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ON SUBGROUPOID LATTICES OF SOME FINITE GROUPOID

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Abstract. We investigate finite commutative groupoids $\mathcal{G} = \langle G, \circ \rangle$ such that $g \circ h \neq g$ for all elements g, h of $\mathcal{G}$. First, we show that for any such groupoid, its weak (i.e. partial) subgroupoid lattice uniquely determines its subgroupoid lattice. Next, we characterize the lattice of all weak subgroupoids of such a groupoid. This is a distributive finite lattice satisfying some combinatorial conditions concerning its atoms and join–irreducible elements.

In [**5**] we proved that for any (total) locally finite unary algebra of finite type (i.e. with finitely many unary operations), its weak subalgebra lattice uniquely determines its strong subalgebra lattice. Here we generalize this result for some finite commutative groupoids. Next, in the second part of this paper, necessary and sufficient conditions are found for a lattice to be isomorphic to the weak subgroupoid lattice of such a groupoid. The classical subgroupoids are sometimes called strong as opposed to the other kind of partial subgroupoids, called weak, considered in this paper. Recall that a partial groupoid $\mathcal{H}$ is a weak subgroupoid of a (partial) groupoid $\mathcal{G}$ iff the carrier of $\mathcal{H}$ is contained in the carrier of $\mathcal{G}$, and for any elements g, h of $\mathcal{H}$, if the product $g \circ h$ is defined in $\mathcal{H}$, then this is also define in $\mathcal{G}$ and these two products are equal. The lattices of all weak and strong subgroupoids of a groupoid $\mathcal{G}$ are denoted by $\mathcal{S}_w(\mathcal{G})$ and $\mathcal{S}_s(\mathcal{G})$, respectively. (More details on various kinds of partial subalgebras and lattices of such subalgebras can be found e.g. in [**3**] or [**4**]; see also [**6**]).

Theorem 1. *Let $\mathcal{G} = \langle G, \circ \rangle$ be a (total) finite and commutative groupoid such that*

$$(*) \qquad g \circ h \neq g \quad \text{for each} \quad g, h \in G.$$

Let $\mathcal{H} = \langle H, \circ \rangle$ be a partial commutative groupoid such that

$$\mathcal{S}_w(\mathcal{H}) \simeq \mathcal{S}_w(\mathcal{G}).$$

Then the strong subgroupoid lattice $\mathcal{S}_s(\mathcal{H})$ is isomorphic to the strong subgroupoid lattice $\mathcal{S}_s(\mathcal{G})$, and moreover, $\mathcal{H}$ is finite, total and satisfies $(*)$.

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Before the proof observe that for any two elements h_1 and h_2 of a partial commutative groupoid, if $h_1 \circ h_2$ is defined, then $h_2 \circ h_1$ can be also defined (if this is not), and it may be done in the exactly one way. More formally, we assume that the equation $x \circ y \approx y \circ x$ is strongly valid in $\mathcal{H}$, i.e. if one side is defined, then the other also is, and they are equal. For more information on various kinds of partial equations see e.g. [**3**] or [**4**].

Note also that the assumption $(*)$ plays an important role in our proof of this result, but, at this moment, the author do not know any counterexample which would show that Theorem 1 is false without this condition.

To prove the above result we apply hypergraph–algebraic language introduced in [**6**]. Recall that in this paper we defined the directed hypergraph $\mathbf{D}(\mathcal{G})$ representing a groupoid (or more general, any algebra) $\mathcal{G}$. More precisely, the carrier G of $\mathcal{G}$ is the set of vertices, and directed hyperedges are formed by all directed triples $e = \langle g, h, i\rangle$ such that $g \circ h$ is defined and equal to i. Then $\{g, h\}$ is said to be the initial set of e (denoted by $I_1^{\mathbf{D}(\mathcal{G})}(e)$), and i is said to be the final vertex of e (denoted by $I_2^{\mathbf{D}(\mathcal{G})}(e)$). Obviously $\mathbf{D}(\mathcal{G})$ contains only edges and 2–edges (i.e. hyperedges with one– and two–element initial sets). Observe also that if $g \neq h$ and $g \circ h$ and $h \circ g$ are defined, then we have two different 2–edges starting from $\{g, h\}$. But we consider only partial commutative groupoids (i.e. $g \circ h = h \circ g$, and both sides are defined or not). Thus such two 2–edges have the same initial set and the same final vertex. Hence, between any two–vertex set and a vertex, there are none or exactly two 2–edges.

It is easy to see that in $\mathbf{D}(\mathcal{G})$, at most one edge starts from any vertex, and at most two 2–edges start from any two–element set of vertices. Hence, $\mathcal{G}$ is total iff exactly one edge starts (in $\mathbf{D}(\mathcal{G})$) from any vertex, and exactly two 2–edges start from any two–vertex set.

An edge or 2–edge e is said to be regular iff the final vertex of e does not belong to the initial set of e. Otherwise e is a loop or a 2–loop, respectively. It is easily to shown that $\mathcal{G}$ satisfies $(*)$ iff $\mathbf{D}(\mathcal{G})$ has not loops and 2–loops.

Proof. Let $\mathbf{D} = \mathbf{D}(\mathcal{G})$ and $\mathbf{K} = \mathbf{D}(\mathcal{H})$. It is proved in [**6**, Corollary 3.14]

$$\mathcal{S}_w(\mathcal{G}) \simeq \mathcal{S}_w(\mathcal{H}) \quad \Longleftrightarrow \quad \mathbf{D}^* \simeq \mathbf{K}^*,$$

where $\mathbf{D}^*$ and $\mathbf{K}^*$ are (undirected) hypergraphs obtained from $\mathbf{D}$ and $\mathbf{K}$, respectively, by omitting the orientation of all hyperedges (but not hyperedges themselves). More formally, for any (undirected) hyperedge e of $\mathbf{D}^*$, its set of endpoints consists of the initial set $I_1^{\mathbf{D}}(e)$ of e in $\mathbf{D}$ and the final vertex $I_2^{\mathbf{D}}(e)$ of e in $\mathbf{D}$, i.e. $I^{\mathbf{D}^*}(e) = I_1^{\mathbf{D}}(e) \cup \{I_2^{\mathbf{D}}(e)\}$.

First, $\mathbf{D}$ and $\mathbf{K}$ have the same number of vertices. In particular, $\mathcal{H}$ is a finite groupoid. Secondly, since $\mathcal{G}$ is total, $\mathbf{D}$ contains exactly $2 \cdot \frac{M\cdot(M-1)}{2} = M \cdot (M-1)$ 2–edges, and exactly M edges, where M is the number of all vertices of $\mathbf{D}$. Analogously, $\mathbf{K}$ contains at most $M \cdot (M-1)$ 2–edges, and at most M edges. Thirdly, by $(*)$, $\mathbf{D}$ has not loops and 2–loops.

Let $\mathbf{D}_1$ and $\mathbf{K}_1$ be directed hypergraphs obtained from $\mathbf{D}$ and $\mathbf{K}$, respectively, by omitting all regular 2–edges. Let $\mathbf{D}_2$ and $\mathbf{K}_2$ be directed hypergraphs consisting of all vertices and all regular 2–edges of $\mathbf{D}$ and $\mathbf{K}$, respectively. $\mathbf{D}_2$ contains all 2–edges of $\mathbf{D}$, so we first obtain $\mathbf{D}_2$ has exactly $M \cdot (M-1)$ hyperedges. Secondly, $\mathbf{D}_1$ contains all edges of $\mathbf{D}$, so $\mathbf{D}_1$ is the usual directed graph. Moreover, $\mathbf{D}_1$ is a finite total functional directed graph without loops.

The image of a regular 2–edge in $\mathbf{D}^*$ (or $\mathbf{K}^*$) has exactly three endpoints. Conversely, since $\mathbf{D}$ and $\mathbf{K}$ have only edges and 2–edges, each (undirected) hyperedge of $\mathbf{D}^*$ or $\mathbf{K}^*$ with three endpoints is the image of some regular 2–edge. Hence, because $\mathbf{D}^* \simeq \mathbf{K}^*$,

$$\mathbf{D}_1^* \simeq \mathbf{K}_1^* \quad \text{and} \quad \mathbf{D}_2^* \simeq \mathbf{K}_2^*.$$

The second isomorphism implies that $\mathbf{K}_2$ has $M \cdot (M-1)$ 2–edges. Thus $\mathbf{K}_2$ contains all 2–edges of $\mathbf{K}$. Hence first, $\mathbf{K}$ has not 2–loops, and exactly two 2–edge start from any two–element set of vertices. Secondly, $\mathbf{K}_1$ consists of all edges of $\mathbf{K}$. This fact and the first isomorphism imply that $\mathbf{K}_1$ has exactly M edges. Thus exactly one edge starts from any vertex of $\mathbf{K}$. Further, by the same isomorphism, $\mathbf{K}_1$, thus also $\mathbf{K}$, does not contain loops (because $\mathbf{D}_1$ has not loops).

All the above facts imply that $\mathcal{H}$ is a finite total groupoid satisfying $(*)$.

Exactly two 2–edges start from each two–element set of vertices (in $\mathbf{D}$ and $\mathbf{K}$), and moreover, they have the same final vertex. Thus we can replace any such pair of 2–edges by a single 2–edge to obtain two new directed hypergraphs $\overline{\mathbf{D}}$ and $\overline{\mathbf{K}}$. Since $\mathbf{D}$ and $\mathbf{K}$ have only regular edges and 2–edges, and $\mathbf{D}^*$, $\mathbf{K}^*$ are isomorphic, we deduce

$$\overline{\mathbf{D}}^* \simeq \overline{\mathbf{K}}^*.$$

More precisely, since $\mathbf{D}$ (and $\mathbf{K}$) contains only regular hyperedges, we have that 2–edges of $\mathbf{D}$ ($\mathbf{K}$) and hyperedges of $\mathbf{D}^*$ ($\mathbf{K}^*$) with three endpoints are in the bijective correspondence, given by *. Moreover, for any three–element set W of vertices, the set F of all undirected hyperedges with W as the endpoint set consists of all (regular) 2–edges starting from W and ending in W. Hence, first, the number l of all elements of F is even. Secondly, the analogous set of hyperedges for $\overline{\mathbf{D}}^*$ ($\overline{\mathbf{K}}^*$) has exactly $\frac{l}{2}$ elements. These facts imply that any isomorphism between $\mathbf{D}^*$ and $\mathbf{K}^*$ induces an isomorphism from $\overline{\mathbf{D}}^*$ onto $\overline{\mathbf{K}}^*$.

Note that it is not true for directed hypergraphs with 2–loops. For example, take $\mathbf{M}$ with two vertices and two 2–loops, and $\mathbf{N}$ with two vertices and two regular edges forming a directed cycle. Then $\mathbf{M}^* \simeq \mathbf{N}^*$. But $\overline{\mathbf{M}}^*$ and $\overline{\mathbf{N}}^*$ are not isomorphic, because $\overline{\mathbf{M}}$ contains exactly one hyperedge (which is a 2–loop).

Observe also that we can assume

$$\overline{\mathbf{D}}^* = \overline{\mathbf{K}}^*. \tag{1}$$

It is sufficient to transport (by any isomorphism) the structure of the hypergraph $\overline{\mathbf{K}}^*$ onto sets of vertices and hyperedges of $\overline{\mathbf{D}}$. In this way we also transport the structure of the directed hypergraph $\overline{\mathbf{K}}$ onto these two sets.

Let $\overline{\mathbf{D}}_2$ and $\overline{\mathbf{K}}_2$ be directed hypergraphs containing all vertices and all 2–edges of $\overline{\mathbf{D}}$ and $\overline{\mathbf{K}}$, respectively. Note that these two directed hypergraphs are constructed from $\mathbf{D}_2$ and $\mathbf{K}_2$, respectively, in the same way as $\overline{\mathbf{D}}$ and $\overline{\mathbf{K}}$. Note also that $\mathbf{D}_1$

and $\mathbf{K}_1$ consist of all edges of $\overline{\mathbf{D}}$ and $\overline{\mathbf{K}}$. Obviously by (1),

$$\mathbf{D}_1^* = \mathbf{K}_1^* \quad \text{and} \quad \overline{\mathbf{D}}_2^* = \overline{\mathbf{K}}_2^*. \tag{2}$$

Recall (see [**2**, Chapter 3, Theorem 17]), Ore Theorem: all edges of a finite (undirected) graph can be directed to a form of total functional directed graph iff each of its connected components contains exactly one cycle. Recall also that for a finite connected graph with one cycle $(v_1, \ldots, v_n)$ (where $v_1, \ldots, v_n$ are consecutive vertices of the cycle), at most two such directions exist. More precisely, the edges in the cycle have to be directed either from v_i to v_{i+1} for $i = 1, \ldots, n-1$ and from v_n to v_1, or from v_{i+1} to v_i for $i = 1, \ldots, n-1$ and from v_1 to v_n. Moreover, other edges have to be directed towards the cycle. Thus we have exactly two ways (if the cycle is non–trivial, i.e. has at least two vertices) or exactly one (if the cycle is trivial).

Since $\mathbf{D}_1$ and $\mathbf{K}_1$ are finite total functional directed graphs and $\mathbf{K}_1^* = \mathbf{D}_1^*$, the above facts (applying to each connected component separately) we obtain that $\mathbf{K}_1$ is obtained from $\mathbf{D}_1$ by inverting the orientation of some pairwise disjoint directed cycles (more precisely, each of these cycles belongs to another connected component).

Since $\overline{\mathbf{D}}_2^* = \overline{\mathbf{K}}_2^*$, we have that for any 2–edge e,

$$I_1^{\overline{\mathbf{D}}}(e) \cup \{I_2^{\overline{\mathbf{D}}}(e)\} = I_1^{\overline{\mathbf{K}}}(e) \cup \{I_2^{\overline{\mathbf{K}}}(e)\},$$

and each of these sets has exactly three elements.

This equality of sets implies, of course, that exactly one of the following two cases is satisfied:

$$I_2^{\overline{\mathbf{D}}}(e) = I_2^{\overline{\mathbf{K}}}(e) \quad \text{or} \quad I_2^{\overline{\mathbf{D}}}(e) \neq I_2^{\overline{\mathbf{K}}}(e).$$

If the first condition holds, then also $I_1^{\overline{\mathbf{D}}}(e) = I_1^{\overline{\mathbf{K}}}(e)$. Thus all such 2–edges have the same orientation in $\overline{\mathbf{D}}_2$ and in $\overline{\mathbf{K}}_2$.

Now take the set F of all 2–edges f which satisfy the second case, i.e. $I_2^{\overline{\mathbf{K}}}(f) \neq I_2^{\overline{\mathbf{D}}}(f)$. Let us denote $v_f = I_2^{\overline{\mathbf{K}}}(f)$ and $d_f = I_2^{\overline{\mathbf{D}}}(f)$. Then by the above equality of sets, $v_f \in I_1^{\overline{\mathbf{D}}}(f)$. The second vertex of $I_1^{\overline{\mathbf{D}}}(f)$ (i.e. different from v_f) we denote by u_f. Applying again the above equality, since $I_1^{\overline{\mathbf{D}}}(f) = \{v_f, u_f\}$, we obtain $I_1^{\overline{\mathbf{K}}}(f) = \{u_f, d_f\}$.

Obviously $\overline{\mathbf{K}}_2$ is obtained from $\overline{\mathbf{D}}_2$ by inverting the orientation of all 2–edges from F according to the set $\{v_f : \quad f \in F\}$, i.e. in this way that for any $f \in F$, u_f and d_f form the new initial set and v_f forms the new final vertex.

Now we show that F can be divided onto finitely many 2–edge–disjoint subsets in such a way that each of them forms a sequence $(f_1, \ldots, f_n)$, called the quasicycle, such that $\{v_{f_{i+1}}, u_{f_{i+1}}\} = \{u_{f_i}, d_{f_i}\}$, where $f_{n+1} = f_1$. Of course, we can assume that F is non–empty. Our proof is some generalization of one of proofs of Euler Theorem (see [**2**, Chapter 11]).

Take an arbitrary 2–edge f_1 from F. Then there is a 2–edge f_2 of $\overline{\mathbf{D}}_2$ starting from $\{u_{f_1}, d_{f_1}\}$. f_2 has to be contained in F. Otherwise f_2 starts from $\{u_{f_1}, d_{f_1}\}$ also in $\overline{\mathbf{K}}_2$, which is impossible, because this set is the initial set of f_1 in $\overline{\mathbf{K}}_2$ and $f_1 \neq f_2$.

If $\{u_{f_2}, d_{f_2}\} = \{u_{f_1}, v_{f_1}\}$, then (f_1, f_2) forms the desired quasicycle. Thus we can assume that these sets are different. Then first, there is a 2–edge f_3 of $\overline{\mathbf{D}}_2$ starting from $\{u_{f_2}, d_{f_2}\}$. By the assumption, $f_3 \neq f_1$. Moreover, as above it can be proved that $f_3 \in F$. Further, $\{u_{f_3}, d_{f_3}\} \neq \{v_{f_2}, u_{f_2}\}$. To see it assume otherwise that the equality holds, so also $\{u_{f_3}, d_{f_3}\} = \{u_{f_1}, d_{f_1}\}$. But then f_1 and f_3 are two different 2–edges starting from $\{u_{f_3}, d_{f_3}\}$ in $\overline{\mathbf{K}}_2$, which is impossible.

If $\{u_{f_3}, d_{f_3}\} = \{v_{f_1}, u_{f_1}\}$, then the sequence (f_1, f_2, f_3) is a quasicycle. If not, then we can repeat the above procedure; and so on. But $\mathbf{D}$ is finite, so after finite steps this construction have to be finished. Thus we obtain a quasicycle $(f_1, \ldots, f_n)$.

If $f_1, \ldots, f_n$ are all 2–edges of F, then the proof of the fact is complete. If there is a 2–edge $g \in F \setminus \{f_1, \ldots, f_n\}$, then in the same way as above we construct a quasicycle $g_1, \ldots, g_k$ containing g (e.g. $g_1 = g$), and contained in F.

Observe that these two quasicycles are 2–edge–disjoint. To see it assume otherwise. Let $1 \leq l \leq k$ be the greatest number such that g_l is contained in $(f_1, \ldots, f_n)$, and let $1 \leq j \leq n$ be the natural number such that $g_l = f_j$. Take $h = g_{l+1}$ (if $l = k$, then $h = g_1$). Then $h \notin \{f_1, \ldots, f_n\}$, in particular, $h \neq f_j$. By the definition of quasicycle, h starts (in $\overline{\mathbf{D}}_2$) from $\{u_{g_l}, d_{g_l}\}$, and f_{j+1} starts (in $\overline{\mathbf{D}}_2$) from $\{u_{f_j}, d_{f_j}\}$ (if $j = n$, then $f_{j+1} = f_1$). Next, these two sets are equal, because $g_l = f_j$. Thus we obtain two different 2–edges of $\overline{\mathbf{D}}_2$ starting from the same set of two vertices, which is impossible.

Since F is finite, we can repeat this procedure as many times as needed to obtain finitely many pairwise 2–edge–disjoint quasicycles containing all 2–edges from F.

Summarizing, we have shown that directed hypergraph $\overline{\mathbf{K}}$ is obtained from $\overline{\mathbf{D}}$ by inverting the orientation of some pairwise disjoint cycles $c_1, \ldots, c_l$, and by inverting the orientation of some pairwise 2–edge–disjoint quasicycles $q_1, \ldots, q_k$.

Take a strong subhypergraph $\mathbf{M}$ of $\overline{\mathbf{D}}$. (The usual (weak) subhypergraph $\mathbf{M}$ is called strong iff for any edge or 2–edge e of $\overline{\mathbf{D}}$, if the initial set of e is contained in $\mathbf{M}$, then e, thus also the final vertex of e, belongs to $\mathbf{M}$; see Definition 2.4 in [**6**]). Then for any cycle or quasicycle $p \in \{c_1, \ldots, c_l, q_1, \ldots, q_k\}$, p is contained in $\mathbf{M}$, or p and $\mathbf{M}$ are hyperedge–disjoint. Assume that some hyperedge e (or equivalently, its initial set) of p is contained in $\mathbf{M}$. Then all its endpoints belong to $\mathbf{M}$. Thus the initial set of the successor of e is contained in $\mathbf{M}$, which implies that the successor also belongs to $\mathbf{M}$; and so on.

Let p be a quasicycle 2–edge–disjoint with $\mathbf{M}$, and e be a 2–edge of p. Then, in particular, $\{u_e, v_e\}$ is not contained in $\mathbf{M}$. Hence we deduce that $\{u_e, d_e\}$ is also not contained in $\mathbf{M}$, because this set is the initial set of the successor of e (in p).

Let $\widehat{\mathbf{M}}$ be the weak subhypergraph of $\overline{\mathbf{K}}$ consisting of all vertices and hyperedges of $\mathbf{M}$. First, $\widehat{\mathbf{M}}$ is correctly defined, by (1). Observe that $\widehat{\mathbf{M}}$ is just obtained from $\mathbf{M}$ by inverting the orientation of these cycles and quasicycles from the family $\{c_1, \ldots, c_l, q_1, \ldots, q_k\}$ which are contained in $\mathbf{M}$. Secondly, using the above facts it is easy to show that $\widehat{\mathbf{M}}$ is a strong subhypergraph of $\mathbf{K}$. Analogously, for any strong subhypergraph $\mathbf{N}$ of $\overline{\mathbf{K}}$, the weak subhypergraph $\widehat{\mathbf{N}}$ of $\overline{\mathbf{D}}$ consisting of all vertices and hyperedges of $\mathbf{N}$ is a strong subhypergraph of $\overline{\mathbf{D}}$.

Thus we obtain that the function φ, assigning $\widehat{\mathbf{M}}$ to each strong subhypergraph $\mathbf{M}$ of $\overline{\mathbf{D}}$, is a well–defined surjection from the set of all strong subhypergraphs of $\overline{\mathbf{D}}$ onto the set of all strong subhypergraphs of $\overline{\mathbf{K}}$. Recall (Proposition 2.7 in [**6**]) that the set of all strong subhypergraphs forms a complete lattice under (strong subhypergraph) inclusion $\leq_s$. Recall also (Proposition 2.5 (d) and (e) in [**6**]) that for any strong subhypergraphs $\mathbf{O}, \mathbf{P}$, $\mathbf{O} = \mathbf{P}$ iff their vertex sets are equal; $\mathbf{O} \leq_s \mathbf{P}$ iff the vertex set of $\mathbf{O}$ is contained in the vertex set of $\mathbf{P}$. The first fact implies that φ is injective. The second implies that φ and its inverse φ^{-1} preserve inclusion $\leq_s$. Hence we deduce that φ is an isomorphism between the strong subhypergraph lattices of $\overline{\mathbf{D}}$ and $\overline{\mathbf{K}}$. Finally observe that since $\mathbf{D}$ is obtained from $\overline{\mathbf{D}}$ by doubling each 2–edge, the strong subhypergraph lattices of $\mathbf{D}$ and $\overline{\mathbf{D}}$ are isomorphic. More precisely, to each strong subhypergraph $\mathbf{M}$ of $\overline{\mathbf{D}}$, it is sufficient to assign the weak subhypergraph of $\mathbf{D}$ obtained from $\mathbf{M}$ by doubling each of its 2–edges. Similarly as above it can be shown that this function is a lattice isomorphism. Of course, the analogous result for $\mathbf{K}$ and $\overline{\mathbf{K}}$ also holds. This completes the proof of the theorem, because the subgroupoid lattices $\mathcal{S}_s(\mathcal{G})$ and $\mathcal{S}_s(\mathcal{H})$ are isomorphic to the strong subhypergraph lattices of $\mathbf{D}$ and $\mathbf{K}$, respectively (Corollary 3.9 in [**6**]). $\square$

Now necessary and sufficient conditions are found for a lattice to be isomorphic to the weak subgroupoid lattice for some partial commutative groupoid satisfying the non–equality $x \circ y \neq x$. In [**1**] it is proved that a lattice $\mathcal{L}$ is isomorphic to the weak subalgebra lattice for some partial algebra iff $\mathcal{L}$ is distributive algebraic and

(i) every element is a join of join–irreducible elements,
(ii) any non–zero join–irreducible element contains only a finite (and non–empty) set of atoms,
(iii) the set of all non–zero and non–atomic join–irreducible elements is an antichain with respect to the lattice ordering of $\mathcal{L}$.

(Recall that an element l of $\mathcal{L}$ is join–irreducible iff for any elements $k_1, k_2, l = k_1 \vee k_2$ implies $l = k_1$ or $l = k_2$.) Note that a partial algebra is finite iff its weak subalgebra lattice is finite, because each element forms a weak subalgebra. It is also easy to see that any finite distributive lattice is algebraic and satisfies (i). Thus, since only finite groupoids are here considered, we can assume that $\mathcal{L}$ is a finite distributive lattice satisfying (ii) and (iii).

Let $\mathcal{G} = \langle G, \circ \rangle$ be a partial commutative groupoid, such that $g \circ h \neq g$ (if $g \circ h$ is defined) for any $g, h \in G$, and let $\mathcal{H} = \langle H, \circ \rangle$ be a weak subgroupoid of $\mathcal{G}$. Recall first, (see Lemma 3 in [**1**]), that $\mathcal{H}$ is an atom in $\mathcal{S}_w(\mathcal{G})$ iff $\mathcal{H}$ consists of exactly one element and the binary operation $\circ$ is defined nowhere. Secondly, $\mathcal{H}$ is a non–zero non–atomic join–irreducible element in $\mathcal{S}_w(\mathcal{G})$ iff there are $g_1, g_2, h \in G$ such that $H = \{g_1, g_2, h\}$ and $g_1 \circ g_2$ is defined and equal to h, and $\circ$ is defined (in $\mathcal{H}$) onto this directed pair only. Note also $h = g_1 \circ g_2 \neq g_1$. These facts imply that every non–zero non–atomic join–irreducible element in $\mathcal{S}_w(\mathcal{G})$ contains two or three atoms.

Since we consider commutative groupoids in the strong sense, the weak subgroupoid lattice $\mathcal{S}_w(\mathcal{G})$ has some additional property yet. More formally, take an arbitrary non–zero non–atomic join–irreducible element $\mathcal{H} = \langle H, \circ \rangle$ in $\mathcal{S}_w(\mathcal{G})$ containing three atoms. Then $H = \{g_1, g_2, h\}$ and g_1, g_2, h are pairwise different and $g_1 \circ g_2$ is defined and equal to h. Since $g_1 \circ g_2$ is defined also in $\mathcal{G}$, we obtain that $g_2 \circ g_1$ is defined (in $\mathcal{G}$) and equal to h. Hence, the weak subgroupoid $\overline{\mathcal{H}} = \langle \overline{H}, \circ \rangle$ such that $\overline{H} = H$ and $\circ$ is defined onto exactly one pair $\langle g_2, g_1 \rangle$ is a non–zero non–atomic join–irreducible element in $\mathcal{S}_w(\mathcal{G})$. $\overline{\mathcal{H}}$ and $\mathcal{H}$ are different (because their operations are defined onto two different pairs of elements), and $\overline{\mathcal{H}}$ contains the same three atoms as $\mathcal{H}$. (If $\overline{\mathcal{H}}$ contains two atoms, then the non–equality $g_1 \circ g_2 \neq h$ implies $g_1 = g_2$, and thus $\overline{\mathcal{H}} = \mathcal{H}$.) Thus we obtain that for any pairwise different three atoms of $\mathcal{S}_w(\mathcal{G})$, the number of all non–zero non–atomic join–irreducible elements containing these atoms is even.

Summarizing we have the following fact

Proposition 2. *If $\mathcal{L}$ is a lattice isomorphic to the weak subgroupoid lattice $\mathcal{S}_w(\mathcal{G})$ for some partial finite commutative groupoid $\mathcal{G}$ such that $g \circ h \neq g$ for any $g, h \in G$, then*

- (L.1) *$\mathcal{L}$ is finite and distributive,*
- (L.2) *every non–zero non–atomic join–irreducible element contains exactly two or three atoms,*
- (L.3) *for any three–element set A of atoms, the number of all non–zero and non–atomic join–irreducible elements containing A is divided by 2,*
- (L.4) *the set of all non–zero non–atomic join–irreducible elements is an antichain with respect to the lattice ordering of $\mathcal{L}$.*

For any lattice $\mathcal{L}$, by $\mathcal{A}(\mathcal{L})$ we denote the set of all atoms of $\mathcal{L}$, next, the set of all non–zero and non–atomic join–irreducible elements of $\mathcal{L}$ containing exactly two atoms is denoted by $\mathcal{I}_2(\mathcal{L})$, and $\mathcal{I}_3(\mathcal{L})$ denotes the set of all non–zero and non–atomic join–irreducible elements of $\mathcal{L}$ containing exactly three atoms.

Now we can formulate and prove our characterization theorem.

Theorem 3. *Let a lattice $\mathcal{L} = \langle L, \leq_{\mathcal{L}} \rangle$ satisfy (L.1)—(L.4). Then $\mathcal{L}$ is isomorphic to the weak subgroupoid lattice $\mathcal{S}_w(\mathcal{G})$ for some partial finite commutative groupoid $\mathcal{G} = \langle G, \circ \rangle$ such that $g \circ h \neq g$ for each $g, h \in G$ iff*

(i) *For any subset $I \subseteq \mathcal{I}_2(\mathcal{L})$,*

$$|I| \leq \big|\{a \in \mathcal{A}(\mathcal{L})\colon\ \exists_{i \in I}\ a \leq_{\mathcal{L}} i\}\big|.$$

(ii) *For any subset $I \subseteq \mathcal{I}_3(\mathcal{L})$,*

$$|I| \leq 2 \cdot \big|\big\{\{a, b\} \subseteq \mathcal{A}(\mathcal{L})\colon\ a \neq b \ \text{ and } \ \exists_{i \in I}\ a \leq_{\mathcal{L}} i,\ b \leq_{\mathcal{L}} i\big\}\big|.$$

Moreover, $\mathcal{G}$ is total iff the following two additional equalities hold:

$$\big|\mathcal{I}_2(\mathcal{L})\big| = \big|\mathcal{A}(\mathcal{L})\big| \quad \text{and} \quad \big|\mathcal{I}_3(\mathcal{L})\big| = 2 \cdot \big|\mathcal{P}_2\big(\mathcal{A}(\mathcal{L})\big)\big|,$$

where $\mathcal{P}_2\big(\mathcal{A}(\mathcal{L})\big)$ is the family of all two–element subsets of $\mathcal{A}(\mathcal{L})$.

Proof. In the proof of this result we also use hypergraph–algebraic language from [**6**]. Recall that a lattice $\mathcal{L}$ satisfying the conditions (L.1)–(L.4) can be represented (see Definition 3.17 in [**6**]) by the (undirected) hypergraph $\mathbf{U}(\mathcal{L})$ consisting of all atoms of $\mathcal{L}$ as its vertices, and all non–zero non–atomic join–irreducible elements as its hyperedges, and for any hyperedge e, atoms contained in e forms the endpoint set of e. By (L.2), $\mathbf{U}(\mathcal{L})$ has not loops.

By (L.3), for any three-element set W of vertices, the set of all hyperedges with endpoints in W has an even number of hyperedges. Thus this set (if it is non–empty) we can divide onto pairwise disjoint two–element sets. Next, any such two–element set can be replaced by a single hyperedge with W as its endpoints. Obviously we can apply this procedure to each three–element set of vertices. The hypergraph such obtained will be denoted by $\overline{\mathbf{U}(\mathcal{L})}$. More formally, in this way we obtain, in general, many different hypergraphs, but they are isomorphic.

Let $\mathbf{U}_1(\mathcal{L})$ be the hypergraph obtained from $\overline{\mathbf{U}(\mathcal{L})}$ by omitting all hyperedges with three endpoints. Note that $\mathbf{U}_1(\mathcal{L})$ is an usual (undirected) graph with regular edges (i.e. edges having two different endpoints). Let $\mathbf{U}_2(\mathcal{L})$ be the hypergraph consisting of all vertices and all hyperedges with three endpoints of $\overline{\mathbf{U}(\mathcal{L})}$.

Observe first that the conditions (i) and (ii) of Theorem 3 are equivalent, respectively, with the following

(a) Each connected component of the graph $\mathbf{U}_1(\mathcal{L})$ contains at most one (undirected) cycle.

Moreover, $\big|\mathcal{I}_2(\mathcal{L})\big| = \big|\mathcal{A}(\mathcal{L})\big|$ iff each connected component of $\mathbf{U}_1(\mathcal{L})$ has exactly one cycle.

(b) For any set E of hyperedges of $\mathbf{U}_2(\mathcal{L})$, $|E|$ is not greater than the number of all (undirected) pairs $\{v, w\}$ of vertices such that v and w are arbitrary endpoints of some hyperedge e from E (i.e. $v, w \in I^{\mathbf{U}_2(\mathcal{L})}(e)$).

Moreover, $\big|\mathcal{I}_3(\mathcal{L})\big| = 2 \cdot \big|\mathcal{P}_2\big(\mathcal{A}(\mathcal{L})\big)\big|$ iff the number of all hyperedges of $\mathbf{U}_2(\mathcal{L})$ is equal to the number of all (undirected) pairs of vertices.

The equivalence (ii)$\Longleftrightarrow$(b) is trivial.

(i)$\Longleftrightarrow$(a). Of course, the straightforward translation of (i) onto hypergraph language is the following (similarly as (ii)):

(i′) For any subset F of edges of $\mathbf{U}_1(\mathcal{L})$, $|F|$ is not greater than the number of all vertices being endpoints of edges from F.

Moreover, $\big|\mathcal{I}_2(\mathcal{L})\big| = \big|\mathcal{A}(\mathcal{L})\big|$ iff the number of all edges and the number of all vertices are equal.

If (a) holds, then by Ore Theorem (see [**2**], Chapter 3, Theorem 17), all edges of $\mathbf{U}_1(\mathcal{L})$ can be directed to a form of functional directed graph $\mathbf{D}$. Thus (i′) also holds, because from each vertex at most one edge starts. Next, if each connected component of $\mathbf{U}_1(\mathcal{L})$ has exactly one cycle, then, by the same result, $\mathbf{D}$ is a total functional directed graph. Thus obviously the number of all edges of $\mathbf{D}$ and the number of all vertices of $\mathbf{D}$ are equal.

Now assume that some connected component of $\mathbf{U}_1(\mathcal{L})$ contains two different undirected cycles $c_1 = (u_1, \ldots, u_n)$ and $c_2 = (w_1, \ldots, w_m)$ (where $u_1, \ldots, u_n$ and

$w_1, \ldots, w_n$ are consecutive vertices of these cycles). If c_1 and c_2 have common edges, say l edges, then they have at least $l+1$ common vertices. Thus c_1 and c_2 together have $n+m-l$ edges, and at most $n+m-l-1$ vertices. Hence, (i′) does not hold. If c_1 and c_2 are edge–disjoint, then since the cycles belong to one connected component, there is a path $p=(v_1,\ldots,v_k)$, connecting c_1 and c_2 and edge–disjoint with these cycles (we assume here that a single vertex is also a path; in this case our cycles have a common vertex). Let F be the set of all edges of c_1, c_2 and p. Then $|F| = n+m+k-1$. On the other hand c_1, c_2 and p have together at most $n+m+k-2$ vertices, because v_1 and v_k belongs to c_1 and c_2, respectively (if $k=1$, then c_1 and c_2 have at least one common vertex, so they have together at most $n+m-1$ vertices). Thus again (i′) does not hold.

Summarizing we have shown that each connected component of $\mathbf{U}_1(\mathcal{L})$ contains at most one undirected cycle. Thus by Ore Theorem, all edges of $\mathbf{U}_1(\mathcal{L})$ can be directed to a form of functional directed graph $\mathbf{D}$. If we additionally assume that the number of all edges of $\mathbf{U}_1(\mathcal{L})$ is equal to the number of all vertices of $\mathbf{U}_1(\mathcal{L})$, then $\mathbf{D}$ have to be total (recall that these graphs are finite). Hence, applying again Ore Theorem, each connected component of $\mathbf{U}_1(\mathcal{L})$ has exactly one cycle.

We also need the following result from [**6**, Corollary 3.19(b)]

$$(WS) \qquad \mathcal{L} \simeq \mathcal{S}_w(\mathcal{G}) \iff \mathbf{U}(\mathcal{L}) \simeq \mathbf{D}(\mathcal{G})^*.$$

$\Longrightarrow$. For any two–element set W of vertices, if a 2–edge starts from W, then exactly two 2–edges starts from W, and moreover, these two 2–edges have the same initial vertex. Thus we can replace each such pair by a single 2–edge to obtain the new directed hypergraph $\mathbf{D}$. Then (see the second part of the proof of Theorem 1) for any three–element set W of vertices, the set of all hyperedges of $\mathbf{D}(\mathcal{G})^*$ with W as the endpoint set has twice more elements than the analogous set of hyperedges of $\mathbf{D}^*$. Next, for any two–vertex set, sets of edges similarly defined for $\mathbf{D}(\mathcal{G})^*$ and $\mathbf{D}^*$ are equal. It follows from the fact that each hyperedge of $\mathbf{D}(\mathcal{G})$, thus also of $\mathbf{D}$, is regular. Hence, since $\mathbf{D}(\mathcal{G})^* \simeq \mathbf{U}(\mathcal{L})$,

$$\mathbf{D}^* \simeq \overline{\mathbf{U}(\mathcal{L})}.$$

Thus, because $\mathbf{D}$ has not loops and 2–loops,

$$\mathbf{D}_1^* \simeq \mathbf{U}_1(\mathcal{L}) \quad \text{and} \quad \mathbf{D}_2^* \simeq \mathbf{U}_2(\mathcal{L}),$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ consist of all vertices of $\mathbf{D}$ and all edges or 2–edges, respectively.

Since $\mathbf{D}_1$ is a functional directed graph (if $\mathcal{G}$ in total, then $\mathbf{D}$ is also total), by the first isomorphism and Ore Theorem we obtain (a).

By the definition of $\mathbf{D}$, at most one (if $\mathcal{G}$ is total, then exactly one) 2–edge starts from any two–element set of vertices. Take an arbitrary set E of 2–edges of $\mathbf{D}_2$. Then $|E|$ is equal to the number of all pairs from which starts some 2–edge. This fact and the second isomorphism implies (b). Next, if $\mathcal{G}$ is total, then the number of all 2–edges of $\mathbf{D}_2$ is equal to the number of all (undirected) pairs of vertices.

$\Longleftarrow$. Having (WS), it is enough to direct all hyperedges of $\overline{\mathbf{U}(\mathcal{L})}$ to a form of finite directed hypergraph $\mathbf{D}$ in such a way that $\mathbf{D}$ contains only regular edges and regular 2–edges, and at most one edge starts from any vertex, and at most one 2–edge starts from any two–element set of vertices. Because then we can define a groupoid $\mathcal{G} = \langle G, \circ \rangle$ in the following way: The carrier G is the set of all vertices of $\mathbf{D}$. Next, for any $g, h \in G$, $g \circ h$ and $h \circ g$ are defined iff some (directed) hyperedge e starts from $\{g, h\}$ and then $g \circ h$ and $h \circ g$ are equal to the final vertex of e. It is obvious that $\mathcal{G}$ is finite and commutative, and also satisfies $g \circ h \neq g$ (since $\mathbf{D}$ has only regular edges and 2–edges). Further, $\mathbf{D}(\mathcal{G})$ is obtained from $\mathbf{D}$ by doubling each 2–edge. Thus, in the same way as in the proof of $\Longrightarrow$, we obtain that $\mathbf{D}(\mathcal{G})^* \simeq \mathbf{U}(\mathcal{L})$. Note also that $\mathcal{G}$ is total iff exactly one edge (2–edge) starts from each vertex (two–vertex set) of $\mathbf{D}$.

First, by (a) and Ore Theorem, all edges of $\mathbf{U}_1(\mathcal{G})$ can be directed to a form of functional directed graph $\mathbf{D}_1$. Secondly, by (L.2) we have that $\mathbf{U}_1(\mathcal{G})$ has not loops, which implies that $\mathbf{D}_1$ contains regular edges only. Recall also that, if each connected component of $\mathbf{U}_1(\mathcal{G})$ contains exactly one cycle, then $\mathbf{D}_1$ is total.

Thus now it is sufficient to show that all hyperedges of $\mathbf{H} = \mathbf{U}_2(\mathcal{G})$ can be directed to a form of directed hypergraph $\mathbf{D}_2$ in such a way that

(i) $\mathbf{D}_2$ contains regular 2–edges only,

(ii) for any two–element set V, at most one 2–edge starts from V.

We use the induction on the number of hyperedges. If $\mathbf{H}$ has not hyperedges, then it is trivial. Thus we can assume that $\mathbf{H}$ has $x \geq 1$ hyperedges. We also assume that our thesis is true for any hypergraph satisfying (b) and having not greater than $x - 1$ hyperedges.

Let e be a hyperedge of $\mathbf{H}$, and W be the set of all endpoints of e. Let $\mathbf{K}$ be the hypergraph obtained from $\mathbf{H}$ by omitting e. Since $\mathbf{K}$ is a weak subhypergraph of $\mathbf{H}$, and $\mathbf{H}$ satisfies (b), we deduce that $\mathbf{K}$ also satisfies (b). Hence and by the induction hypothesis all hyperedges of $\mathbf{K}$ can be directed to a form of finite directed hypergraph $\mathbf{C}$ satisfying (i) and (ii).

If there is a pair $\{v, w\} \subseteq W$ such that none 2–edge of $\mathbf{C}$ starts from $\{v, w\}$, then we can take this set as the initial set of e and the third vertex of W (different from v and w) as the final vertex of e. Obviously the directed hypergraph such obtained satisfies (i) and (ii).

Thus we can assume that for any two different vertices v and w of W, there is (of course, exactly one) 2–edge of $\mathbf{C}$ starting from $\{v, w\}$.

Let $E_0, E_1, E_2, \dots$ and $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ be sequences of sets such that $E_0 = \emptyset$ and $\mathcal{U}_0$ is the family of all two–element subsets of W, and for any $n \geq 1$, E_n is the set of all 2–edges of $\mathbf{C}$ starting from elements of $\mathcal{U}_{n-1}$, and $\mathcal{U}_n$ is the family of all two–element sets V such that V is contained in the endpoint set of some hyperedge f from E_n (i.e. $V \subseteq I_1^{\mathbf{C}}(f) \cup \{I_2^{\mathbf{C}}(f)\} = I^{\mathbf{H}}(f)$). It is easy to see that $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$, which implies $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{U}_3 \subseteq \dots$.

Hence, since $\mathbf{C}$ is finite, $E_{m+1} = E_m$, thus also $\mathcal{U}_{m+1} = \mathcal{U}_m$, for some non–negative integer $m \in \mathbb{N}$. Let $E = E_m$ and $\mathcal{U} = \mathcal{U}_m$. Observe that the assumption on W implies $\mathcal{U}_0 \subseteq \mathcal{U}_1$, so $\mathcal{U}_0 \subseteq \mathcal{U}$. Thus, since $\mathbf{H}$ satisfies (b) and $e \notin E$, we infer

$$|E| + 1 = \big|E \cup \{e\}\big| \leq |\mathcal{U}|,$$

so

$$|E| \leq |\mathcal{U}| - 1.$$

Since $\mathbf{C}$ satisfies (ii) and the family of initial sets (in $\mathbf{C}$) of all 2–edges from E is contained in $\mathcal{U}$, this inequality implies that there is a two–element set $V = \{v_1, v_2\} \in \mathcal{U}$ such that none 2–edge of E starts from V. Hence it follows that there is not 2–edge of $\mathbf{C}$ starting from V. (Otherwise this 2–edge would have to belong to E_{m+1}, but $E_{m+1} = E$.)

Let n be the least number such that $V \in \mathcal{U}_n$, of course, $n \geq 1$ (by the assumption on W). By the definition of $\mathcal{U}_n$, there is a 2–edge $f \in E_n$ such that $v_1 \in I_1^{\mathbf{C}}(f)$ and $v_2 = I_2^{\mathbf{C}}(f)$ (or conversely, but it does not matter). First, $I_1^{\mathbf{C}}(f) \in \mathcal{U}_{n-1}$. Secondly, we can change the orientation of f in this way that V forms the new initial set of f and the second vertex w of $I_1^{\mathbf{C}}(f)$ (different from v_1) forms the new final vertex of f. The directed hypergraph $\mathbf{C}_1$ such obtained satisfies (i) and (ii), moreover, none 2–edge of $\mathbf{C}_1$ starts from $V_1 = \{v_1, w\}$, and $V_1 \in \mathcal{U}_{n-1}$. If $n - 1 \geq 1$, then we can apply the above construction to $\mathbf{C}_1$ and V_1. Repeating this procedure as many times as needed we obtain that the orientation of some 2–edges in $\mathbf{C}$ can be changed in such a way that a new directed hypergraph $\mathbf{D}_2'$ also satisfies (i) and (ii), and additionally, none 2–edge starts from some two–element subset V of W. Then, it has been shown earlier, V can be taken as the initial set of e and the vertex of W outside V can be taken as the final vertex of e. Thus we obtain that all hyperedges of $\mathbf{H}$ can be directed to a form of directed hypergraph $\mathbf{D}_2$ satisfying (i) and (ii). This completes the proof of the induction step.

Finally observe that if the number of all 2–edges of $\mathbf{D}$ is equal to the number of all pairs of vertices, then by (ii), exactly one 2–edge starts from each two–vertex set. □

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