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ON BOUNDED MODULE MAPS BETWEEN HILBERT MODULES OVER LOCALLY C^*-ALGEBRAS

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Abstract. Let A be a locally C^*-algebra and let E be a Hilbert A-module. We show that the algebra $B_A(E)$ of all bounded A-module maps on E is a locally m-convex algebra which is algebraically and topologically isomorphic to $LM(K_A(E))$, the algebra of all left multipliers of $K_A(E)$, where $K_A(E)$ is the locally C^*-algebra of all "compact" A-module maps on E. Also we show that $b(B_A(E))$, the algebra of all bounded elements in $B_A(E)$, is a Banach algebra which is isometrically isomorphic to $B_{b(A)}(b(E))$.

1. Introduction

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_i\}_i$ converges to 0 if and only if the net $\{p(a_i)\}_i$ converges to 0 for every continuous C^*-seminorm p on A. In fact a locally C^*-algebra is an inverse limit of C^*-algebras.

Hilbert modules over locally C^*-algebras generalize the notion of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra. In [**9**], Phillips showed that many results about multipliers of a C^*-algebra are valid for multipliers of a locally C^*-algebra. Thus, he proved that $M(A)$, the multiplier algebra of a locally C^*-algebra A, is a locally C^*-algebra in the topology of seminorm [**9**, Theorem 3.14]. In this note we show that any left multiplier of a locally C^*-algebra A is automatically continuous (Proposition 3.4) and $LM(A)$, the algebra of left multipliers of A, is a complete locally m-convex algebra in the topology of seminorm (Theorem 3.5). Also, Phillips shows that if E is a Hilbert module over a locally C^*-algebra A, then the locally C^*-algebra $L_A(E)$ of all adjointable maps on E is isomorphic to $M(K_A(E))$, where $K_A(E)$ is the locally C^*-algebra of all "compact" A-module maps on E [**9**, Theorem 4.2]. This result is a generalization of Theorem 1 of [**5**] for Hilbert module over locally C^*-algebras. We show that the locally m-convex algebra $B_A(E)$ of all bounded A-module maps is isomorphic to $LM(K_A(E)$ (Theorem 3.6). This result generalizes Theorem 1.5

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of [**6**] in the context of Hilbert modules over locally C^*-algebras. Finally we prove that if E and F are Hilbert modules over a locally C^*-algebra A, then $b(B_A(E,F))$, the set of all bounded elements in $B_A(E,F)$, is a Banach space in the norm $\|\cdot\|_\infty$ which is isometrically isomorphic to $B_{b(A)}(b(E),b(F))$, the Banach space of all bounded $b(A)$-module maps from $b(E)$ to $b(F)$ (Theorem 3.7). In particular, $b(B_A(E))$ is a Banach algebra which is isometrically isomorphic to $B_{b(A)}(b(E))$ and $b(L_A(E))$ is a C^*-algebra which is isomorphic to $L_{b(A)}(b(E))$.

2. Preliminaries

If A is a locally C^*-algebra and $S(A)$ is the set of all continuous C^*-seminorms on A, then for each $p \in S(A)$, $A_p = A/\ker(p)$ is a C^*-algebra in the norm induced by p and $A = \lim\limits_{p\leftarrow} A_p$ (see, for example, [**9**]). The canonical map from A onto A_p, $p \in S(A)$ is denoted by π_p and the image of a in A under π_p by a_p. The connecting maps of the inverse system $\{A_p\}_{p\in S(A)}$ are denoted by π_{pq}, $q,p \in S(A)$, with $p \geq q$.

Now we recall some facts about Hilbert modules over locally C^* -algebras from [**9**].

Definition 2.1. A pre-Hilbert A-module is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot,\cdot \rangle : E \times E \to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:

(i) $\langle x,y\rangle^* = \langle y,x\rangle$ for every $x,y \in E$;
(ii) $\langle x,x\rangle \geq 0$ for every $x \in E$;
(iii) $\langle x,x\rangle = 0$ if and only if $x = 0$.

We say that E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\overline{p}_E(x) = \sqrt{p(\langle x,x\rangle)}$, $x \in E$, $p \in S(A)$.

Given a Hilbert A-module E, for each $p \in S(A)$, $N_p^E = \ker(\overline{p}_E)$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert A_p-module with $(x+ N_p^E)\pi_p(a) = xa + N_p^E$ and $\left\langle x + N_p^E, y + N_p^E\right\rangle = \pi_p(\langle x,y\rangle)$. The canonical map from E onto E_p is denoted by σ_p^E, and the image of x in E under σ_p^E by x_p, $p \in S(A)$.

For each $p,q \in S(A)$ with $p \geq q$ there is a canonical surjective linear map $\sigma_{pq}^E : E_p \to E_q$ such that $\sigma_{pq}^E(x_p) = x_q$, $x \in E$. Then $\{E_p; A_p; \sigma_{pq}^E, p \geq q,\ p,q \in S(A)\}$ is an inverse system of Hilbert C^*-modules in the following sense:

- $\sigma_{pq}^E(x_p a_p) = \sigma_{pq}^E(x_p)\pi_{pq}(a_p)$ for every $x_p \in E_p$ and for every $a_p \in A_p$;
- $\left\langle \sigma_{pq}^E(x_p), \sigma_{pq}^E(y_p)\right\rangle = \pi_{pq}(\langle x_p,y_p\rangle)$for every $x_p,y_p \in E_p$;
- $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$, $p \geq q \geq r$;
- $\sigma_{pp}^E = \mathrm{id}_{E_p}$;

and $\lim\limits_{p\leftarrow} E_p$ is a Hilbert A-module with $((x_p)_p)((a_p)_p) = (x_p a_p)_p$ and $\langle (x_p)_p, (y_p)_p\rangle = (\langle x_p, y_p\rangle)_p$. Moreover, $\lim\limits_{p\leftarrow} E_p$ can be identified with E.

We recall that an element a in A respectively x in E is bounded if

$$\|a\|_\infty = \sup\{p(a); p \in S(A)\} < \infty$$

respectively

$$\|x\|_\infty = \sup\{\overline{p}_E(x); p \in S(A)\} < \infty$$

The set of all bounded elements in A respectively in E will be denoted by $b(A)$ respectively $b(E)$. We know that $b(A)$ is a C^*-algebra in the C^*-norm $\|\cdot\|_\infty$, and $b(E)$ is a Hilbert $b(A)$-module.

3. Bounded modules maps

Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules. An A-module map $T : E \to F$ is said to be bounded if for each $p \in S(A)$, there is $K_p > 0$ such that $\overline{p}_F(Tx) \leq K_p \overline{p}_E(x)$ for all $x \in E$. The set of all bounded A-module maps from E to F is denoted by $B_A(E, F)$ and we write $B_A(E)$ for $B_A(E, E)$.

Clearly, for each $p \in S(A)$, the map $\widetilde{p}$ defined by

$$\widetilde{p}(T) = \sup\{\overline{p}_F(Tx); x \in E \text{ and } \overline{p}_E(x) \leq 1\}, \quad T \in B_A(E, F)$$

is a seminorm on $B_A(E, F)$.

Proposition 3.1. *Let A be a locally C^*-algebra and let E and F be two Hilbert A-modules. Then we have:*

1. *$B_A(E, F)$ with the topology determined by the family of seminorms $\{\widetilde{p}\}_{p \in S(A)}$ is a complete locally convex space.*
2. *$B_A(E)$ with the topology determined by the family of seminorms $\{\widetilde{p}\}_{p \in S(A)}$ is a complete locally m-convex algebra.*

Proof. (1): Let $p, q \in S(A)$ with $p \geq q$ and let $S \in B_{A_p}(E_p, F_p)$. Since

$$\begin{aligned} \left\langle \sigma_{pq}^F\left(S\left(\sigma_p^E(x)\right)\right), \sigma_{pq}^F\left(S\left(\sigma_p^E(x)\right)\right)\right\rangle &= \pi_{pq}\left(\left\langle S\left(\sigma_p^E(x)\right), S\left(\sigma_p^E(x)\right)\right\rangle\right) \\ &\leq \|S\|_p \, \pi_{pq}\left(\left\langle \sigma_p^E(x), \sigma_p^E(x)\right\rangle\right) \text{ cf. [\mathbf{7}, 2.8]} \\ &= \|S\|_p \left\langle \sigma_q^E(x), \sigma_q^E(x)\right\rangle \end{aligned}$$

for all $x \in E$, where $\|\cdot\|_p$ is the norm on $B_{A_p}(E_p, F_p)$, we can define $(\pi_{pq})_*(S) : E_q \to F_q$ by $(\pi_{pq})_*(S)\left(\sigma_q^E(x)\right) = \sigma_{pq}^F\left(S\left(\sigma_p^E(x)\right)\right)$. It is easy to see that $(\pi_{pq})_*(S)$ is a bounded A_q-module map from E_q to F_q. Thus we have obtained a map $(\pi_{pq})_*$ from $B_{A_p}(E_p, F_p)$ to $B_{A_q}(E_q, F_q)$. Also it is easy to see that $\{B_{A_p}(E_p, F_p); (\pi_{pq})_*, p \geq q, p, q \in S(A)\}$ is an inverse system of Banach spaces.

We will show that the locally convex spaces $B_A(E, F)$ and $\lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$ are isomorphic.

Let $p \in S(A)$ and let $T \in B_A(E, F)$. Since $T(N_p^E) \subseteq N_p^F$ there is a unique linear map $T_p : E_p \to F_p$ such that $\sigma_p^F \circ T = T_p \circ \sigma_p^E$. Moreover, T_p is a bounded A_p -module map. Thus we can define a map $(\pi_p)_* : B_A(E, F) \to B_{A_p}(E_p, F_p)$ by $(\pi_p)_*(T) = T_p$, where $\sigma_p^F \circ T = T_p \circ \sigma_p^E$. Clearly $(\pi_p)_*$ is a continuous linear map and $(\pi_{pq})_* \circ (\pi_p)_* = (\pi_q)_*$ for all $p, q \in S(A)$ with $p \geq q$. Therefore we can define a map Φ from $B_A(E, F)$ to $\lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$ by $\Phi(T) = \left((\pi_p)_*(T)\right)_p$. It is not difficult to check that Φ is linear and $\|\Phi(T)\|_p = \widetilde{p}(T)$ for all $T \in B_A(E, F)$. To show that Φ is surjective, let $(T_p)_p \in \lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$. Define $T : E \to F$ by $T(x) = \left(T_p\left(\sigma_p^E(x)\right)\right)_p$. Since $\sigma_{pq}^F\left(T_p\left(\sigma_p^E(x)\right)\right) = (\pi_{pq})_*(T_p)\left(\sigma_q^E(x)\right) = T_q\left(\sigma_q^E(x)\right)$ for all $p, q \in S(A)$ with $p \geq q$, T is well-defined. It is not difficult to check that T is a bounded A -module map and $\Phi(T) = (T_p)_p$. Hence Φ is surjective.

Thus we showed that the topological spaces $B_A(E, F)$ and $\lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$ are isomorphic, and since $\lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$ is complete, $B_A(E, F)$ is complete.

(2): It is not difficult to check that $\widetilde{p}$ is a submultiplicative seminorm on $B_A(E)$ for all $p \in S(A)$ and $\{B_p(E_p); (\pi_{pq})_*,\ p \geq q,\ p, q \in S(A)\}$ is an inverse system of Banach algebras. Also it is easy to check that the map $\widetilde{\Phi}$ from $B_A(E)$ to $\lim\limits_{p\leftarrow} B_{A_p}(E_p)$ defined by $\widetilde{\Phi}(T) = \left((\pi_p)_*(T)\right)_p$ is an isomorphism of topological algebras, and since $\lim\limits_{p\leftarrow} B_{A_p}(E_p)$ is complete, the assertion is proved. □

Remark 3.2. *If A is a locally C^*-algebra and E and F are Hilbert A-modules, then the locally convex spaces $B_A(E, F)$ and $\lim\limits_{p\leftarrow} B_{A_p}(E_p, F_p)$ as well as the locally m-convex algebras $B_A(E)$ and $\lim\limits_{p\leftarrow} B_{A_p}(E_p)$ can be identified.*

A map T from E to F is adjointable if there is a map T^* from F to E such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all x in E and for all y in F. Any adjointable map from E into F is a bounded A-module map (cf. [**11**]). The set of all adjointable maps from E into F is denoted by $L_A(E, F)$, and we write $L_A(E)$ for $L_A(E, E)$. For x in E and for y in F the map $\theta_{y,x} : E \to F$ defined by $\theta_{y,x}(z) = y\langle x, z\rangle$ is adjointable. The closed subspace of $L_A(E, F)$ generated by $\{\theta_{y,x}; x \in E,\ y \in F\}$ is denoted by $K_A(E, F)$, and we write $K_A(E)$ for $K_A(E, E)$. It is easy to verify that $(\pi_{pq})_*\left(L_{A_p}(E_p, F_p)\right) \subseteq L_{A_q}(E_q, F_q)$ and $(\pi_{pq})_*\left(K_{A_p}(E_p, F_p)\right) \subseteq K_{A_q}(E_q, F_q)$ for all $p, q \in S(A)$ with $p \geq q$. Then the restriction of Φ on $L_A(E, F)$ is exactly the same map as defined in Proposition 4.7 of [**9**]. Therefore the restriction of Φ on $L_A(E, F)$ is an isomorphism between the locally convex spaces $L_A(E, F)$ and $\lim\limits_{p\leftarrow} L_{A_p}(E_p, F_p)$, and the restriction of Φ on $K_A(E, F)$ is an isomorphism between the locally convex spaces $K_A(E, F)$ and $\lim\limits_{p\leftarrow} K_{A_p}(E_p, F_p)$ [**9**, Proposition 4.7]. Also the restriction of $\widetilde{\Phi}$ on $L_A(E)$ is an isomorphism between the locally C^*-algebras $L_A(E)$ and $\lim\limits_{p\leftarrow} L_{A_p}(E_p)$, and the restriction of $\widetilde{\Phi}$ on $K_A(E)$ is an isomorphism between the locally C^*-algebras $K_A(E)$ and $\lim\limits_{p\leftarrow} K_{A_p}(E_p)$ [**9**, Theorem 4.2].

In [**9**, Theorem 4.2], Phillips shows that the locally C^*-algebras $L_A(E)$ and $M(K_A(E))$, the multiplier algebra of $K_A(E)$, are isomorphic. We will prove here that the locally m-convex algebras $B_A(E)$ and $LM(K_A(E))$, the algebra of left multipliers of $K_A(E)$, are isomorphic.

If A is a locally C^*-algebra, we recall that a left multiplier of A is a linear map $l : A \to A$ such that $l(ab) = l(a)b$ for all a and b in A. We know that any left multiplier of a C^*-algebra is automatically continuous. We will show that this result is still valid for left multipliers of a locally C^*-algebra. Recall that in

[**11**], Weinder showed that the multipliers of a locally C^*-algebra are automatically continuous.

Lemma 3.3. *Let a be an element of a locally C^*-algebra A. If $0 < \alpha < 1$, then there is an element u in A such that $a = u\,|a|^{\alpha}$, where $|a|^2 = aa^*$.*

Proof. We know that for each p in $S(A)$, there is an element u_p in A_p such that $\pi_p(a) = u_p|\pi_p(a)|^{\alpha}$. Moreover, $u_p = \lim_n \pi_p(a)\left(\frac{1}{n} + |\pi_p(a)|^2\right)^{\frac{-1}{2}} |\pi_p(a)|^{1-\alpha}$ (see, for example, [**8**, 1.4.6]).

To show that $(u_p)_p$ is a coherent sequence in $A_p,\ p \in S(A)$, let $p, q \in S(A)$ with $p \geq q$. Since π_{pq} preserves spectral functions, we have

$$\begin{aligned} \pi_{pq}\left(u_p\right) &= \lim_n \pi_{pq}\left(\pi_p(a)\left(\frac{1}{n} + |\pi_p(a)|^2\right)^{\frac{-1}{2}} |\pi_p(a)|^{1-\alpha}\right) \\ &= \lim_n \pi_q(a)\left(\frac{1}{n} + |\pi_q(a)|^2\right)^{\frac{-1}{2}} |\pi_q(a)|^{1-\alpha} \\ &= u_q. \end{aligned}$$

Hence $(u_p)_p$ is a coherent sequence in $A_p,\ p \in S(A)$. Let u in A be such that $\pi_p(u) = u_p$ for all $p \in S(A)$. Then, since $\pi_p(|a|^{\alpha}) = |\pi_p(a)|^{\alpha}$ for all $p \in S(A)$ (see [**9**] or [**2**]), we have $a = u\,|a|^{\alpha}$. ☐

Proposition 3.4. *Any left multiplier of a locally C^*-algebra A is automatically continuous.*

Proof. Let l be a left multiplier of A, let $p \in S(A)$ and $a \in \ker(p)$. By Lemma 3.3, there is $u \in A$ such that $a = u|a|^{\frac{1}{2}}$, and then

$$p(l(a)) = p(l(u)|a|^{\frac{1}{2}}) \leq p(l(u))p(a)^{\frac{1}{2}}$$

whence we conclude that $l(a) \in \ker(p)$. Hence there is a unique linear map $l_p : A_p \to A_p$ such that $\pi_p \circ l = l_p \circ \pi_p$. Moreover, l_p is a left multiplier of A_p and so it is continuous (see, for example, [**8**, 3.12.2]). From these facts we conclude that l is continuous and the proposition is proved. ☐

We consider on $LM(A)$, the set of all left multipliers of A, the seminorm topology (that is the topology determined by that family of seminorms $\{\widetilde{p}\}_{p\in S(A)}$, where $\widetilde{p}(l) = \sup\{p(l(a)),\, a \in A \text{ and } p(a) \leq 1\}$).

Theorem 3.5. *Let A be a locally C^*-algebra. Then we have:*

(1) *$LM(A)$ is a complete locally m-convex algebra.*
(2) *If $A = \lim\limits_{\lambda\in\Lambda\leftarrow} A_\lambda$ and the canonical maps $\pi_\lambda : A \to A_\lambda$ are all surjective, then the locally m-convex algebras $LM(A)$ and $\lim\limits_{\lambda\in\Lambda\leftarrow} LM(A_\lambda)$ are isomorphic.*

Proof. To prove this theorem we use the same arguments as in the proof of Theorem 3.14 of [**9**].

(1): Let $p, q \in S(A)$ with $p \geq q$. Since π_{pq} is surjective, there is a unique morphism $\pi''_{pq} : A''_p \to A''_q$ which extends π_{pq} and $\pi''_{pq}(LM(A_p)) \subseteq LM(A_q)$ (see, for example, [**8**, 3.7.7 and 3.12]). Then $\{LM(A_p); \pi''_{pq}|_{LM(A_p)}, p \geq q, p, q \in S(A)\}$ is an inverse system of Banach algebras. It is not difficult to check that the map $\Psi : LM(A) \to \lim\limits_{p\leftarrow} LM(A_p)$ defined by $\Psi(l) = (l_p)_p$, where $\pi_p \circ l = l_p \circ \pi_p$ for all $p \in S(A)$, is an isomorphism of locally m-convex algebras.

(2): Exactly as in the proof of Theorem 3.14 of [**9**] we show that the inverse systems $\{LM(A_\lambda)\}_{\lambda\in\Lambda}$ and $\{LM(A_p)\}_{p\in S(A)}$ have the same inverse limit and thus the assertion is proved. □

The following theorem is a generalization of Theorem 1.5 of [**6**] in the context of Hilbert modules over locally C^*-algebras.

Theorem 3.6. *Let A be a locally C^*-algebra and let E be a Hilbert A-module. Then the locally m-convex algebras $B_A(E)$ and $LM(K_A(E))$ are isomorphic.*

Proof. Let $p, q \in S(A)$ with $p \geq q$. Since $(\pi_{pq})_* (\theta_{y,x}) = \theta_{\sigma_{pq}(y),\sigma_{pq}(x)}$ for all $x, y \in E_p$, and since the map σ_{pq} from E_p to E_q is surjective, the morphism $(\pi_{pq})_*$ from $K_{A_p}(E_p)$ to $K_{A_q}(E_q)$ is surjective. Then according to Theorem 3.5 (2), the locally m-convex algebras $LM(K_A(E))$ and $\lim\limits_{p\leftarrow} LM(K_{A_p}(E_p))$ are isomorphic.

For each $p \in S(A)$, the map $\Phi_p : B_{A_p}(E_p) \to LM(K_{A_p}(E_p))$ defined by $\Phi_p(T_p)(S_p) = T_p \circ S_p$ is an isometric isomorphism of Banach algebras [**6**, Theorem 1.5]. It is easy to check that $(\Phi_p)_p$ is an inverse system of isometric isomorphisms of Banach algebras. Then $\lim\limits_{p\leftarrow} \Phi_p$ is an isomorphism of locally m -convex algebras from $\lim\limits_{p\leftarrow} B_{A_p}(E_p)$ onto $\lim\limits_{p\leftarrow} LM(K_{A_p}(E_p))$ and the theorem is proved. □

We say that an element T of $B_A(E, F)$ is bounded in $B_A(E, F)$ if there is $M > 0$ such that $\widetilde{p}(T) \leq M$ for all $p \in S(A)$ and denote by $b(B_A(E, F))$ the set of all bounded elements in $B_A(E, F)$. It is clear that the map $\|\cdot\|_\infty : b(B_A(E, F)) \to [0, \infty)$ defined by

$$\|T\|_\infty = \sup\{\widetilde{p}(T); p \in S(A)\}$$

is a norm on $b(B_A(E, F))$.

Theorem 3.7. *If E and F are Hilbert A-modules, then $b(B_A(E, F))$ is a Banach space in the norm $\|\cdot\|_\infty$. Moreover, $b(B_A(E, F))$ is isometrically isomorphic to $B_{b(A)}(b(E), b(F))$.*

Proof. Let $T \in b(B_A(E, F))$. Then, since

$$\overline{p}_F(Tx) \leq \|T\|_\infty \|x\|_\infty$$

for every $x \in b(E)$ and for every $p \in S(A)$, $T(b(E)) \subseteq b(F)$ and it is easy to see that the restriction $T|_{b(E)}$ of T on $b(E)$ is an element in $B_{b(A)}(b(E), b(F))$. Moreover, $\left\|T|_{b(E)}\right\| \leq \|T\|_\infty$. On the other hand, since $b(E)$ is dense in E [**4**, Proposition 3.1], and since

$$\left\langle T|_{b(E)}x, T|_{b(E)}x\right\rangle \leq \left\|T|_{b(E)}\right\|^2 \langle x, x\rangle$$

for every $x \in b(E)$ (cf. [**7**, 2.8]), we have $\|T\|_\infty \leq \left\|T|_{b(E)}\right\|$. Hence $\|T\|_\infty = \left\|T|_{b(E)}\right\|$. Define $\Psi : b(B_A(E,F)) \to B_{b(A)}(b(E), b(F))$ by

$$\Psi(T) = T|_{b(E)}.$$

Clearly Ψ is an isometric morphism from $b(B_A(E,F))$ to $B_{b(A)}(b(E), b(F))$. To show that Ψ is surjective, let $S \in L(b(E), b(F))$. Since

$$\langle Sx, Sx\rangle \leq \|S\|^2 \langle x, x\rangle$$

for all x in $b(E)$ (cf. [**7**, 2.8]) and $b(E)$ is dense in E, S can be extended to a bounded A-module map $\widetilde{S}$ from E to F. Moreover, since $\widetilde{p}(\widetilde{S}) \leq \|S\|$ for all $p \in S(A)$, $\widetilde{S}$ is a bounded element in $B_A(E,F)$. Hence Ψ is surjective.

Thus we showed that $b(B_A(E,F))$ is isometrically isomorphic to $B_{b(A)}(b(E), b(F))$, and so $b(B_A(E,F))$ is a Banach space. □

It is easy to check that an element T in $b(B_A(E,F))$ is adjointable if and only if $T|_{b(E)}$ is adjointable.

Remark 3.8. *The restriction of Ψ on $b(L_A(E,F))$ is an isometric isomorphism from $b(L_A(E,F))$ onto $L_{b(A)}(b(E), b(F)))$.*

Knowing that for each $p \in S(A)$, $\widetilde{p}$ is a submultiplicative seminorm on $B_A(E)$ and $\widetilde{p}|_{L_A(E)}$ is a C^*-seminorm on $L_A(E)$, it is easy to see that $\|\cdot\|_\infty$ is a submultiplicative norm on $b(B_A(E))$ and a C^*-norm on $b(L_A(E))$.

Corollary 3.9. *Let A be a locally C^*-algebra and let E be a Hilbert A-module. Then we have:*

(1) *$b(B_A(E))$ with the norm $\|\cdot\|_\infty$ is a Banach algebra which is isometrically isomorphic to $B_{b(A)}(b(E))$.*

(2) *$b(L_A(E))$ with the norm $\|\cdot\|_\infty$ is a C^*-algebra which is isomorphic to $L_{b(A)}(b(E))$* [**4**, Theorem 3.3].

Proof. Putting $F = E$ in Theorem 3.7, it is easy to verify that Ψ is an isometric isomorphism from $b(B_A(E))$ onto $B_{b(A)}(b(E))$ and the restriction Ψ on $b(L_A(E))$ is an isomorphism from $b(L_A(E))$ onto $L_{b(A)}(b(E))$. □

Remark 3.10. Let E and F be two Hilbert A-modules. In general, $b(K_A(E,F))$ is not isomorphic to $K_{b(A)}(b(E), b(F))$.

Example. Let $A = C(\mathbb{Z}^+)$, the $*$-algebra of all complex valued functions on $\mathbb{Z}^+$. It is not difficult to see that A is just $\prod_{n=1}^{\infty} \mathbb{C}$. Also it is not difficult to check that A with the topology determined by the family of C^*-seminorms $\{p_n\}_n$, where $p_n((a_n)_n) = \sup\{|a_k|; 1 \leq k \leq n\}$, is a locally C^*-algebra, and A_{p_n} can be identified with the product of the first n factors of A for each n.

Let $E = \prod_{n=1}^{\infty} \mathbb{C}^n$. We make E into a Hilbert A-module via $(\xi_n)_n (a_n)_n = (\xi_n a_n)_n$ and $\langle (\xi_n)_n, (\eta_n)_n\rangle = (\langle \xi_n, \eta_n\rangle_n)_n$, where $\langle \cdot, \cdot\rangle_n$ denotes the usual $\mathbb{C}$-inner product on $\mathbb{C}^n$. Clearly E is not finitely generated as Hilbert A-module. Moreover, E_{p_n}

can be identified with the product of the first n factors of E for each n. Therefore, $L_{A_{p_n}}(E_{p_n}) = K_{A_{p_n}}(E_{p_n})$ for each n. This implies that $L_A(E) = K_A(E)$ [**9**, Example 4.9], and by Corollary 3.9, $b(K_A(E))$ is isomorphic with $L_{b(A)}(b(E))$.

Suppose that $b(K_A(E))$ is isomorphic with $K_{b(A)}(b(E))$. Then the C^*-algebras $K_{b(A)}(b(E))$ and $L_{b(A)}(b(E))$ are isomorphic. This implies that $b(E)$ is finitely generated as Hilbert $b(A)$-module [**10**] and so E is finitely generated as Hilbert A-module, a contradiction. Therefore $b(K_A(E))$ is not isomorphic with $K_{b(A)}(b(E))$.

Remark 3.11. If A is a locally C^*-algebra then A is a Hilbert A-module with $\langle a, b\rangle = a^*b$, $a, b \in A$ and the locally C^*-algebras $L_A(A)$ and $M(A)$, where $M(A)$ is the set of all multipliers of A, are isomorphic [**9**]. Putting $E = A$ in Corollary 3.9, we deduce that the C^*-algebras $M(b(A))$ and $b(M(A))$ are isomorphic, a result obtained independently by Bhatt and J. Karia [**1**, Theorem 5.1] and the author [**3**, Theorem 2].

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