Acta Math. Univ. Comenianae
Vol. LXX, 2(2001), pp. 193–196

ON A PROBLEM OF FUJII CONCERNING RIEMANN'S ζ-FUNCTION

J.-C. PUCHTA

Abstract. We improve bounds of A. Fujii concerning the distribution of zeros of Riemann's ζ-function with respect to logarithms of prime numbers.

H. Rademacher ([**5**, p. 456]) posed the problem of the distribution of the zeros of $\zeta(s) \pmod 1$. Especially he asked for the distribution of those zeros $\rho = \sigma + i\gamma$, such that for a prime p and an integer k we have

$$\left\| \gamma \frac{k \log p}{2\pi} - \frac{1}{2} \right\| < \frac{1}{4}.$$

Defining $\Xi(p_k)$ by

$$\Xi(p_k) := \left\{ \gamma > 0 : \left\| \gamma \frac{\log p_k}{2\pi} - \frac{1}{2} \right\| < \frac{1}{4}. \right\}$$

A. Fujii[**2**] proved that

$$\{\gamma > 0\} = \bigcup_{k=1}^{\infty} \Xi(p_k).$$

Thus the following function $M(T)$ is well defined:

$$M(T) := \min \left\{ K : \{0 < \gamma < T\} \subseteq \bigcup_{k=1}^{K} \Xi(p_k). \right\}$$

In a subsequent article [**3**] he showed that

$$\frac{\log \log T}{\log \log \log T} \ll M(T) \ll e^{A \log^2 T}.$$

Furthermore, under the Riemannian hypothesis, the right hand side can be replaced by $T^2 \log^4 T$. He asked whether these bounds can be improved. The aim of this note is to give such an improvement.

Theorem 1. *With the notation as above, we have*

$$M(T) \ll T^{18/13+\epsilon}.$$

Received March 24, 2000.
2000 *Mathematics Subject Classification.* Primary 11M26, 11N05.

If we assume the Riemannian Hypothesis, we have

$$M(T) \ll T \log^{3+\epsilon} T.$$

Theorem 2. *With the notation as above we have*

$$M(T) \gg \frac{\log^{1/2} T}{\log_2^{3/4} T}.$$

If we assume that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + o\left(\frac{\log T}{\log_2 T}\right)$$

where $N(T)$ denotes the number of zeros of ζ with imaginary part $\leq T$, we even have

$$M(T) \geq \frac{1}{9} \log T$$

for arbitrary large T.

A probabilistic argument shows that one should expect $M(T) \asymp \log T$, thus Theorem 2 seems to be closer to the truth than Theorem 1.

The proof of the upper bound will be based on the following estimate on gaps between primes.

Theorem 3 (Selberg, Heath-Brown)**.** *We have*

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll x^{23/18+\epsilon}.$$

If we assume RH, the right hand side can be replaced by $x \log^{3+\epsilon} x$.

Proof. The estimate under RH was given by Selberg[**6**], the unconditional case was treated by Heath-Brown[**4**]. □

For the lower bound we need an estimate for the distribution of the zeros of ζ. Define as usual $S(t) = \arg \xi(1/2 + it) - \arg \xi(2 + it)$.

Theorem 4. *For any integer $k \geq 1$ and any $h < 1$ we have*

$$\int_0^T (S(t+h) - S(t))^{2k} \, dt \ll T\big(ck^4 \log(3 + h \log T)\big)^k.$$

Proof. This was proven by Fujii[**1**]. □

Now we can give the upper bound stated in Theorem 1. Let $x < T$ be some real number, and assume that for all $p_k \leq M$ we have $\|x \log p_k\| < \frac{1}{4}$. If p_k runs through the interval $[M/2, M]$, $x \log p_k$ runs through an interval of length $(\log 2 + o(1))x$. Thus in this interval there are $\gg x$ primes p_k, such that $x \log p_{k+1} - x \log p_k > \frac{1}{2}$, from which we deduce $(p_{k+1} - p_k) \gg \frac{M}{x}$. Hence we obtain the bound

$$\sum_{p_n \leq M} (p_{n+1} - p_n)^2 \gg x \left(\frac{M}{x}\right)^2 = M^2 x^{-1}.$$

If we estimate the left hand side using Theorem 3, we obtain $M^{23/18+\epsilon}$ resp. $M\log^{3+\epsilon} M \ll M^2x^{-1}$ under RH. Solving for M gives the claimed upper bound.

To prove the second estimate we note that by the pigeon-hole-principle for any given $0<\epsilon<1/2$ and real numbers $\alpha_1,\dots,\alpha_n$ there is some sequence $0<t_1<\dots<t_N<N\epsilon^{-n}$, such that $\|t_i\alpha_j\|<\epsilon$ for all i and j and $|t_i-t_j|>1$. Set $\alpha_j=\log p_j$, $\epsilon=1/8$. Then there are N disjoint intervals I of length $\gg \log^{-1} p_n$, such that for all $t\in I$ we have $\|t\log p_j\|<\frac{1}{4}, j\leq n$, and all these intervals are contained in $[0,N8^n]$. Now we define $n=[c(\log T)^{1/2}\log_2^{-3/4} T]$, $N=[T8^{-n}]$ where c is a sufficiently small constant. Since $n=o(\log T)$, this implies that $N>T^{2/3}$ for T sufficiently large, thus the total length of all intervals is $>2\sqrt{T}$, thus at least $N/2$ of the intervals are contained within $[\sqrt{T},T]$. Now assume that one of these intervals contains the imaginary part γ of a zero of ζ. Then we have $\|\gamma\log p_j\|<1/2$ for all $j\leq n$, thus $M(T)>n$. If on the other hand none of these intervals contains a zero, we can give a lower bound for

$$\int_0^T \left(S\left(t+\frac{c}{\log n}\right)-S(t)\right)^{2k}\,dt.$$

We choose c such that $\frac{c}{\log n}$ is half the length of an interval I. Then on at least one half of I we have $\left|S\left(t+\frac{c}{\log n}\right)-S(t)\right|\gg\frac{\log T}{\log n}$, and there are at least N such intervals. Thus the integral becomes

$$\gg N\cdot\left(c\frac{\log T}{\log n}\right)^{2k}$$

and by Theorem 4 we get the inequality

$$N\cdot\left(c\frac{\log T}{\log n}\right)^{2k}\ll T\bigl(ck^4\log(3+h\log T)\bigr)^k.$$

Since $N\gg T8^{-n}$, we get by taking the $2k$-th root

$$\begin{aligned}\frac{\log T}{\log n} &\ll 8^{n/2k}k^2\log_2^{1/2} T\\ \frac{\log T}{\log_2^{1/2} T} &\ll 8^{n/2k}k^2\log n.\end{aligned}$$

By choosing $k=n$ the right hand side becomes $n^2\log n$, thus by definition of n the inequality $\frac{\log T}{\log_2^{1/2} T}<c_1c\frac{\log T}{\log_2^{1/2} T}$, which becomes wrong for c sufficiently small. Hence there is some positive constant c, such that under the given circumstances the inequality $n<c\frac{\log^{1/2} T}{\log_2^{3/4} T}$ implies $M(T)>n$. Thus for T sufficiently large we have $M(T)>c\frac{\log^{1/2} T}{\log_2^{3/4} T}-1$ proving our theorem.

If we finally assume $S(T)=o\left(\frac{\log t}{\log_2 t}\right)$, the argument above may be simplified by choosing $N=1$. Then if $M(T)<n, 8^n<T$, we get a gap between consecutive

zeros of length $\gg \frac{1}{\log_2 T}$, however, the assumption on S implies that the maximal gap length is $o\left(\frac{1}{\log_2 t}\right)$. Thus we get $M(T) \geq \frac{\log T}{\log 8} - 2$.

Acknowledgment. I would like to thank the referee of some other journal for sharpening Theorem 2.

References

1. Fujii A., *On the distribution of the zeros of the Riemann zeta function in short intervals*, Bull. Am. Math. Soc. **81** (1975), 139–142.

2. ———, *On a problem and comment of Rademacher's*, Comment. Math. Univ. St. Paul. **44** (1995), 69–92.

3. ———, *The distribution of the zeros of the Riemann zeta function and Baker's theorem*, In Number theory. Diophantine, computational and algebraic aspects (Gyoery, Kalman et al., eds.), Proceedings of the international conference, Eger, Hungary, July 29-August 2, 1996.

4. Heath-Brown D. R., *The differences between consecutive primes III*, J. London Math. Soc. **II**(Ser. 20) (1979), 177–178.

5. Rademacher H., *Fourier analysis in number theory*, In Collected Papers of H. Rademacher, Vol. **II** (1974), 434–459.

6. Selberg A., *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **B47**(6) (1943), 1–19.

J.-C. Puchta, Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstr. 1, 79104 Freiburg, Germany