wendroff scheme (3.4) and its two-step versions — the Richtmyer scheme [**86**] and the MacCormack scheme [**75**]. For convex scalar conservation laws with algebraic growth flux functions, it was proved in [**16**] that these schemes converge to the weak solutions satisfying appropriate entropy inequalities, provided the dissipation dominates the dispersion, that is, β is large enough to control the dispersion. The proof was based on detailed L^p estimates of the approximate solutions, H^{-1} compactness estimates of the corresponding entropy dissipation measures, and a compensated compactness framework in Chen-Lu [**17**]. Then these techniques were generalized to study the convergence problem for the nonconvex scalar case and hyperbolic systems of conservation laws.

Another related class of second-order shock capturing schemes, MUSCL-type schemes, was carefully analyzed in Lions-Souganidis [**66**], Yang [**104**], Osher [**82**], and the references cited therein.

3.3. Spectral Viscosity Methods

We now discuss the spectral viscosity in the spectral viscosity methods, formulated in Chen-Du-Tadmor [**11**], for (3.1) subject to initial data

$$u(x,0) \equiv u_0(x) \in L^\infty(T^d[0,2\pi]), \tag{3.5}$$

and augmented with the Lax entropy inequality (1.14).

We approximate the spectral/pseudo-spectral projection of the exact entropy solution, $P_N u(\cdot,t)$, using an N-trigonometric polynomial, $u_N(x,t) = \sum_{|\xi|\le N} \hat{u}_\xi(t) e^{i\xi\cdot x}$, which is governed by the semi-discrete approximation

$$\partial_t u_N(x,t) + \nabla\cdot P_N \mathbf{f}(u_N(x,t)) = \varepsilon_N \sum_{j,k=1}^{d} \partial^2_{jk} Q_N^{j,k}(x,t) * u_N(x,t). \tag{3.6}$$

Together with one's favorite ODE solver, (3.6) gives a fully discrete method for the approximate solutions of (3.1) and (3.5).

The left-hand side of (3.6) is the standard Fourier approximation of (3.1). Although this part of the approximation is spectrally accurate with the conservation law (3.1), it lacks **entropy dissipation**, which is inconsistent with the entropy condition (1.14). Consequently, the standard Fourier approximation of (3.1) supports spurious Gibbs oscillations (once shocks are formed), which prevent strong convergence to the entropy solution of (3.1). To suppress these oscillations, without sacrificing the overall spectral accuracy, we augmented the standard Fourier approximation on the right-hand side of (3.6) by **spectral viscosity**, which consists of the following three ingredients:

- A vanishing viscosity amplitude, ε_N, of size
$$\varepsilon_N \sim N^{-\theta}, \quad \theta < 1.$$
- A viscosity-free spectrum of size $m_N >> 1$,
$$m_N \sim \frac{N^{\frac{\theta}{2}}}{(\log N)^{\frac{d}{2}}}, \quad \theta < 1.$$
- A family of viscosity kernels, $Q_N^{j,k}(x,t) = \sum_{|\xi|=m_N}^{N} \hat{Q}_\xi^{j,k}(t)e^{i\xi\cdot x}$, $1 \le j,k \le d$, activated only on high wavenumbers $|\xi| \ge m_N$, which can be conveniently implemented in the Fourier space as
$$\varepsilon_N \sum_{j,k=1}^{d} \partial_{jk} Q_N^{j,k} * u_N(x,t) \equiv -\varepsilon_N \sum_{|\xi|=m_N}^{N} < \hat{Q}_\xi \xi, \xi > \hat{u}_\xi(t) e^{i\xi\cdot x},$$
$$\left\langle \hat{Q}_\xi \xi, \xi \right\rangle \equiv \sum_{j,k=1}^{d} \hat{Q}_\xi^{j,k}(t)\xi_j\xi_k.$$

The viscosity kernels $Q_N^{j,k}(x,t)$ are assumed to be spherically symmetric, that is, $\hat{Q}_\xi^{j,k} = \hat{Q}_p^{j,k}$, for all $|\xi| = p$, with monotonically increasing Fourier coefficients, $\hat{Q}_p^{j,k}$, that satisfy
$$|\hat{Q}_p^{j,k} - \delta_{jk}| \le \text{Const.}\, \frac{m_N^2}{p^2}, \quad \text{for all } p \ge m_N.$$

The main purpose of the spectral viscosity is to achieve a compromise between two conflicting requirements. It is known (cf. Gottlieb-Orszag [**47**]) that the use of the spectral/pseudo-spectral projections yields a spectrally small error in the sense that

$$\|(I - P_N)f(u_N)(\cdot,t)\|_{L^2} \le \text{Const.}\, N^{-s}\|\partial_x^s u_N(\cdot,t)\|_{L^2}, \quad \text{for all } s \ge 0. \tag{3.7}$$

The additional spectral viscosity is also spectrally small, since
$$\varepsilon_N \| \sum_{j,k=1}^{d} \partial_{jk} Q_N^{j,k} * u_N(\cdot,t)\|_{L^2} \le \text{Const.}\, N^{-\frac{s\theta}{2}} \|\partial_x^s u_N(\cdot,t)\|_{L^2}, \quad \text{for all } s \ge 2.$$

Thus, on the one hand, the spectral viscosity is small enough to retain the formal spectral accuracy of the overall approximation; and, on the other hand, the spectral viscosity is sufficiently large to enforce the correct amount of entropy dissipation that is missing in the standard Fourier method, that is, $\varepsilon_N = 0$. In fact, the smallest scale of the spectral viscosity approximation (3.6) is order ϵ_N. It follows that, because of the presence of the spectral viscosity in (3.6), the spectral decay of the truncation error on the left of (3.7) is **independent** of the smoothness of the underlying solution. The spectral viscosity, although spectrally small, is only an L^p-bounded perturbation of the standard vanishing viscosity. This fact yields that the spectral viscosity solution remains uniformly bounded and that its weak limit is a measure-valued solution consistent with the entropy condition (1.14). Hence, DiPerna's uniqueness theorem [**34**] combining with the finiteness of propagation speed implies that u_N converges to the unique entropy solution of (3.1) and (3.5). An alternative, independent convergence proof of the spectral viscosity method was derived from its total-variation boundedness, provided the total variation of the initial data is bounded. An L^1-convergence rate estimate of the usual optimal order one-half was also achieved in Chen-Du-Tadmor [**11**].

3.4. Upwind Schemes

In this section, we show through a simple upwind scheme in [**5, 6**] that upwind schemes not only generate numerical viscosity to capture shocks, contact discontinuities, and vortices, but also have led the discovery of a new type of nonlinear hyperbolic waves, called smoothed Delta shocks in [**5, 7**], which had been missing in mathematical fluid dynamics.

To illustrate the ideas clearly, we first formulate such a scheme for the two-dimensional linear scalar equation:

$$\partial_t u + a\partial_x u + b\partial_y u = 0, \qquad a, b \text{ are constants.} \tag{3.8}$$

If a difference scheme $u_{i,j}^{n+1} = Lu_{i,j}^n$ satisfies that

$$u_{i,j}^{n+1} \text{ is a convex combination of } u_{i,j}^n, u_{i-1,j}^n, u_{i,j-1}^n, u_{i-1,j-1}^n, a > 0, b > 0,$$

or

$$u_{i,j}^{n+1} \text{ is a convex combination of } u_{i,j}^n, u_{i+1,j}^n, u_{i,j-1}^n, u_{i+1,j-1}^n, a < 0, b > 0,$$

or

$$u_{i,j}^{n+1} \text{ is a convex combination of } u_{i,j}^n, u_{i-1,j}^n, u_{i,j+1}^n, u_{i-1,j+1}^n, a > 0, b < 0,$$

or

$$u_{i,j}^{n+1} \text{ is a convex combination of } u_{i,j}^n, u_{i+1,j}^n, u_{i,j+1}^n, u_{i+1,j+1}^n, a < 0, b < 0.$$

We call the scheme an upwind averaging scheme based on four points.

For the case $a > 0$ and $b > 0$, the first-order upwind averaging scheme based on four points can be formulated as

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n - a\lambda^x(u_{i,j}^n - u_{i-1,j}^n) - b\lambda^y(u_{i,j}^n - u_{i,j-1}^n) \\ &\quad + ab\lambda^x\lambda^y(u_{i,j}^n - u_{i-1,j}^n - u_{i,j-1}^n + u_{i-1,j-1}^n), \qquad 0 < a\lambda^x, b\lambda^y \le 1, \end{aligned}$$

where $\lambda^x = \frac{\Delta t}{\Delta x}$ and $\lambda^y = \frac{\Delta t}{\Delta y}$. Then the second-order upwind scheme can be formulated as

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n - a\lambda^x\Delta^x u_{i-\frac12,j}^n - \frac12 a\lambda^x(1-a\lambda^x)\Delta_-^x[\phi_{i+\frac12,j}^{x,-}\Delta^x u_{i+\frac12,j}^n] \\ &\quad - b\lambda^y\Delta^y u_{i,j-\frac12}^n - \frac12 b\lambda^y(1-b\lambda^y)\Delta_-^y[\phi_{i,j+\frac12}^{y,-}\Delta^y u_{i,j+\frac12}^n] \\ &\quad + \frac12\alpha ab\lambda^x\lambda^y\Delta^x\Delta^y u_{i-\frac12,j-\frac12}^n, \qquad 0 < a\lambda^x, b\lambda^y \le 1, \end{aligned} \tag{3.9}$$

where $\Delta^x u_{i+\frac{1}{2},j} = u_{i+1,j} - u_{i,j}$, $\Delta^y u_{i,j+\frac{1}{2}} = u_{i,j+1} - u_{i,j}$, $\phi^{x,-}_{i+\frac{1}{2},j} = \phi(r^{x,-}_{i+\frac{1}{2},j})$, $\phi^{y,-}_{i,j+\frac{1}{2}} = \phi(r^{y,-}_{i,j+\frac{1}{2}})$, $r^{x,-}_{i+\frac{1}{2},j} = \frac{\Delta^x u_{i+\frac{3}{2},j}}{\Delta^x u_{i+\frac{1}{2},j}}$, $r^{y,-}_{i,j+\frac{1}{2}} = \frac{\Delta^y u_{i,j+\frac{3}{2}}}{\Delta^y u_{i,j+\frac{1}{2}}}$, and $\alpha \in [0,1]$ can be adjusted depending on the need of computations. When $\alpha = 0$, this scheme is of second-order accuracy in space variables and of first-order accuracy in time. When $\alpha = 1$, this scheme is of second-order accuracy in both space and time variables. One can easily check that this scheme is upwind averaging provided that

$$
\begin{aligned}
&\phi(r)|_{r\le 0} = 0,\\
&0 \le \phi(r) \le \min\Big(\frac{2(1-2b\lambda^y)}{1-a\lambda^x}, \frac{2(1-2a\lambda^x)}{1-b\lambda^y}\Big),\\
&0 \le \frac{\phi(r)}{r} \le \min\Big(\frac{1}{a\lambda^x} - \frac{1-\alpha b\lambda^y}{1-a\lambda^x}, \frac{1}{b\lambda^y} - \frac{1-\alpha a\lambda^x}{1-b\lambda^y}\Big),\\
&|a\lambda^x|, |b\lambda^y| \le \frac{1}{4}.
\end{aligned}
\tag{3.10}
$$

Based on the simple idea described above, a second-order upwind scheme for the two-dimensional compressible Euler equations was formulated in [**5, 6**]. We also refer the reader to [**63, 74**] for Lax-Liu's effective, robust positive schemes for gas dynamics.

With the aid of the upwind scheme in [**5, 6**], the Riemann problem for the two-dimensional compressible Euler equations in gas dynamics was systematically analyzed in Chang-Chen-Yang [**5, 6, 7**]. The central point at this issue is the dynamical interaction of shocks, centered rarefaction waves, and contact discontinuities that connect two neighboring constant initial states in the quadrants. Noting the essential difference between contact discontinuities J^+ and J^- distinguished by the sign of the vorticity and necessary compatibility conditions of initial data, we classified the Riemann data from sixteen cases in [**105**] to eighteen genuinely different cases. For each configuration, the structure of the Riemann solution was analyzed by using the method of characteristics, and corresponding numerical solution was illustrated by contour plots by using the upwind scheme.

In [**5, 6**], the main focus was on the interaction of shocks and rarefaction waves. We proved that there are two subcases for each of the two cases of the interaction of rarefaction waves. The numerical solutions are considerably coincident with conjectures in [**105**] except the case of interaction of rarefaction waves propagating in the opposite direction. For the latter, the numerical solution clearly shows that two compressive waves, even shock waves, appear in the solution. This phenomenon can be explained as the effect of compression of the flow characteristics.

In [**5, 7**], our focus was on the interaction of contact discontinuities, which consists of two genuinely different cases. For one case, the four contact discontinuities role up and generate a vortex, and the density monotonically decreases to zero at the center of the vortex along the stream curves. For the other, two shocks are formed and, in the subsonic region between two shocks, a vortex is generated for one subcase, and a new kind of nonlinear hyperbolic waves (called smoothed Delta shocks) was first discovered.

Related interesting phenomena and configurations can be found in Glimm, Klingenberg, McBryan, Plohr, Sharp, and Yaniv [**42**] (also see Chern, Glimm, McBryan, Plohr, and Yaniv [**20**]) by using their front-tracking algorithms, and Lax and Liu [**63, 74**] by using their positive schemes. Also see Schulz-Rinne, Collins, and Glaz [**87**], and Kurganov and Tadmor [**57**]. We remark that most of the numerical results in [**5, 6, 7**] are strikingly consistent with calculation in [**42, 57, 63, 74, 87**] via several different schemes.

4. Shock Capturing and Existence Theory

Effective numerical shock capturing algorithms not only provide excellent numerical solutions, but also yield a mathematical existence theory to construct rigorously global entropy solutions for the compressible Euler equations. The Glimm scheme [**41**] is an excellent example (see [**29, 68, 72, 88, 91**]; also [**1**] for the wave-front tracking algorithm first proposed by Dafermos [**30**]). The Lax-Friedrichs scheme and the Godunov scheme are another example (see [**9, 31, 32**]). In this section, we discuss a recent example to yield a new existence theory of global solutions with geometrical structure for the compressible Euler equations of isentropic gas dynamics (1.8).

Consider the spherically symmetric solutions outside a solid core ($|\mathbf{x}| \geq 1$):

$$\rho(\mathbf{x},t) = \rho(r,t), \quad \mathbf{m}(\mathbf{x},t) = m(r,t)\frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}|. \tag{4.1}$$

Then $(\rho, m)(r,t)$ is determined by the equations:

$$\begin{cases} \partial_t \rho + \partial_r m = -\frac{A'(r)}{A(r)} m, \\ \partial_t m + \partial_r(\frac{m^2}{\rho} + p(\rho)) = -\frac{A'(r)}{A(r)}\frac{m^2}{\rho}, \quad r > 1, \end{cases} \tag{4.2}$$

$$\begin{cases} m|_{r=1} = 0, \\ (\rho, m)|_{t=0} = (\rho_0, m_0)(r), \quad r > 1, \end{cases} \tag{4.3}$$

where $A(r) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} r^{d-1}$ is the surface area of d-dimensional sphere.

Although (4.2) is derived through a spherically symmetric flow, it also describes many important physical flows such as transonic nozzle flows with variable cross-sectional area $A(r) \geq c_0 > 0$ (see Courant-Friedirchs [**27**] and Whitham [**103**]).

The eigenvalues of (4.2) are

$$\lambda_{\pm} = \frac{m}{\rho} \pm c = c(M \pm 1),$$

where $c = \sqrt{p'(\rho)}$ is the sound speed, and $M = \frac{m}{\rho c}$ is the Mach number. Notice that

$$\lambda_+ - \lambda_- = 2c(\rho) = 2\rho^{\frac{\gamma-1}{2}} \to 0$$

as $\rho \to 0$, which implies that the system in (4.2) is nonstrictly hyperbolic near $\rho = 0$. On the other hand, the geometric source speed is zero, and the eigenvalues

$\lambda_\pm$ are also zero near $M \approx \pm 1$, which indicates that there is also a nonlinear resonance between the geometrical source terms and the characteristic modes.

The insights we sought for this problem are: (a) whether the solution has the same geometrical structure globally; (b) whether the solution blows up to infinity in a finite time, especially the density. These questions are not easily understood in physical experiments and numerical simulations, especially for the second question, due to a limited capacity of available instruments and computers. The central difficulty of this problem in the unbounded domain is the reflection of waves from infinity and their strengthening as they move radially inward. Another difficulty is that the associated steady-state equations change type from elliptic to hyperbolic at the sonic point; such steady-state solutions are fundamental building blocks in the approach in Chen-Glimm [**12**].

Consider the steady-state solutions:

$$\begin{cases} m_r = -\frac{A'(r)}{A(r)}m, \\ (\frac{m^2}{\rho} + p(\rho))_r = -\frac{A'(r)}{A(r)}\frac{m^2}{\rho}, \\ (\rho, m)|_{r=r_0} = (\rho_0, m_0). \end{cases} \tag{4.4}$$

The first equation can be directly integrated to get

$$A(r)m = A(r_0)m_0. \tag{4.5}$$

The second equation can be rewritten as

$$(A(r)\frac{m^2}{\rho})_r + A(r)p(\rho)_r = 0.$$

Hence, using (4.5) and $\theta = \frac{\gamma-1}{2}$,

$$\rho^{2\theta}(\theta M^2 + 1) = \rho_0^{2\theta}(\theta M_0^2 + 1). \tag{4.6}$$

Then (4.5)-(4.6) become

$$\left(\frac{\rho}{\rho_0}\right)^{\theta+1} = \frac{A(r_0)M_0}{A(r)M}, \quad \left(\frac{\rho}{\rho_0}\right)^{2\theta} = \frac{\theta M_0^2 + 1}{\theta M^2 + 1}. \tag{4.7}$$

Eliminating ρ in (4.7), we obtain

$$F(M) = \frac{A(r_0)}{A(r)}F(M_0), \tag{4.8}$$

where

$$F(M) = M\left(\frac{1+\theta}{1+\theta M^2}\right)^{\frac{\theta+1}{2\theta}}$$

satisfies

$$\begin{cases} F(0) = 0, F(1) = 1; \quad F(M) \to 0, \text{ when } M \to \infty, \\ F'(M)(1-M) > 0, \text{ when } M \in [0, \infty), \\ F'(M)(1+M) > 0, \text{ when } M \in (-\infty, 0]. \end{cases}$$

Thus we see that, if $A(r) < A(r_0)|F(M_0)|$, no smooth solution exists because the right-hand side of (4.8) exceeds the maximum values of $|F|$. If $A(r) > A(r_0)|F(M_0)|$, there are two solutions of (4.8), one with $|M| > 1$ and the other with $|M| < 1$ since the line $F = \frac{A(r_0)}{A(r)}F(M_0)$ intersects the graph of $F(M)$ at two points.

For $A'(r) = 0$, the system becomes the one-dimensional isentropic Euler equations; the convergence of shock capturing numerical schemes was handled in Chen [**9**] and Ding-Chen-Luo [**31, 32**] (also see Chen-LeFloch [**14**]).

For $A'(r) \neq 0$, the existence of global solutions for the transonic nozzle flow problem was established in Liu [**69**] by first incorporating the steady-state building blocks with the Glimm method [**41**], provided that the initial data have small total variation and are bounded away from both sonic and vacuum states. A generalized random choice method was introduced to compute transient gas flows in a Laval nozzle in [**40, 44**]. A global weak entropy solution with spherical symmetry was constructed in [**78**] for $\gamma = 1$, and the local existence of such a weak solution for $1 < \gamma \leq \frac{5}{3}$ was also discussed in [**79**]. Also see Liu [**69, 70, 71**], Embid-Goodman-Majda [**35**], and Fok [**38**].

In Chen-Glimm [**12**], we developed a numerical shock capturing scheme for (4.2) and applied it to constructing global solutions of (1.8) with geometrical structure and large initial data in L^∞ for $1 < \gamma \leq 5/3$, including both the spherical symmetric flows and the transonic nozzle flow. The case $\gamma \geq 5/3$ was treated in [**19**]. We proved that the solutions do not blow up to infinity in a finite time. More precisely, we proved that there exists a family of numerical shock capturing approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ of (4.2) such that

(i) $0 \leq \rho^\varepsilon(r,t) \leq C$, $|\frac{m^\varepsilon(r,t)}{\rho^\varepsilon(r,t)}| \leq C$;

(ii) $\partial_t\eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon)$ is compact in $H^{-1}_{\text{loc}}(\Omega)$ for any weak entropy pair (η, q), where $\Omega \subset \mathbb{R}^2_+$ or $\Omega \subset \mathbb{R}_+ \times (1,\infty)$.

Furthermore, there is a convergent subsequence $(\rho^{\epsilon_k}, m^{\epsilon_k})$ in the approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ such that

$$(\rho^{\varepsilon_k}, m^{\varepsilon_k})(r,t) \to (\rho, m)(r,t), \text{ a.e.}$$

which is a global entropy solution of (4.2) in L^∞ and satisfies

$$0 \leq \rho(r,t) \leq C, \quad \left|\frac{m(r,t)}{\rho(r,t)}\right| \leq C.$$

Moreover, for the initial-boundary value problem (4.2)-(4.3), the vector function $(\rho, \mathbf{m})(\mathbf{x},t)$, defined in (4.1) through $(\rho, m)(r,t)$, is a global entropy solution of (1.8) in L^∞ with spherically symmetric initial data.

The approach in Chen-Glimm [**12**] for the construction of the family of numerical approximate solutions $(\rho^\epsilon, m^\epsilon)$ above is to combine the shock capturing ideas with the fractional-step techniques to develop a first-order Godunov-type shock capturing scheme, with piecewise constant building blocks replaced by piecewise smooth ones. The main point is to use the steady-state solutions governing the large-time asymptotic states, which incorporate the main geometrical source terms, to modify the wave strengths in the Riemann solutions, in order to reduce the errors in large number time-steps in the approximate solutions. This construction yields better approximate solutions and permits a uniform L^∞ bound. There are two technical difficulties to achieve this, both due to transonic phenomena. One is that no smooth steady-state solution exists in each cell in general. This problem can be solved by introducing a standing shock. The other is that the constructed steady-state solution in each cell must satisfy the following requirements:

(a) The oscillation of the steady-state solution around the Godunov value must be of the same order as the cell length to obtain the L^∞ estimate for the convergence arguments;

(b) The difference between the average of the steady-state solution over each cell and the Godunov value must be higher than first-order in the cell length to ensure the consistency of the corresponding approximate solutions with the Euler equations. That is,

$$\frac{1}{\Delta r}\int_{(j-\frac{1}{2})\Delta r}^{(j+\frac{1}{2})\Delta r} \mathbf{u}(r, n\Delta t - 0)\,dr = \mathbf{u}_j^n(1 + O(|\Delta r|^{1+\delta})), \quad \delta > 0.$$

These requirements are naturally satisfied by smooth steady-state solutions that are bounded away from the sonic state in the cell. The general case must include the transonic steady-state solutions. The sonic difficulty was overcome, as in experimental physics, by introducing an additional standing shock with continuous mass and by adjusting its left state and right state in the density and its location to control the growth of the density. These requirements can yield the H^{-1} compactness estimates for entropy dissipation measures

$$\partial_t\eta(\rho^\varepsilon, m^\varepsilon) + \partial_r q(\rho^\varepsilon, m^\varepsilon)$$

and the strong compactness of approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ with the aid of a compensated compactness framework in [**9, 14, 31, 67**].

5. Shock Capturing and the Navier-Stokes Equations

The shock capturing ideas are also very useful for designing numerical viscosity and heat-conductivity terms for the Navier-Stokes equations and for establishing the global existence of solutions with large discontinuous initial data. In this section, we give such two examples.

Example 1. Consider the one-dimensional compressible Navier-Stokes equations:

$$\begin{aligned} &\partial_t\tau - \partial_x v = 0,\\ &\partial_t v + \partial_x p(\tau, e) = \partial_x\left(\frac{\epsilon\partial_x v}{\tau}\right),\\ &\partial_t\left(e + \frac{v^2}{2}\right) + \partial_x(vp(\tau, e)) = \partial_x\left(\frac{\epsilon v\partial_x v + \lambda\partial_x e}{\tau}\right). \end{aligned} \tag{5.1}$$

Here ϵ and λ are fixed positive viscosity parameters; and x is the Lagrangian coordinate, so that x =constant corresponds to a particle trajectory. We assume that e, v, and p are related by the equation of state of a polytropic gas in (1.5)-(1.6).

For concreteness, we focus on the initial boundary-value problem for (5.1); thus $0 < x < 1$ without loss of generality, and the boundary conditions

$$v(i,t) = 0, \quad e_x(i,t) = 0, \qquad i = 0, 1, \tag{5.2}$$

are to hold for $t > 0$. Let initial data

$$(\tau, v, e)|_{t=0} = (\tau_0, v_0, e_0)(x), \quad 0 \le x \le 1, \tag{5.3}$$

be given, satisfying

$$C_0^{-1} \le \tau_0(x) \le C_0, \quad e_0(x) \ge C_0^{-1}, \quad \|v_0\|_{L^4} + \|e_0\|_{L^2} + \mathrm{TV}(\tau_0) \le C_0, \tag{5.4}$$

for a constant $C_0 > 0$. On the other hand, our existence results can be extended with little difficulty to the Cauchy problem.

Let h be an increment in x such that $Kh = 1$ for $K \in \mathbb{Z}_+$, $x_k = kh$ for $k \in \{0, 1, \dots, K\}$, and $x_j = jh$ for $j \in \{\frac{1}{2}, \frac{3}{2}, \dots, K - \frac{1}{2}\}$. Time-dependent approximations $(\tau_j(t), v_k(t), e_j(t))$ to $(\tau(x_j,t), v(x_k,t),\ e(x_j,t))$ are then constructed as follows:

$$\dot{\tau}_j = \delta v_j, \tag{5.5}$$

$$\dot{v}_k + \delta p_k = \epsilon \delta \left(\frac{\delta v}{\tau}\right)_k, \tag{5.6}$$

$$\dot{e}_j + p_j \delta v_j = \epsilon \frac{(\delta v_j)^2}{\tau_j} + \lambda \delta \left(\frac{\delta e}{\tau}\right)_j. \tag{5.7}$$

Here $p_j = p(\tau_j, e_j)$, $e_j = e(\tau_j, e_j)$, τ_k is taken to be the average $\tau_k = \frac{\tau_{k+\frac{1}{2}} + \tau_{k-\frac{1}{2}}}{2}$ with $j \in \{\frac{1}{2}, \frac{3}{2}, \dots, K - \frac{1}{2}\}$, $k \in \{0, 1, \dots, K\}$, and δ is the operator defined by $\delta w_l = \frac{w_{l+\frac{1}{2}} - w_{l-\frac{1}{2}}}{h}, l = k$, or j. The initial data $(\tau_j(0), v_k(0), e_j(0))$ for the ordinary differential equations specified above satisfy that

$$v_0 = v_K = 0, \qquad \delta e_0 = \delta e_K = 0, \tag{5.8}$$

$$C_0^{-1} \le \tau_j(0) \le C_0, \quad e_j(0) \ge C_0^{-1}, \quad \textstyle\sum_k v_k^4(0) h + \sum_j e_j^2(0) h \le C_0, \tag{5.9}$$

and that there are distinguished points $0 < x_{k_1} < x_{k_2} < \dots < x_{k_N} < 1, N = N(h), N^4 h \le 1$, such that $\sum_{k=k_i} |[\tau_{k_i}(0)]| + \sum_{k \ne k_i} |\delta \tau_k(0)|^2 h \le C_0$. Now, the standard theory of ordinary differential equations applies to guarantee that the initial-value problem (5.5)-(5.9) has a unique solution $(\tau_j(t), v_k(t), e_j(t))$, defined at least for small time.

In Chen-Hoff-Trivisa [**13**], we first derived the apriori bounds to show that these approximate solutions exist **globally** in time and to provide sufficient compactness both to extract limiting solutions as $h \to 0$ as well as to determine their asymptotic behavior. Then we proved the global existence of weak solutions (τ, v, e) to the Navier-Stokes equations (5.1) with **large, discontinuous** initial data such that $\tau, v \in C([0,\infty); L^2)$, $e \in C((0,\infty); L^2)$ with $e(\cdot,t) \rightharpoonup e_0$ weakly in L^2 as

$t \to 0$. Furthermore, we showed that there is a constant M depending on C_0, but independent of $t > 0$, such that the following hold:

$$\begin{aligned} M^{-1} \le \tau(x,t) \le M, \qquad & M^{-1} \le e(x,t) \le M\sigma^{-1}(t), \\ \mathrm{TV}_{[0,1]}\,(\tau(\cdot,t)) \le M, \qquad & \|\tau(\cdot,t') - \tau(\cdot,t)\| \le M|t'-t|^{1/2}, \\ \|v_x(\cdot,t)\| \le M\sigma^{-1/2}(t), \qquad & \|e_x(\cdot,t)\| \le M\sigma^{-1}(t), \\ \|v(\cdot,t)\|_{L^\infty} \le M\sigma^{-1/4}(t), \qquad & \mathcal{E}(t) + \mathcal{F}(t) \le M, \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}(t) = & \sup_{0\le s\le t} \big(\sigma(s)\|v_x(\cdot,s)\|^2 + \sigma^2(s)\|e_x(\cdot,s)\|^2\big) \\ & + \int_0^t (\|v_x(\cdot,s)\|^2 + \|e_x(\cdot,s)\|^2 + \sigma(s)\|v_t(\cdot,s)\|^2 + \sigma^2(s)\|e_t(\cdot,s)\|^2)\,ds, \\ \mathcal{F}(t) = & \sup_{0\le s\le t} \big(\sigma^2(s)\|v_t(\cdot,s)\|^2 + \sigma^3(s)\|e_t(\cdot,s)\|^2\big) \\ & + \int_0^t \sigma^2(s)\|v_{xt}(\cdot,s)\|^2 ds + \int_0^t \sigma^3(s)\|e_{xt}(\cdot,s)\|^2\,ds, \end{aligned}$$

with $\sigma(t) = \min\{t, 1\}$, and $\|\cdot\|$ the norm in $L^2(0,1)$.

These a-priori estimates for the weak solutions are **independent of large time** so that their asymptotic behavior can be determined. In particular, we showed that, as time goes to infinity, the solution tends to a constant state determined by the initial mass and energy, and that the magnitudes of singularities in the solution decay to zero. The regularity and stability for the weak solutions were also established.

Example 2. Consider the multi-dimensional Navier-Stokes equations for compressible heat-conducting flow in $\mathbb{R}^d (d \ge 2)$;

$$\begin{aligned} & \partial_t\rho + \mathrm{div}(\rho\mathbf{v}) = 0, \\ & \partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) + \nabla p = \mu\Delta\mathbf{v} + (\lambda+\mu)\nabla(\mathrm{div}\,\mathbf{v}), \\ & \partial_t(\rho(e + \frac{|\mathbf{v}|^2}{2})) + \mathrm{div}(\mathbf{v}(\rho(e + \frac{|\mathbf{v}|^2}{2}) + p)) \\ & \qquad = \Delta(\kappa e + \frac{\mu}{2}|\mathbf{v}|^2) + \lambda\,\mathrm{div}((\mathrm{div}\,\mathbf{v})\mathbf{v}) + \mu\,\mathrm{div}((\nabla\mathbf{v})\mathbf{v}). \end{aligned} \tag{5.10}$$

Here λ and μ are the constant viscosity coefficients, $\mu > 0, \lambda + 2\mu/n \ge 0$; and $\kappa > 0$ is the ratio of the heat conductivity coefficient over the heat capacity. We focus on polytropic fluids (1.5)-(1.6).

At $t = 0$:

$$\begin{aligned} & (\rho,\mathbf{v},e)|_{t=0} = (\rho_0,\mathbf{v}_0,e_0)(\mathbf{x}) = (\rho_0(r), v_0(r)\frac{\mathbf{x}}{r}, e_0(r)), \\ & (\rho_0,v_0,e_0)(r) \in W^{1,2}(0,r_0), \quad v_0(0) = 0, \\ & 0 < \rho_0(r) \le C_0, \quad e_0(r) \ge C_0^{-1}, \qquad 1 < r = |\mathbf{x}| < r_0. \end{aligned} \tag{5.11}$$

On the fixed boundary $r = 1$:

$$\mathbf{v}|_{r=1} = 0, \quad \partial_r e|_{r=1} = 0. \tag{5.12}$$

The conditions on the free boundary $r = r(t)$ specified later so that the free boundary connects the compressible heat-conducting fluids to the vacuum state with free normal stress and zero normal heat flux, which represent a natural physical situation that the fluids are free to expand into the vacuum region without normal resistance from the vacuum. The fluids are initially assumed to fill with a finite volume and zero density at the free boundary, and with bounded positive density and temperature (equivalently, internal energy) between the solid core and the initial position of the free boundary. One of the main analytical difficulties to handle this problem is the singularity of solutions near the free boundary.

Unlike the stability of solutions with the vacuum states for the compressible Euler equations (see [**10**]), solutions with the vacuum states for the compressible Navier-Stokes equations are not stable in general (see Hoff-Serre [**52**]). Hence it is essential to estimate first a relative positive lower bound of the density between the solid core and the free boundary. Another physical important question is whether the free boundary expands with a finite speed. The positive answer to these two questions is also essential for establishing the existence theorem of global solutions without the vacuum states between the solid core and the free boundary for this problem.

To solve this problem, we look for spherically symmetric solutions $(\rho, \mathbf{v}, e)$:

$$\rho(\mathbf{x}, t) = \rho(r, t), \quad \mathbf{v}(\mathbf{x}, t) = v(r, t)\frac{\mathbf{x}}{r}, \quad e(\mathbf{x}, t) = e(r, t). \tag{5.13}$$

It is more convenient to reduce the problem in Eulerian coordinates (r, t) to the problem in Lagrangian coordinates (x, t) moving with the fluid, via the transformation:

$$x = \int_1^{r(x,t)} s^{d-1}\rho(s, t)\, ds, \ \text{ or, } \ r(x, t) = r(x, 0) + \int_0^t v(r(x, \tau), \tau)\, d\tau. \tag{5.14}$$

It is easy to check that $x = \int_1^{r(x,0)} s^{d-1}\rho_0(s)\, ds$. Without loss of generality, we assume that $\int_1^{r_0} s^{d-1}\rho_0(s)\, dx = 1$, so that the region $\{(r, t) \,|\, 1 \le r \le r(t), 0 \le t \le T\}$ under consideration is transformed into the region $\{(x, t) \,|\, 0 \le x \le 1, 0 \le t \le T\}$. With this change of coordinates, then $(\rho, v, e)(x, t)$ is determined by

$$\begin{aligned} &\partial_t \rho = -\rho^2 \partial_x (r^{d-1} v), \\ &\partial_t v = r^{d-1}\partial_x \sigma, \\ &\partial_t e = \kappa \partial_x \left(\rho r^{2d-2}\partial_x e\right) + \sigma \partial_x (r^{d-1} v) - 2\mu(d-1)\partial_x (r^{d-2} v^2), \end{aligned} \tag{5.15}$$

where $\sigma = (\lambda + 2\mu)\rho\partial_x(r^{d-1}v) - p$. Then the initial conditions are in the form

$$(\rho, v, e)|_{t=0} = (\rho_0, v_0, e_0)(x), \qquad 0 \le x \le 1. \tag{5.16}$$

The free boundary conditions and the solid boundary conditions then are specified as

$$v(0,t)=0, \quad (\rho\partial_x e)(0,t)=(\rho\partial_x e)(1,t)=0, \quad \sigma(1,t)=0. \tag{5.17}$$

To articulate the assumption that there is no initial cavity with the fluid residing in the bounded region, we assume that there exist a decreasing nonnegative function $\Lambda(x)$, $\Lambda(1)=0$, $\int_0^1 \Lambda^{-1}(x)\,dx<\infty$, and a constant $C_0>0$ such that

$$C_0^{-1}\Lambda(x)\le \rho_0(x)\le C_0\Lambda(x) \tag{5.18}$$

holds uniformly for all $x\in[0,1]$.

We use the following a space-discrete difference scheme to handle the singularity of solutions near the free boundary. Let h be an increment in x such that $Kh=1$ for $K\in\mathbb{Z}_+$, $x_j=jh$ for $j\in\{0,1,\dots,K\}$. For each integer K, we construct the following time-dependent functions:

$$(\rho_j(t),v_j(t),e_j(t)), \quad j=0,\dots,K, \tag{5.19}$$

that form a discrete approximation to $(\rho(x_j,t),v(x_j,t),e(x_j,t))$ for $j=0,\dots,K$.

First, $(\rho_i(t),v_j(t),e_j(t))$, $i=0,\dots,K-1$, $j=1,\dots,K-1$, are determined by the following system of $3K-2$ differential equations:

$$\begin{aligned}
&\dot\rho_i=-\rho_i^2\delta(r_i^{d-1}v_i),\\
&\dot v_j=r_j^{d-1}\delta\sigma_j,\\
&\dot e_j=\kappa\delta(\rho_{j-1}r_j^{2d-2}\delta e_{j-1})+\sigma_j\delta(r_{j-1}^{d-2}v_{j-1})-2\mu(d-1)\delta(r_{j-1}^{d-2}v_{j-1}^2),
\end{aligned}\tag{5.20}$$

where δ is the operator defined by $\delta w_j=\frac{w_{j+1}-w_j}{h}$, and

$$\begin{aligned}
&\sigma_j(t)=(\lambda+2\mu)\rho_{j-1}\delta(r_{j-1}^{d-1}v_{j-1})-p(\rho_{j-1},e_j),\\
&r_0=1, \quad r_j^d(t)=1+d\sum_{i=0}^{j-1}\frac{h}{\rho_i}, \quad j=1,\dots,K.
\end{aligned}\tag{5.21}$$

The corresponding initial data are defined by

$$\begin{aligned}
&\rho_{K-1}(0)=\rho_{K-2}(0),\\
&(\rho_i,v_j,e_j)(0)=\Big(\frac1h\int_{ih}^{(i+1)h}\rho_0(x)\,dx,\ \frac1h\int_{(j-1)h}^{jh}v_0(x)\,dx,\\
&\qquad\qquad \frac1h\int_{(j-1)h}^{jh}e_0(x)\,dx\Big),\\
&\qquad\qquad i=0,\dots,K-2;\ j=1,\dots,K-1.
\end{aligned}\tag{5.22}$$

Then the initial data $(\rho_j,v_j,e_j)(0)$ satisfy

$$\begin{aligned}
&C_0^{-1}\Lambda(x_{j+1})\le\rho_j(0)\le C_0\Lambda(x_j), \qquad e_j(0)\ge C_0^{-1},\\
&\sum_{j=0}^{K-1}(|(\rho_j,v_j,e_j)|^2+|(\delta\rho_j,\delta v_j,\delta e_j)|^2)(0)h\le C_1,
\end{aligned}\tag{5.23}$$

which also imply $\sum_{j=0}^{K-1}v_j^4(0)h\le C_1$, where $C_1>0$ is independent of K.

These are coupled with additional five boundary conditions:

$$\begin{aligned} &\rho_K(t) = \rho_{K-1}(t), \qquad v_0(t) = 0, \\ &e_0(t) = e_1(t), \quad e_K(t) = e_{K-1}(t), \quad \sigma_K(t) = 0. \end{aligned} \tag{5.24}$$

These boundary conditions are consistent with the initial data. Condition $\sigma_K(t) = 0$ determines $v_K(t)$.

The basic theory of differential equations guarantees the local existence of smooth solutions $(\rho_i, v_j, e_j)(t), i = 0, \ldots, K-1, j = 1, \ldots, K-1$, to the Cauchy problem (5.20)-(5.23) in some time interval $(0, T)$.

In Chen-Krakta [**18**], we derived the apriori bounds to guarantee that the solutions are actually globally defined in $[0, \infty)$, that is, the scheme $(\rho_j, v_j, e_j)(t), j = 0, 1, \ldots, K$, in (5.19)-(5.24) is globally well defined. Then we constructed global solutions of the multi-dimensional Navier-Stokes equations for compressible heat-conducting flow, with spherically symmetric initial data of large oscillation between a static solid core and a free boundary connected to a surrounding vacuum state. The approach we employed is to combine an effective difference scheme (5.19)-(5.24) to construct approximate solutions with the energy methods and the pointwise estimate techniques to deal with the singularity of solutions near the free boundary and to obtain the bounded estimates of the solutions and the free boundary as time evolves. The convergence of the difference scheme was established. It was also proved that no vacuum develops between the solid core and the free boundary, and the free boundary expands with a finite speed.

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