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NEARLY CONTINUOUS MULTIFUNCTIONS

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Abstract. The aim of this paper is to present and study the notion of nearly continuous multifunctions. The concept of nearly continuous multifunction is defined and basic characterizations and basic properties of nearly continuous multifunctions are obtained.

1. Introduction

Strong and weak forms of continuity has been recently of major interest among general topologists. The aim of this paper is to introduce and study the notion of nearly continuous multifunctions.

A topological space (X, τ) is called nearly compact [**8**] if every cover of X by regular open sets has a finite subcover.

Let (X, τ) be a topological space and let A be a subset of X. If every cover of A by regular open subsets of (X, τ) has a finite subfamily whose union covers A, then A is called N-closed (relative to X) [**4**]. Sometimes, such sets are called N-sets or α-nearly compact. The class of N-closed sets is important in the study of functions with strongly closed graphs [**5**].

Basic observation about N-closed sets involves the fact that every compact set is N-closed.

In this paper, a multifunction $F : X \to Y$ from a topological space X to a topological space Y is a point to set correspondence and is assumed that $F(x) \neq \emptyset$ for all $x \in X$, where $\emptyset$ denotes the empty set.

If A is a subset of a topological space, then $\mathrm{cl}(A)$ denote the closure of A and $\mathrm{int}(A)$ denote the interior of A and $\mathrm{co}(A)$ denote the complement of A relative to the topological space.

For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), we denote the upper and lower inverse of a subset V of Y by $F^+(V)$ and $F^-(V)$, respectively; $F^+(V) = \{x \in X : F(x) \subseteq V\}$ and $F^-(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ [**1**]. If $U \subseteq X$, we denote $F(U) := \bigcup\{F(x) : x \in U\}$ the image of the set U.

A multifunction $F : X \to Y$ is said to be (i) upper semi continuous at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(U) \subseteq V$; and (ii) lower semi continuous at a point $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(a) \cap V \neq \emptyset$ for every $a \in U$ [**7**].

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For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ [**9**].

The graph multifunction G_F of the multifunction $F : X \to Y$ is said to be strongly closed if for each $(x, y) \notin G_F(x)$, there exists open sets U and V containing x and containing y respectively such that $(U \times \mathrm{cl}(V)) \cap G_F(x) = \emptyset$ [**2**].

For a multifunction $F : X \to Y$, $G_F^+(A \times B) = A \cap F^+(B)$ and $G_F^-(A \times B) = A \cap F^-(B)$ where $A \subseteq X$ and $B \subseteq Y$ [**6**].

2. Nearly continuous multifunctions

Definition 1. Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then it is said that F is lower (upper) nearly continuous if for each $x \in X$ and for each open set V having N-closed complement such that $x \in F^-(V)$ ($x \in F^+(V)$), there exists an open set U containing x such that $U \subseteq F^-(V)$ ($U \subseteq F^+(V)$).

The following theorem give us some characterizations of lower (upper) nearly continuous multifunction.

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subseteq X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Theorem 2. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent.*

(i) *F is lower (upper) nearly continuous multifunction,*

(ii) *For each $x \in X$ and for each open set V having N-closed complement such that $F(x) \cap V \neq \emptyset$ ($F(x) \subseteq V$), there exists an open set U containing x such that if $y \in U$, then $F(y) \cap V \neq \emptyset$ ($F(y) \subseteq V$),*

(iii) *$F^-(V)$ ($F^+(V)$) is an open set for any open set $V \subseteq Y$ having N-closed complement,*

(iv) *$F^+(\mathrm{co}(V))$ ($F^-(\mathrm{co}(V))$) is a closed set for any open set $V \subseteq Y$ having N-closed complement,*

(v) *For each $x \in X$ and for each net (x_α) which converges to x in X and for each open set $V \subseteq Y$ having N-closed complement such that $x \in F^-(V)$ ($x \in F^+(V)$), the net (x_α) is eventually in $F^-(V)$ ($F^+(V)$).*

Proof. (i)$\Leftrightarrow$(ii). Clear.

(i)$\Leftrightarrow$(iii). Let $x \in F^-(V)$ and let V be an open set having N-closed complement. From (i), there exists an open set U containing x such that $U \subseteq F^-(V)$. It follows that $x \in \mathrm{int}(F^-(V))$ and (iii) holds.

The converse can be shown easily.

(iii)$\Leftrightarrow$(iv). Since $F^-(\mathrm{co}(V)) = \mathrm{co}(F^+(V))$ and $F^+(\mathrm{co}(V)) = \mathrm{co}(F^-(V))$, the proof is clear.

(i)$\Rightarrow$(v). Let (x_α) be a net which converges to x in X and let $V \subseteq Y$ be any open set having N-closed complement such that $x \in F^-(V)$. When F is lower nearly continuous multifuction, it follows that there exists an open set $U \subseteq X$ containing x such that $U \subseteq F^-(V)$. Since (x_α) converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subseteq F^-(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^-(V)$.

(v)$\Rightarrow$(i). Suppose that (i) is not true. There exists a point x and an open set V having N-closed complement with $x \in F^-(V)$ such that $U \nsubseteq F^-(V)$ for each open set $U \subseteq X$ containing x. Let $x_U \in U$ and $x_U \notin F^-(V)$ for each open set $U \subseteq X$ containing x. Then for the neighborhood net (x_U), $x_U \to x$, but (x_U) is not eventually in $F^-(V)$. This is a contradiction. Thus, F is lower nearly continuous multifunction.

The proof of the upper nearly continuity of F is similar to the above. □

Remark 3. For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), it is obtained the following implication:

F is lower (upper) semi continuous multifunction $\Rightarrow$ F is lower (upper) nearly continuous multifunction.

However the converse are not true in general by the following examples.

Example 4. Define τ_{FCT} to be the collection consisting of $\emptyset$ together with all subsets of $\mathbb{R}$ whose complements in $\mathbb{R}$ are finite. Then, it is known that τ_{FCT} is a topology on $\mathbb{R}$, called the finite complement topology.

Take the discrete topology τ_D on $\mathbb{R}$. We define the multifunction as follows; $F : (\mathbb{R}, \tau_{FCT}) \to (\mathbb{R}, \tau_D)$, $F(x) = \{x\}$ for each $x \in X$. Then F is lower (upper) nearly continuous multifunction, but F is not lower (upper) semi continuous.

The following theorem give us the condition for the converse.

Theorem 5. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Suppose that Y has a base of neighbourhoods such that complement of each set of the base of neighbourhoods is a closed N-closed set. If F is lower nearly continuous multifunction, then F is lower semi continuous.*

Proof. Let $x \in X$ and let V be any open set such that $x \in F^-(V)$. Since Y has a base of neighbourhoods such that complement of each set of the base of neighbourhoods is a closed N-closed set, then we have $V = \bigcup_{i \in I} G_i$ where complement of G_i is a closed N-closed set for $i \in I$. Since $x \in F^-(\bigcup_{i \in I} G_i)$, then there exists $i_0 \in I$ such that $x \in F^-(G_{i_0})$. Since F is lower nearly continuous multifunction, it follows that there exists an open set K containing x such that $K \subseteq F^-(G_{i_0})$ and hence $K \subseteq F^-(V)$. Thus, we obtain that F is lower semi continuous multifunction. □

Suppose that (X, τ), (Y, υ) and (Z, ω) are topological spaces. If $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the composite multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 6. *Let (X, τ), (Y, υ), (Z, ω) be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is upper (lower) semi continuous multifunction and $G : Y \to Z$ is upper (lower) nearly continuous multifunction, then $G \circ F : X \to Z$ is upper (lower) nearly continuous multifunction.*

Proof. Let $V \subseteq Z$ be any open set having N-closed complement. From the definition of $G \circ F$, we have $(G \circ F)^{+}(V) = F^{+}(G^{+}(V))$ $((G \circ F)^{-}(V) = F^{-}(G^{-}(V)))$. Since G is upper (lower) nearly continuous multifunction, it follows that $G^{+}(V)$ $(G^{-}(V))$ is an open set. Since F is upper (lower) semi continuous multifunction, it follows that $F^{+}(G^{+}(V))$ $(F^{-}(G^{-}(V)))$ is an open set. It shows that $G \circ F$ is upper (lower) nearly continuous multifunction. □

Theorem 7. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ) and let $U \subseteq X$ be a nonempty subset. If F is lower (upper) nearly continuous multifunction, then the restriction multifunction $F|_U : U \to Y$ is lower (upper) nearly continuous multifunction.*

Proof. Suppose that $V \subseteq Y$ is an open set having N-closed complement. Let $x \in U$ and let $x \in (F|_U)^{-}(V)$. Since F is lower nearly continuous multifunction, it follows that there exists an open set G such that $x \in G \subseteq F^{-}(V)$. From here we obtain that $x \in G \cap U$ and $G \cap U \subseteq (F|_U)^{-}(V)$. Thus, we show that the restriction multifunction $F|_U$ is a lower nearly continuous.

The proof of the upper nearly continuity of $F|_U$ is similar to the above. □

Theorem 8. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Let $\{H_\alpha : \alpha \in \Phi\}$ be an open cover of X. If the restriction multifunction $F_\alpha = F|_{H_\alpha}$ is a lower (upper) nearly continuous multifunction for each $\alpha \in \Phi$, then F is a lower (upper) nearly continuous.*

Proof. Let $V \subseteq Y$ be an open set having N-closed complement. Since F_α is lower nearly continuous for each α, from Theorem 2, $F_\alpha^{-}(V) \subseteq \mathrm{int}_{H_\alpha}(F_\alpha^{-}(V))$ and since H_α is open, we have $F^{-}(V) \cap H_\alpha \subseteq \mathrm{int}_{H_\alpha}(F^{-1}(V) \cap H_\alpha)$ and $F^{-}(V) \cap H_\alpha \subseteq \mathrm{int}(F^{-1}(V)) \cap H_\alpha$. Since $\{H_\alpha : \alpha \in \Phi\}$ is an open cover of X, it follows that $F^{-}(V) \subseteq \mathrm{int}(F^{-}(V))$. Hence, from Theorem 2, we obtain that F is a lower nearly continuous multifunction.

The proof of the upper nearly continuity of F is similar to the above. □

Theorem 9. *Let $A_i \subseteq X_i$ be nonempty for each $i \in I$. Set $A = \prod_{i \in I} A_i$ and $X = \prod_{i \in I} X_i$. Then A is N-closed relative to X if and only if each A_i is N-closed relative to X_i* [**3**].

Theorem 10. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ) and let X be a nearly companct space. If the graph function of F is lower (upper) nearly continuous multifunction, then F is lower (upper) nearly continuous multifunction.*

Proof. Let $x \in X$ and let $V \subseteq Y$ be an open set having N-closed complement such that $x \in F^{-}(V)$. We obtain that $x \in G_F^{-}(X \times V)$ and that $X \times V$ is an open set having N-closed complement relative to $X \times Y$. Since graph multifunction G_F is lower nearly continuous, it follows that there exists an open set $U \subseteq X$ containing x such that $U \subseteq G_F^{-}(X \times V)$. Since $U \subseteq G_F^{-}(X \times V) = X \cap F^{-}(V) = F^{-}(V)$, we obtain that $U \subseteq F^{-}(V)$. Thus, F is lower nearly continuous multifunction.

The proof of the upper nearly continuity of F is similar to the above. □

Theorem 11. *Suppose that (X,τ) and (X_α,τ_α) are topological spaces and (X_α,τ_α) is nearly compact for each $\alpha \in J$. Let $F : X \to \prod_{\alpha\in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha\in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If F is upper (lower) nearly continuous multifunction, then $P_\alpha \circ F$ is upper (lower) nearly continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be an open set having N-closed complement in $(X_{\alpha_0},\tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P^+_{\alpha_0}(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ (respectively, $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P^-_{\alpha_0}(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$). Since F is upper (lower) nearly continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha$ is an open set having N-closed complement to $\prod_{\alpha\in J} X_\alpha$, it follows that $F^+(V_{\alpha_0}\times \prod_{\alpha\neq\alpha_0} X_\alpha)$ (respectively, $F^-(V_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$) is open in (X,τ). Therefore, $P_{\alpha_0} \circ F$ is upper (lower) nearly continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is upper (lower) nearly continuous multifunction for each $\alpha \in J$. □

Theorem 12. *Suppose that for each $\alpha \in J$, (X_α,τ_α), $(Y_\alpha,\upsilon_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha\in J} X_\alpha \to \prod_{\alpha\in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha\in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha\in J} X_\alpha$ to the product space $\prod_{\alpha\in J} Y_\alpha$. If F is upper (lower) nearly continuous multifunction and Y_α is nearly compact for each $\alpha \in J$, then each F_α is upper (lower) nearly continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subseteq Y_\alpha$ be an open set having N-closed complement. Then $V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta$ is an open set having N-closed complement relative to $\prod_{\alpha\in J} Y_\alpha$. Since F is upper (lower) nearly continuous multifunction, it follows that
$F^+(V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta) = F^+_\alpha(V_\alpha) \times \prod_{\alpha\neq\beta} X_\beta$ $(F^-(V_\alpha \times \prod_{\alpha\neq\beta} Y_\beta) = F^-_\alpha(V_\alpha) \times \prod_{\alpha\neq\beta} X_\beta)$
is an open set. Consequently we obtain that $F^+_\alpha(V_\alpha)$ $(F^-_\alpha(V_\alpha))$ is an open set. Thus, we show that F_α is upper (lower) nearly continuous multifunction. □

Theorem 13. *Suppose that (X_1,τ_1), (X_2,τ_2), (Y_1,υ_1) and (Y_2,υ_2) are topological spaces and $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are multifunctions. Let $F_1\times F_2 : X_1\times X_2 \to Y_1\times Y_2$ be a multifunction which is defined by $(F_1\times F_2)(x,y) = F_1(x)\times F_2(y)$ for each $(x,y) \in X_1 \times X_2$. If $F_1 \times F_2$ is upper (lower) nearly continuous multifunction, then F_1 and F_2 are upper (lower) nearly continuous multifunctions.*

Proof. Let $K \subseteq Y_1$, $H \subseteq Y_2$ be open sets having N-closed complements. It is known that $K \times H$ is an open set having N-closed complements relative to $Y_1\times Y_2$ and $(F_1\times F_2)^+(K\times H) = F^+_1(K)\times F^+_2(H)$. Since $F_1\times F_2$ is upper nearly continuous multifunction, it follows that $F^+_1(K) \times F^+_2(H)$ is open set. From here,

$F_1^+(K)$ and $F_2^+(H)$ are open sets. Hence, it is obtained that F_1 and F_2 are upper nearly continuous multifunctions.

The proof of the lower nearly continuity of F_1 and F_2 is similar to the above. □

Theorem 14. *Suppose that (X,τ), (Y,υ), (Z,ω) are topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) nearly continuous multifunction, then F_1 and F_2 are upper (lower) nearly continuous multifunctions.*

Proof. Let $x \in X$ and let $K \subseteq Y$, $H \subseteq Z$ be open sets having N-closed complements such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subseteq K$ and $F_2(x) \subseteq H$ and consequently, $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subseteq K \times H$ that means $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper nearly continuous multifunction, it follows that there exists an open set U containing x such that $U \subseteq (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subseteq F_1^+(K)$ and $U \subseteq F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper nearly continuous multifunctions.

The proof of the lower nearly continuity of F_1 and F_2 is similar to the above. □

Theorem 15. *Let $F : X \to Y$ be upper nearly continuous multifunction. If every two distinct points of Y are contained in disjoint open sets such that one of them may be chosen to have N-closed complement, then the graph multifunction G_F is strongly closed.*

Proof. Suppose that $(x,y) \notin G_F$. Then $y \notin F(x)$. Due to the assumption, there exist disjoint open sets H and K such that $F(x) \subset H$, $y \in K$ and H has N-closed complement. Since F is upper nearly continuous multifunction, it follows that $U = F^+(H)$ is an open set containing x and $F(U) \subset H \subset Y \backslash K$. We have $x \in U$ and $y \in K$. Consequently, $U \times cl(K)$ does not contain any points of G_F. Thus, we proved that G_F is strongly closed. □

Definition 16. Let (X,τ) be a topological space. X is said to be a N-normal space if for every disjoint closed subsets K and F of X, there exists two open sets U and V having N-closed complements such that $K \subseteq U$, $F \subseteq V$ and $U \cap V = \emptyset$.

Example 17. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then (X,τ) is a N-normal space.

Theorem 18. *Let F and G be upper nearly continuous and point closed multifunctions from a topological space X to a N-normal topological space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is closed in X.*

Proof. Let $x \in X\backslash K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are point closed multifunctions and Y is a N-normal space, it follows that there exists disjoint open sets U and V having N-closed complements containing $F(x)$ and $G(x)$ respectively. Since F and G are upper nearly continuous, then the sets $F^+(U)$ and $G^+(V)$ are open and contain x. Let $H = F^+(U) \cup G^+(V)$. Then H is an open set containing x and $H \cap K = \emptyset$. Hence, K is closed in X. □

Theorem 19. *Let $F : X \to Y$ be an upper nearly continuous multifunction and point closed from a topological space X to a N-normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a Hausdorff space.*

Proof. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a N-normal space, it follows that there exists disjoints open sets U and V having N-closed complement containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint open sets containing x and y respectively. Thus, it is obtained that X is Hausdorff. □

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