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TOPOLOGICAL TRANSITIVITY AND STRONG TRANSITIVITY

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Abstract. We discuss the relation between (topological) transitivity and strong transitivity of dynamical systems. We show that a transitive and open self-map of a compact metric space satisfying a certain expanding condition is strongly transitive. We also prove a couple of results for interval maps; for example it is shown that a transitive piecewise monotone interval map is strongly transitive.

1. Introduction

In this paper, we investigate (topological) transitivity of dynamical systems. We discuss the relation between topological transitivity and strong transitivity for continuous maps satisfying certain expanding condition and interval maps.

We prove

Theorem 1. *Let X be a compact metric space, and $f : X \to X$ a transitive continuous map. Suppose f is an open map satisfying*

(C) *for some compatible metric d of X, there exists $c > 0$ such that $d(x, y) < c$ implies $d(x, y) \leq d(f(x), f(y))$.*

Then f is strongly transitive.

The condition (C) is weaker than positive expansiveness. (f is positively expansive if there is a constant $c > 0$ such that $x \neq y$ implies $d(f^n(x), f^n(y)) > c$ for some positive integer n.) Therefore a transitive, positively expansive, open, and continuous map is strongly transitive. This is a generalization of a known fact that a transitive subshift of finite type is strongly transitive. Note that a one-sided subshift is open if and only if it is of finite type ([**6**]). We prove Theorem 1 in Section 2.

We discuss the strong transitivity of interval maps in Section 3. Let $I = [0, 1]$. We prove

Theorem 2. *Let $f : I \to I$ be a transitive piecewise continuous interval map. Then $I - \bigcup_{k=0}^{\infty} \mathrm{int} f^k(J)$ contains at most finite points for any nonempty open interval $J \subset I$.*

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We use the notation int$J$ to refer the interior of J in the relative topology (for example, int$I = I$).

Theorem 3. *Let $f : I \to I$ be a piecewise monotone interval map. If f is transitive, then f is strongly transitive.*

Coven and Mulvey [**3**] proved that if a continuous piecewise monotone interval map is transitive, then it is strongly transitive. Theorem 3 is a generalization of their result. By Parry's results [**6**], a strongly transitive piecewise monotone map is topologically conjugate to a piecewise linear map.

2. Transitive maps satisfying expanding condition

Definition 1. Let X be a topological space, and $f : X \to X$ a continuous map. We say f is *(topologically) transitive* if for any nonempty open sets $U, V \subset X$, there exists $k > 0$ such that $U \cap f^k(V) \neq \emptyset$. We say f is *strongly transitive* if for any nonempty open set $U \subset X$, $X = \bigcup_{k=0}^{s} f^k(U)$ for some $s > 0$.

Remark 1. Let $f : X \to X$ be a continuous map.

1. Suppose that X is a compact metric space. If f is transitive, then f is surjective and moreover the set of points $x \in X$ with $\overline{\{x, f(x), \dots\}} = X$ is dense in X.
2. It is easily seen that $X = \bigcup_{k=0}^{\infty} f^k(U)$ for any nonempty open set $U \subset X$ if and only if $\bigcup_{k=0}^{\infty} f^{-k}(x)$ is dense in X for any $x \in X$.
3. Suppose X is the unit interval $I = [0, 1]$. Then f is transitive if and only if for any nonempty open sets $U, V \subset X$, there exists $k > 0$ such that $U \cap \mathrm{int} f^k(V) \neq \emptyset$.

For more detail about transitivity, see Kolyada and Snoha's survey [**5**].

Proof of Theorem 1. Suppose that $f : X \to X$ satisfies the assumption in Theorem 1. To see the strong transitivity of f, it suffices to show that $X = \bigcup_{k=0}^{\infty} f^k(U)$ for any nonempty open set $U \subset X$ in virtue of the openness of f and the compactness of X. We will prove that by contradiction. We assume that there exists a point $a \in X$ such that $S = \overline{\bigcup_{k=0}^{\infty} f^{-k}(a)} \neq X$ on account of Remark 1-(2).

By the openness of f, we have $f^{-1}(S) \subset S$, or equivalently $f(X - S) \subset X - S$p. Indeed, by $f^{-1}(\bigcup_{k=0}^{\infty} f^{-k}(a)) \subset \bigcup_{k=0}^{\infty} f^{-k}(a)$, we have $X - \bigcup_{k=0}^{\infty} f^{-k}(a) \supset \supset f(X - \bigcup_{k=0}^{\infty} f^{-k}(a)) \supset f(X - S)$. Thus $X - S = \mathrm{int}(X - \bigcup_{k=0}^{\infty} f^{-k}(a)) \supset \supset f(X - S)$.

We have $x \in X - S$ with dense orbit by Remark 1-(1). Hence $d(f^k(x), S) > 0$ for $k \geq 0$ and $\inf_{0 \leq k \leq m} d(f^k(x), S) \to 0$ as $m \to \infty$. For $i = 1, 2, \dots$, let $n(i)$ be the positive integer such that $d(f^k(x), S) > 1/i$ for $0 \leq k \leq n(i) - 1$ and $d(f^{n(i)}(x), S) \leq 1/i$. We extract a subsequence $\{n(i_j)\}$ such that $f^{n(i_j)}(x)$ converges to a point $y \in S$.

We denote by K the set of accumulation points of $\{f^{n(i_j)-1}(x)\}_{j \geq 1}$. Then $K \subset f^{-1}(y)$. For each $z \in K$, take the $c/2$-neighborhood $U_z = \{p \,|\, d(p, z) < c/2\}$. Write $V = \{p \,|\, d(p, y) < \beta/2\}$, where $\beta = \sup\{b \,|\, \{p \,|\, d(p, y) < b\} \subset \bigcap_{z \in K} f(U_z)\}$. Note that $\beta > 0$ for $\# f^{-1}(y) < \infty$ by (C).

Let $x_j \in S$ be a point such that $d(x_j, f^{n(i_j)}(x)) = d(S, f^{n(i_j)}(x))$. For sufficiently large j, $f^{n(i_j)}(x) \in V$ and $f^{n(i_j)-1}(x) \in U_{z_0}$ for some $z_0 \in K$. Then

$$\begin{aligned} d(x_j, y) &\leq d(x_j, f^{n(i_j)}(x)) + d(f^{n(i_j)}(x), y) \\ &= d(S, f^{n(i_j)}(x)) + d(f^{n(i_j)}(x), y) \quad \leq \quad 2d(y, f^{n(i_j)}(x)) \quad < \quad \beta, \end{aligned}$$

and hence $x_j \in V \subset \bigcap_{z \in K} f(U_z)$. Take $x'_j \in f^{-1}(x_j) \cap U_{z_0}$. Since x'_j and $f^{n(i_j)-1}(x)$ belong to U_{z_0}, we have $d(x'_j, f^{n(i_j)-1}(x)) < c$, and by (C)

$$d(S, f^{n(i_j)-1}(x)) \leq d(x'_j, f^{n(i_j)-1}(x)) \leq d(x_j, f^{n(i_j)}(x)) = d(S, f^{n(i_j)}(x)).$$

This contradicts the definition of $n(i)$. □

Corollary 4. *Let $f : X \to X$ be a transitive continuous map. If f is positively expansive and open, then f is strongly transitive.*

Proof. A positively expansive map of a compact metric space satisfies (C). In fact, Reddy [**7**] proved that if f is a positively expansive map of a compact metric space, then there exist a compatible metric d and constants $c > 0, \lambda > 1$ such that $d(x, y) \leq c$ implies $d(f(x), f(y)) \geq \lambda d(x, y)$. □

3. Interval maps

Coven and Mulvey [**3**] showed the following. We denote by I the unit interval $[0, 1]$.

Theorem 5. *Let $f : I \to I$ be a transitive continuous piecewise monotone interval map. Then for any nonempty open interval $J \subset I$, there exists $n > 0$ such that $f^n(J) \cup f^{n+1}(J) = I$.*

We prove a similar results for piecewise monotone interval maps with discontinuities.

Definition 2. We say $f = (f_1, f_2, \dots, f_N) : I \to I$ is a *piecewise continuous map* if there exist *discontinuities* $0 < c_1 < c_2 < \cdots < c_{N-1} < 1$ such that each $f_i : [c_{i-1}, c_i] \to I,\ i = 1, 2, \dots, N$ is continuous, where $c_0 = 0, c_N = 1$. Moreover, if each $f_i : [c_{i-1}, c_i] \to I,\ i = 1, 2, \dots, N$ is strictly monotone, then we say f is *piecewise monotone.*

For a subset $J \subset I$, we write $f(J) = \bigcup_{i=1}^N f_i(J \cap [c_{i-1}, c_i])$, and $f^{-1}(J) = = \{x \in I \mid f(\{x\}) \cap J \neq \emptyset\}$. The iteration $f^n(J)$ is inductively defined for $n \in \mathbb{Z}$. If $J = \{x\}$, we write $f^n(\{x\}) = f^n(x)$.

Remark that even if f is piecewise continuous, f^k is not always piecewise continuous for $k \geq 2$.

Definition 3. Let $f : I \to I$ be a piecewise continuous interval map. We say f is *(topologically) transitive* if for any nonempty open intervals $K, L \subset I$, there exists $n \geq 0$ such that $K \cap \mathrm{int} f^n(L) \neq \emptyset$. We say f is *strongly transitive* if for any nonempty open interval $J \subset I$, $I = \bigcup_{k=0}^s f(J)$ for some $s > 0$.

Remark 2. If a piecewise continuous map $f : I \to I$ is transitive, then there is no subinterval J such that $f|J$ is constant.

Proof of Theorem 2. Let f be a transitive piecewise continuous map. Let $J \subset I$ be a nonempty open interval. We write $U = \bigcup_{k=0}^{\infty} \mathrm{int} f^k(J)$. By transitivity, U is dense in I.

Let $\mathcal{U}$ be the set of connected components of U. We show that $\mathcal{U}$ is a finite set.

1. Let $H \subset U$ be an interval without discontinuities. We show $\mathrm{int} f(H) \subset U$. Let $x \in \mathrm{int} f(H)$. Then there exists $y \in H$ such that $f(K)$ is a neighborhood of x for any neighborhood K of y. Take $n \geq 0$ such that $y \in \mathrm{int} f^n(J)$, and $x \in \mathrm{int} f^{n+1}(J) \subset U$.
2. We denote by C the set of discontinuities. Let $A \in \mathcal{U}$. For each connected component H of $A - C$, we have $\mathrm{int} f(H) \subset U$. Since $\mathrm{int} f(H)$ is connected, it is included in some $B \in \mathcal{U}$. Therefore
$$\#\{B \in \mathcal{U} \mid \mathrm{int} f(A) \cap B \neq \emptyset\} \leq \#(A \cap C) + 1.$$
Thus
$$\mathcal{U}_0 = \{A \in \mathcal{U} \mid \#\{B \in \mathcal{U} \mid \mathrm{int} f(A) \cap B \neq \emptyset\} > 1\}$$
is a finite set. We inductively define $\mathcal{U}_1, \mathcal{U}_2, \ldots$ by
$$\mathcal{U}_{k+1} = \{B \in \mathcal{U} \mid \mathrm{int} f(A) \cap B \neq \emptyset \text{ for some } A \in \mathcal{U}_k\} - \bigcup_{i=0}^{k} \mathcal{U}_i.$$
3. We show that $\mathcal{U} = \bigcup_{i=0}^{\infty} \mathcal{U}_i$. Let $A \in \mathcal{U}$. By transitivity, $\mathrm{int} f^k(B) \cap A \neq \emptyset$ for some $k \geq 0$ and some $B \in \mathcal{U}_0$. We prove that $A \in \mathcal{U}_i$ for some i by induction with respect to k. If $k = 0$, then $A = B \in \mathcal{U}_0$. Suppose $k > 0$. Since $\mathrm{int} f^k(B) \cap A$ is open and $f^{k-1}(B)$ is a finite union of intervals, $f^{-1}(\mathrm{int} f^k(B) \cap A) \cap f^{k-1}(B)$ includes a nonempty open interval. It follows from the denseness of U that this interval intersects some $A' \in \mathcal{U}$. By the inductive assumption, $A' \in \mathcal{U}_i$ for some i. Hence $A \in \bigcup_{j=0}^{i+1} \mathcal{U}_j$.
4. By transitivity, for any $A \in \mathcal{U}_1$ there exists the minimal $k = k(A)$ such that $f^k(A) \cap B \neq \emptyset$ for some $B \in \mathcal{U}_0$. It is easily seen that $\mathcal{U}_k$ is empty for $k > n = \max_{A \in \mathcal{U}_1} k(A)$. Since $\#\mathcal{U}_i < \infty$ for every i, we have $\#\mathcal{U} < \infty$.

Now we know that U is a dense set which is a finite union of open intervals, so we obtain $\#(I - U) < \infty$. $\square$

Remark 3. If $f : I \to I$ is a transitive continuous interval map, then there exists $S \subset (0,1)$ with $\#S \leq 1$ such that for any nonempty open interval $J \subset I$, $I - (S \cup \{0,1\}) \subset \bigcup_{k=0}^{\infty} \mathrm{int} f^k(J)$.

This is an immediate consequence of Barge and Martin's results. Indeed, let $J \subset I$ be a nonempty open interval. If f^2 is transitive, then a compact set in $(0,1)$ is included in $f^k(J)$ for any large k ([**2**], Theorem 6). If f^2 is not transitive, then there exists $p \in (0,1)$ such that $f([0,p]) = [p,1], f([p,1]) = [0,p]$ and $f^2|[0,p], f^2|[p,1]$, are transitive ([**1**], Lemma 2).

For the proof of Theorem 3, we show a stronger statement. Note that a transitive piecewise monotone interval map has at most finitely many k-periodic points for each $k \geq 1$.

Theorem 6. *Let $f : I \to I$ be a piecewise continuous interval map such that $\#\{x \in I \,|\, x \in f^k(x)\} < \infty$ for any $k > 0$. If f is transitive, then f is strongly transitive.*

Definition 4. For $x \in I$, an interval of the form $(x-\epsilon, x], \epsilon > 0$ ($[x, x+\epsilon), \epsilon > 0$) is said to be a *$-$-neighborhood* (*$+$-neighborhood*) of x.

Proof of Theorem 6. Let $f : I \to I$ be a transitive piecewise continuous interval map, and $J \subset I$ a nonempty open interval. Write $U = \bigcup_{k=0}^{\infty} \mathrm{int} f^k(J)$. Suppose $\#\{x \in I \,|\, x \in f^k(x)\} < \infty$ for any $k > 0$. By Theorem 2, $S = I - U$ is finite.

Let S_- (S_+) be the set of $x \in S$ such that $f^n(J)$ includes a $-$- ($+$-) neighborhood of x for some $n \geq 0$. Set

$$S^* = \{(x, \mathfrak{t}) \in S \times \{-, +\} \,|\, x \notin S_{\mathfrak{t}}\} - \{(0, -), (1, +)\}.$$

If $S^* = \emptyset$, then f is strongly transitive.

For $(x, \mathfrak{t}) \in I \times \{-, +\} - \{(0, -), (1, +)\}$, set

$$F^n(x, \mathfrak{t}) = \bigcap_{K:\mathfrak{t}\text{-neighborhood of } x} \overline{f^{-n}(K - \{x\})}.$$

For $(y, \mathfrak{u}) \in I \times \{-, +\} - \{(0, -), (1, +)\}$, let $f^*(y, \mathfrak{u})$ be the set of $(x, \mathfrak{t}) \in f(y) \times$ $\times \{-, +\}$ such that $f(K)$ includes a $\mathfrak{t}$-neighborhood of x for any $\mathfrak{u}$-neighborhood K of y. Then $f^{*n}(y, \mathfrak{u}) \subset f^n(y) \times \{-, +\}$ is inductively defined for $n \in \mathbb{N}$, and we write $f^{*-n}(x, \mathfrak{t}) = \{(y, \mathfrak{u}) \,|\, (x, \mathfrak{t}) \in f^{*n}(y, \mathfrak{u})\}$.

We show that $f^*(y, \mathfrak{u})$ is a singleton in S^* for any $(y, \mathfrak{u}) \in S^*$ and that $f^{*-1}(x, \mathfrak{t})$ is a singleton in S^* for any $(x, \mathfrak{t}) \in S^*$.

1. If $(x, \mathfrak{t}) \in f^*(y, \mathfrak{u})$ and $(x, \mathfrak{t}) \in S^*$, then $(y, \mathfrak{u}) \in S^*$. Indeed, otherwise, $f^m(J)$ includes a $\mathfrak{u}$-neighborhood of y for some $m \geq 0$, so that $f^{m+1}(J)$ includes a $\mathfrak{t}$-neighborhood of x.
2. Let $(x, \mathfrak{t}) \in S^*$. Then for $y \in F^n(x, \mathfrak{t}), n \geq 1$, there exists $\mathfrak{u} \in \{-, +\}$ such that $(y, \mathfrak{u}) \in S^* \cap f^{*-n}(x, \mathfrak{t})$. To this end, let $n = 1$. Take a sequence $x_1, x_2, \ldots$ in a $\mathfrak{t}$-neighborhood of x which converges to x and $y_i \in f^{-1}(x_i)$ which converges to y. Then $x \in f(y)$. Let $\mathfrak{u} \in \{-, +\}$ be such that any $\mathfrak{u}$-neighborhood K of y satisfies $\#(K \cap \{y_i\}) = \infty$. By (1), $(y, \mathfrak{u}) \in S^*$. Thus the case $n = 1$ is true, and the general cases are proved by induction.
3. If $(y, \mathfrak{u}) \in S^*$, then $\#(f^*(y, \mathfrak{u}) \cap S^*) \leq 1$. For the proof, suppose that distinct $(x_i, \mathfrak{t}_i) \in S^*, i = 1, 2$ are contained in $f^*(y, \mathfrak{u})$. Then $x_1 = x_2 = x$ and $\{\mathfrak{t}_1, \mathfrak{t}_2\} =$ $= \{-, +\}$. It follows that $f(K)$ includes a neighborhood of x for any $\mathfrak{u}$-neighborhood K of y. Then $\#(K \cap F^1(x, \mathfrak{t})) = \infty$. That contradicts the finiteness of S^*.
4. The facts (2) and (3) proved above together with the finiteness of S^* imply the existence and the uniqueness of $(x, \mathfrak{t}) \in f^*(y, \mathfrak{u}) \cap S^*$ for any $(y, \mathfrak{u}) \in S^*$. It is also seen that $f^{*-1}(x, \mathfrak{t})$ is a singleton in S^* for any $(x, \mathfrak{t}) \in S^*$. Now it is clear that $f^*(y, \mathfrak{u}) \subset S^*$ for $(y, \mathfrak{u}) \in S^*$

Thus the mapping $f^* : S^* \to S^*$ is well-defined and S^* consists of periodic points of f^*. Suppose $S^* \neq \emptyset$. Let $(x, \mathfrak{t}) \in S^*$ with period p. Without loss of generality, we assume $\mathfrak{t} = +$. Let $\epsilon_0 > 0$ be a positive number such that $(x, x + \epsilon_0)$ contains no point of $\{x \,|\, x \in f^p(x)\}$ and no discontinuity. Moreover, there exists $0 < \epsilon_1' \leq \epsilon_0$ such that $f((x, x+\epsilon_1'))$ contains no discontinuity. Indeed, if $f^{-1}(c) \cap [x, x+\epsilon_0]$ accumulates at x for some discontinuity c, then $f^*(x,+) = (c, \mathfrak{s})$ and $\#([x, x+\epsilon_0] \cap F^1(c, \mathfrak{s})) = \infty$, which contradicts the finiteness of S^*. Then we can take $0 < \epsilon_1 \leq \epsilon_0$ by induction such that $f^i((x, x+\epsilon_1))$ contains no discontinuity for $i = 0, 1, \ldots, p-1$. Hence $f^i|(x, x+\epsilon_1)$ is considered as a continuous map for $i = 1, 2, \ldots, p$.

Since $(x, x+\epsilon_1)$ has no p-periodic point, by transitivity $f^p(z) > z$ whenever $z \in (x, x+\epsilon_1)$. There exists $0 < \epsilon \leq \epsilon_1$ such that $f^{-p}((x, x+\epsilon)) \subset (x, x+\epsilon)$. Indeed, otherwise, we obtain an element of $f^{*-p}(x,+)$ other than $(x,+)$ by using the arguments of (2), which contradicts the fact that $f^{*-p}(x,+)$ is a singleton.

Let $J' = (a, b)$ be an open interval with $x < a < b < x+\epsilon$. There exists the minimal $k > 0$ such that $(x, a) \cap f^k(J') \neq \emptyset$. Since $f^{-p}((x, a)) \subset (x, a)$, that is a contradiction. Thus S^* is empty, and the proof is completed. □

The following result was announced beforehand in [**4**], Remark 3.18. An piecewise monotone interval map with codability (i.e. the existence of a semiconjugacy from some subshift) and non-recurrence can be decomposed into finite strongly transitive components. Moreover, such a map is topologically conjugate to an expanding piecewise linear map. Actually, by [**4**], Theorem 2.4, a piecewise monotone map $f : I \to I$ satisfying the assumption of Corollary 7 is expanding; by Parry's result [**6**], a strongly transitive piecewise monotone map is topologically conjugate to a piecewise linear map.

Corollary 7. *Let $f : I \to I$ be a piecewise monotone map with discontinuities $0 = c_0 < c_1 < c_2 < \cdots < c_{N-1} < c_N = 1$. Write $I_i = [c_{i-1}, c_i]$. Suppose f satisfies the following:*

1. *(codability) For any nonempty open interval K in I_i, there exists $n > 0$ such that $\#\{j \mid I_j \cap \mathrm{int} f^n(K) \neq \emptyset\} \geq 2$.*
2. *(non-recurrence) Any discontinuity $c \in C$ is not an accumulation point of $\bigcup_{n=0}^{\infty} f^n(C)$, where C is the set of discontinuities.*

Then there exist open sets $J_1, J_2, \ldots, J_s$ in I which are mutually disjoint and are finite unions of open intervals such that (i) $f(J_i) \subset \overline{J_i}$, (ii) $f|J_i$ is extended to a strongly transitive piecewise monotone map $f' : \overline{J_i} \to \overline{J_i}$, and (iii) for any nonempty open interval $K \subset I$, $\bigcup_{k=0}^{n} f^k(K)$ includes J_i for some $n > 0$ and some i.

Proof. Let A_i^+ (A_i^-) be a $+$-neighborhood ($-$-neighborhood) of c_i such that $(A_i^{\pm} - \{c_i\}) \cap \bigcup_{n=0}^{\infty} f^n(C) = \emptyset$ for $i = 1, 2, \ldots, N-1$. By a method similar to the argument to prove the finiteness of $\mathcal{U}$ in Theorem 2, we can see that $J_i^{\mathfrak{t}} = \mathrm{int}\overline{\bigcup_{k \geq 0} \mathrm{int} f^k(A_i^{\mathfrak{t}})}$ is a finite union of open intervals. It is evident that $f(J_i^{\mathfrak{t}}) \subset \overline{J_i^{\mathfrak{t}}}$.

Let $K \subset I$ be a nonempty open interval. It is easily seen that there exist $n > 0$, i, and $\mathfrak{t}$ such that $f^n(K) \supset A_i^{\mathfrak{t}}$. Thus $\bigcup_{k=0}^{\infty} f^k(K) \supset J_i^{\mathfrak{t}}$. Hence if $J_{i_1}^{\mathfrak{t}_1} \cap J_{i_2}^{\mathfrak{t}_2} \neq \emptyset$, then there exist i and $\mathfrak{t}$ such that $J_i^{\mathfrak{t}} \subset J_{i_1}^{\mathfrak{t}_1} \cap J_{i_2}^{\mathfrak{t}_2}$. Let $J_1, J_2, \ldots, J_s$ be the minimal $J_i^{\mathfrak{t}}$'s, that is, the $J_i^{\mathfrak{t}}$'s such that there does not exist $J_{i'}^{\mathfrak{t}'}$ properly included in $J_i^{\mathfrak{t}}$.

Clearly, every extended piecewise monotone map $f' : \overline{J_i} \to \overline{J_i}$ is transitive. By Theorem 3, each f' is strongly transitive. Thus for any nonempty open interval $K \subset I$, $\bigcup_{k=0}^{n} f^k(K)$ includes J_i for some $n > 0$ and some i. □

Barge and Martin constructed a transitive continuous interval map which is not strongly transitive (see [**2**], Example 3). Modifying this example, we can construct a piecewise continuous interval map $f : I \to I$ which is not strongly transitive but satisfies $\bigcup_{k=1}^{\infty} f^k(J) = I$ for any nonempty open interval $J \subset I$. The detail is left to the reader.

For continuous maps, we have a favorable result. In fact, we prove the following.

Proposition 8. *Let $f : I \to I$ be a continuous interval map such that $\bigcup_{k=0}^{\infty} f^k(J) = I$ for any nonempty open interval $J \subset I$. Then for any nonempty open interval $J \subset I$, there exists $n > 0$ such that $f^n(J) \cup f^{n+1}(J) = I$.*

Proof. Let $J \subset I$ be a nonempty open interval. Let $p \in I$ be a fixed point of f. Then there exists $n_0 > 0$ such that $p \in f^{n_0}(J)$. The fixed point p is contained in $f^n(J)$ whenever $n \geq n_0$.

Take $q \in f^{-1}(0)$. Suppose $0 \leq q \leq p$. Let $n_1 \geq n_0$ be the positive number with $0 \in f^{n_1}(J)$. Then $[0, p] \subset f^n(J)$ for $n \geq n_1$. Take $n \geq n_1$ such that $1 \in f^n(J)$, and $f^n(J) = I$. If $p < q \leq 1$, then take $n \geq n_0$ such that $1 \in f^n(J)$, and $f^n(J) \cup f^{n+1}(J) = I$. □

We have seen that if $f : I \to I$ is strongly transitive continuous interval map, then

$$\inf\{\#N \mid N \subset \mathbb{N}, \bigcup_{n \in N} f^n(J) = I\} \leq 2$$

for any nonempty open interval $J \subset I$. In the case where f is not continuous, however, the number is not necessarily bounded. Indeed, consider $f = (f_1, f_2)$ with $f_1(x) = x + t\,(0 \leq x \leq 1 - t)$, $f_2(x) = x + t - 1\,(1 - t \leq x \leq 1)$ for an irrational number $t \in (0, 1)$.

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