Acta Math. Univ. Comenianae
Vol. LXXV, 2(2006), pp. 219–225

ON THE COMPUTATION OF MINIMAL REDUCTION

E. BOĎA AND I. ŠUŇAL

Let $P := k\,[X, Y, Z]$ be a polynomial ring over an algebraic closed field k and $(X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f)\cdot k\,[X, Y, Z]$ an (X, Y, Z)-primary ideal in P, (m, n, l, a, b, c, d, e, f are integers). The ideal $Q = (X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f)\cdot R$ is $(X, Y, Z)\cdot R$-primary ideal in the local ring $R = k\,[X, Y, Z]_{(x,y,z)}$. In this short note we give a formula for the calculation of Samuel multiplicity $e_0(Q, R)$ of the ideal Q in R. Remark, that the multiplicity $e_0(Q, R)$ is the leading coefficient in the Hilbert-Samuel polynomial $P(n) = l(R/Q^n)$, where $l(R/Q^n)$ is the length of the R-module R/Q^n. We use the notion of a reduction of ideal for the proof of a main theorem. We say, that the ideal $\overline{q}$ is a reduction of the m-primary ideal q in the local ring (A, m), if $\overline{q} \subset q$ and for same integer $n \in N$ it holds $\overline{q}\cdot q^n = q^{n+1}$. If $\overline{q}$ is the reduction of the ideal q in A then we know that $e_0(q, A) = (\overline{q},\ A)$ [**5**, Theorem 1].

Let's formulate the first statement of this note. For the monomial ideal $Q = (X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f)\cdot R$ we set

$$Q_1 = (X^m, Y^n, Z^l)\cdot R,$$
$$Q_2 = (X^m + Y^n, X^m + Z^l, X^aY^bZ^c)\cdot R,$$
$$Q_3 = (X^m + Y^n, X^m + Z^l, X^dY^eZ^f)\cdot R,$$
$$Q_4 = (X^m - Y^n, X^m - X^dY^eZ^f, Z^l - X^aY^bZ^c)\cdot R,$$
$$Q_5 = (Y^n - Z^l, Y^n - X^aY^bZ^c, X^m - X^dY^eZ^f)\cdot R,$$
$$Q_6 = (X^m - Z^l, X^m - X^aY^bZ^c, Y^n - X^dY^eZ^f)\cdot R.$$

Further define

$$\alpha_1 = mnl,$$
$$\alpha_2 = nla + mlb + mnc,$$
$$\alpha_3 = nld + mle + mnf,$$
$$\alpha_4 = nld + mle + mnc + n(af - cd) + m(bf - ce),$$
$$\alpha_5 = nld + mlb + mnc + n(af - cd) + l(ae - bd),$$
$$\alpha_6 = nla + mle + mnc + m(bf - ce) + l(bd - ae)$$

Received May 12, 2005.
2000 *Mathematics Subject Classification.* Primary 13H15; Secondary 13A17.
Key words and phrases. Multiplicity, minimal reduction.

and finally $\mathfrak{M} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$.

Theorem 1. *Let $Q = (X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f) \cdot R$ be an $(X, Y, Z) \cdot R$-primary ideal in the local polynomial ring $R = k\,[X, Y, Z]_{(x,y,z,)}$ (where $m > a > d$, $n > b \geq e$, $l > f > c$ w.l.g.). For all $i \in \{1, 2, 3, 4, 5, 6\}$ we have:*

$$\text{If } \alpha_i = \min \mathfrak{M} \text{ then } Q_i \text{ is a reduction of } Q.$$

Proof. Let $\alpha_1 = \min \mathfrak{M}$. Hence

$$\alpha_1 \leq \alpha_2, \quad \alpha_1 \leq \alpha_3.$$

Further let

$$G_1(T_1, T_2, T_3, T_4, T_5) = T_4^{\alpha_2} - X^{a(\alpha_2-\alpha_1)}Y^{b(\alpha_2-\alpha_1)}Z^{c(\alpha_2-\alpha_1)} \cdot T_1^{nla}T_2^{mlb}T_3^{mnc},$$
$$G_2(T_1, T_2, T_3, T_4, T_5) = T_5^{\alpha_3} - X^{d(\alpha_3-\alpha_1)}Y^{e(\alpha_3-\alpha_1)}Z^{f(\alpha_3-\alpha_1)} \cdot T_1^{nld}T_2^{mle}T_3^{mnf}$$

be the polynomials of $R[T_1, T_2, T_3, T_4, T_5]$. It is clear that

$$G_1(X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f) = G_2(X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f) = 0.$$

Let's idetify G_i^* with $G_i \cdot R[T_1, T_2, T_3, T_4, T_5]/m$, so

$$G_1^*(T_1, T_2, T_3, T_4, T_5) = T_4^{\alpha_2} \qquad (\text{resp. } T_4^{\alpha_2} - T_1^{nla}T_2^{mlb}T_3^{mnc} \text{ if } \alpha_1 = \alpha_2),$$
$$G_2^*(T_1, T_2, T_3, T_4, T_5) = T_5^{\alpha_3} \qquad (\text{resp. } T_5^{\alpha_3} - T_1^{nld}T_2^{mle}T_3^{mnf} \text{if } \alpha_1 = \alpha_3).$$

Let $I = (G_1^*, G_2^*) \cdot k[T_1, T_2, T_3, T_4, T_5]$. Then for all possibility of the choice of G_i^* the ideal $I + (T_1, T_2, T_3) \cdot k[T_1, T_2, T_3, T_4, T_5]$ is $(T_1, T_2, T_3, T_4, T_5)$-primary. Hence the ideal $Q_1 = (X^m, Y^n, Z^l) \cdot R$ is the reduction of Q by Proposition of [**2**]. For the rest five cases we denote

$$G_3 = T_1^{nla}T_2^{mlb}T_3^{mnc} - X^{a(\alpha_1-\alpha_2)}Y^{b(\alpha_1-\alpha_2)}Z^{c(\alpha_1-\alpha_2)} \cdot T_4^{\alpha_2}$$
$$G_4 = T_5^{na} - Z^{(\alpha_5-\alpha_2)} \cdot T_2^{(ae-bd)}T_3^{(na-ae+bd-nd)} \cdot T_4^{nd} \qquad \text{if } ae \geq bd$$
$$G_5 = T_5^{mb} - Z^{(\alpha_6-\alpha_2)} \cdot T_1^{(bd-ae)}T_3^{(mb-bd+ae-me)} \cdot T_4^{me} \qquad \text{if } ae \leq bd$$
$$G_6 = T_1^{nld}T_2^{mle}T_3^{mnf} - X^{d(\alpha_1-\alpha_3)}Y^{e(\alpha_1-\alpha_3)}Z^{f(\alpha_1-\alpha_3)} \cdot T_4^{\alpha_3}$$
$$G_7 = T_4^{\alpha_2} - X^{a(\alpha_2-\alpha_1)}Y^{b(\alpha_2-\alpha_1)}Z^{c(\alpha_2-\alpha_1)} \cdot T_1^{nla}T_2^{mlb}T_3^{mnc} \qquad \text{if } \alpha_2 \geq \alpha_1$$
$$G_8 = T_4^{(mn-nd-me)} - Z^{(\alpha_4-\alpha_3)} \cdot T_1^{(na-nd+bd-ae)}T_2^{(mb-me+ae-bd)}T_5^{(mn-na-mb)} \qquad \text{if } \alpha_1 \geq \alpha_2$$
$$G_9 = T_1^{(na-nd+bd-ae)}T_2^{(mb-me+ae-bd)}T_5^{(mn-na-mb)} - Z^{\alpha_3-\alpha_4}T_4^{(mn-nd-me)}$$
$$G_{10} = T_3^{(mb-me+ae-bd)}T_4^{me} - Z^{\alpha_5-\alpha_4} \cdot T_1^{(ae-bd)}T_5^{mb} \qquad \text{if } ae \geq bd$$
$$G_{11} = T_3^{(na-nd+bd-ae)}T_4^{nd} - Z^{\alpha_6-\alpha_4} \cdot T_2^{(bd-ae)}T_5^{na} \qquad \text{if } ae \leq bd$$
$$G_{12} = T_2^{(ae-bd)}T_3^{(na-ae+bd-nd)}T_4^{nd} - Z^{\alpha_2-\alpha_5} \cdot T_5^{na}$$

$$G_{13} = T_1^{(ae-bd)} T_5^{mb} - Z^{\alpha_4-\alpha_5} \cdot T_3^{(mb-me+ae-bd)} T_4^{me}$$
$$G_{14} = T_1^{(bd-ae)} T_3^{(mb-bd+ae-me)} T_4^{me} - Z^{\alpha_2-\alpha_6} T_5^{mb}$$
$$G_{15} = T_2^{(bd-ae)} T_5^{na} - Z^{\alpha_4-\alpha_6} \cdot T_3^{(na-nd+bd-ae)} T_4^{nd}.$$

With the notions as in first part of the proof we concetrate the steps in the rest parts in the following table

$\min \mathfrak{M}$	G_i	I	Q_i
α_2	G_3, G_4, G_5	(G_3^*, G_4^*), resp. (G_3^*, G_5^*)	$Q_2=(X^m+Y^n, X^m+Z^l, X^aY^bZ^c)\cdot R$
α_3	G_6, G_7, G_8	(G_6^*, G_7^*), resp. (G_6^*, G_8^*)	$Q_3=(X^m+Y^n, X^m+Z^l, X^dY^eZ^f)\cdot R$
α_4	G_9, G_{10}, G_{11}	(G_9^*, G_{10}^*), resp. (G_9^*, G_{11}^*)	$Q_4=(X^m-Y^n, X^m-X^dY^eZ^f, Z^l-X^aY^bZ^c)\cdot R$
α_5	G_{12}, G_{13}	(G_{12}^*, G_{13}^*)	$Q_5=(Y^n-Z^l, Y^n-X^aY^bZ^c, X^m-X^dY^eZ^f)\cdot R$
α_6	G_{14}, G_{15}	(G_{14}^*, G_{15}^*)	$Q_6=(X^m-Z^l, X^m-X^aY^bZ^c, Y^n-X^dY^eZ^f)\cdot R$

what completes the proof. □

Let's prove the main theorem of this note.

Theorem 2. *Let* $Q = (X^m, Y^n, Z^l, X^aY^bZ^c, X^dY^eZ^f)\cdot R$ *be an* $m = (X,Y,Z)\cdot R$*-primary ideal in the local polynomial ring* $R = k[X,Y,Z]_{(x,y,z,)}$ *(where* $m > a > d$, $n > b \geq e$, $l > f > c$ *w.l.g.). Then*

$$e_0(Q,R) = \min \mathfrak{M}$$

Proof. We prove, that $e_0(Q_i,R) = \alpha_i$ for all $i \in \{1,2,3,4,5,6\}$. For $i = 1$

$$e_0(Q_1,R) = e_0((X^m,Y^n,Z^l)\cdot R, R) = mnl = \alpha_1$$

by [**4**, Chapter 7, Theorem 7]. For $i = 2$ we have

$$\begin{aligned} e_0(Q_2,R) &= e_0((X^m+Y^n, X^m+Z^l, X^aY^bZ^c)\cdot R, R) \\ &= e_0((Y^n,Z^l,X^a)\cdot R,R) + e_0((X^m,Z^l,Y^b)\cdot R,R) \\ &\quad + e_0((Y^n,X^m,Z^c)\cdot R,R) \\ &= nla + mlb + mnc = \alpha_2 \end{aligned}$$

by [**4**, Chapter 7, Theorem 7]. The same argument is applicable for $i = 3$. Now the case $i = 4$. For the ideal Q_4 we have

$$\begin{aligned}
e_0(Q_4, R) &= e_0((X^m - Y^n, X^m - X^d Y^e Z^f, Z^l - X^a Y^b Z^c) \cdot R, R) \\
&= e_0(X^m - Y^n, X^d(X^{m-d} - Y^e Z^f), Z^c(Z^{l-c} - X^a Y^b) \cdot R, R) \\
&= e_0((X^m - Y^n, X^d, Z^c) \cdot R, R) \\
&\quad + e_0((X^m - Y^n, X^{m-d} - Y^e Z^f, Z^c) \cdot R, R) \\
&\quad + e_0((X^m - Y^n, X^d, Z^{l-c} - X^a Y^b) \cdot R, R) \\
&\quad + e_0((X^m - Y^n, X^{m-d} - Y^e Z^f, Z^{l-c} - X^a Y^b) \cdot R, R) \\
&= ndc + n(m-d)c + nd(l-c) \\
&\quad + e_0((X^m - Y^n, X^{m-d} - Y^e Z^f, Z^{l-c} - X^a Y^b) \cdot R, R)
\end{aligned}$$

by the argument above. Let's obserbe the polynomial $X^m - Y^n$. Let $r = \gcd(m, n)$, $m = \overline{m} \cdot r$, $n = \overline{n} \cdot r$. As the filed k is algebraic closed there are $\varsigma_1, \varsigma_2, ..., \varsigma_r \in k$ such that

$$X^m - Y^n = (X^{\overline{m}})^r - (Y^{\overline{n}})^r = \prod_{i=1}^{r} (X^{\overline{m}} - \varsigma_i \cdot Y^{\overline{n}}).$$

As

$$\begin{aligned}
&e_0(\prod_{i=1}^{r} (X^{\overline{m}} - \varsigma_i \cdot Y^{\overline{n}}), X^{m-d} - Y^e Z^f, Z^{l-c} - X^a Y^b) \cdot R, R) \\
&\qquad = \sum_{i=1}^{r} e_0((X^{\overline{m}} - \varsigma_i \cdot Y^{\overline{n}}, X^{m-d} - Y^e Z^f, Z^{l-c} - X^a Y^b) \cdot R, R),
\end{aligned}$$

we can assume the integers m and n are not divisible. Then the surface given by $X^m - Y^n = 0$ have the following parametric representation

$$\begin{aligned}
X &= t^n \\
Y &= t^m \\
Z &= s.
\end{aligned}$$

Now the module $k\,[s, t]$ is finite over $k\,[t^n, t^m, s]$ (as s and t are integral over $k\,[t^n, t^m, s]$). Further $s, t \in k(t^n, t^m, s)$ (as m and n are not divisible), so $k(t^n, t^m, s) = k(s, t)$. By Proposition 3 below then we have

$$\begin{aligned}
&e_0((X^m - Y^n, X^{m-d} - Y^e Z^f, Z^{l-c} - X^a Y^b) \cdot R, R) \\
&\qquad = e_0((t^{n(m-d)} - t^{me} s^f, s^{l-c} - t^{na+mb}) \cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)}) \\
&\qquad = e_0((t^{me}(s^f - t^{n(m-d)-me}), s^{l-c} - t^{na+mb}) \cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)}) \\
&\qquad = m \cdot e \cdot (l-c) + e_0((s^f - t^{n(m-d)-me}, t^{na+mb} - s^{l-c}) \\
&\qquad\quad \cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)}) \\
&\qquad = me \cdot (l-c) + \min\{f(na+mb), (l-c)(mn - nd - me)\} \\
&\qquad = me \cdot (l-c) + f \cdot (na+mb).
\end{aligned}$$

by [**1**, Theorem 3]. The unequality $f(na+mb) < (l-c)(mn-nd-me)$ is equivalent to $nld + mle + mnc + n(af-cd) + m(bf-ce) < mnl$ and this is true because $\alpha_4 = \min \mathfrak{M}$. So we have

$$\begin{aligned} e_0(Q_4, R) &= ndc + (m-d)\cdot nc + nd\cdot(l-c) + me\cdot(l-c) + f\cdot(na+mb) \\ &= nld + mle + mnc + n(af-cd) + m(bf-ce) = \alpha_4. \end{aligned}$$

For $i = 5$ we have

$$\begin{aligned} e_0(Q_5, R) &= e_0((Y^n - Z^l, Y^n - X^aY^bZ^c, X^m - X^dY^eZ^f)\cdot R, R) \\ &= e_0((Y^n - Z^l, Y^b, X^d)\cdot R, R) + e_0((Y^n - Z^l, X^aZ^c - Y^{n-b}, X^d)\cdot R, R) \\ &\quad + e_0((Y^n - Z^l, Y^b, Y^eZ^f - X^{m-d})\cdot R, R) \\ &\quad + e_0((Y^n - Z^l, X^aZ^c - Y^{n-b}, Y^eZ^f - X^{m-d})\cdot R, R) \\ &= lbd + ld\cdot(n-b) + lb\cdot(m-d) \\ &\quad + e_0((Y^n - Z^l, X^aZ^c - Y^{n-b}, Y^eZ^f - X^{m-d})\cdot R, R). \end{aligned}$$

Let's observe the surface $Y^n - Z^l = 0$. With the same argument as before we may assume that $\gcd(n,l) = 1$, so the rational parametrisation of this surface is given by

$$\begin{aligned} X &= s \\ Y &= t^l \\ Z &= t^n \end{aligned}$$

Now (as the condition of Proposition 3 are satisfied) we have

$$\begin{aligned} &e_0((Y^n - Z^l, X^aZ^c - Y^{n-b}, Y^eZ^f - X^{m-d})\cdot R) \\ &\qquad = e_0((s^at^{nc} - t^{l(n-b)}, t^{le+nf} - s^{m-d})\cdot k[s,t]_{(s,t)}) \\ &\qquad = e_0((t^{nc}(s^a - t^{l(n-b)-nc}), t^{le+nf} - s^{m-d})\cdot k[s,t]_{(s,t)}) \\ &\qquad = nc(m-d) + e_0((s^a - t^{l(n-b)-nc}, t^{le+nf} - s^{m-d})\cdot k[s,t]_{(s,t)}) \\ &\qquad = nc(m-d) + \min\{a(le+nf), (m-d)(nl-lb-nc)\} \\ &\qquad = nc(m-d) + a(le+nf), \end{aligned}$$

as

$$a(le+nf) < (m-d)(nl-lb-nc)$$

(equivalent to $nld + mlb + mnc + n(af-cd) + l(ae-bd) < mnl$). So for the multiplicity of Q_5 it holds

$$\begin{aligned} e_0(Q_5, R) &= lbd + ld\cdot(n-b) + lb\cdot(m-d) + nc\cdot(m-d) + a\cdot(le+nf) \\ &= nld + mlb + mnc + n(af-cd) + l(ae-bd) = \alpha_5. \end{aligned}$$

We finish the proof with the last case $i = 6$.

$$\begin{aligned} e_0(Q_6, R) = {} & e_0((X^m - Z^l, X^m - X^aY^bZ^c, Y^n - X^dY^eZ^f) \cdot R, R) \\ = {} & e_0((X^m - Z^l, X^a, Y^e) \cdot R, R) \\ & + e_0((X^m - Z^l, X^a, X^dZ^f - Y^{n-e}) \cdot R, R) \\ & + e_0((X^m - Z^l, Y^bZ^c - X^{m-a}, Y^e) \cdot R, R) \\ & + e_0((X^m - Z^l, Y^bZ^c - X^{m-a}, X^dZ^f - Y^{n-e}) \cdot R, R) \\ = {} & lae + la \cdot (n-e) + le \cdot (m-a) \\ & + e_0((X^m - Z^l, Y^bZ^c - X^{m-a}, X^dZ^f - Y^{n-e}) \cdot R, R). \end{aligned}$$

Applying the former method (for the surface $X^m - Z^l = 0$) we obtain

$$\begin{aligned} & e_0((X^m - Z^l, Y^bZ^c - X^{m-a}, X^dZ^f - Y^{n-e}) \cdot R, R) \\ & = e_0((s^bt^{mc} - t^{l(m-a)}, t^{ld+mf} - s^{n-e}) \cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)}) \\ & = e_0((t^{mc}(s^b - t^{l(m-a)-mc}), t^{ld+mf} - s^{n-e}) \cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)}) \\ & = mc(n-e) + \min\{b(ld+mf), (n-e)(ml - la - mc)\} \\ & = mc(n-e) + b(ld+mf), \end{aligned}$$

as $b(ld+mf) < (n-e)(ml-la-mc)$, so we have

$$\begin{aligned} e_0(Q_6, R) & = lae + la \cdot (n-e) + le \cdot (m-a) + mc(n-e) + b(ld+mf) \\ & = nla + mnc + mle + m(bf - ce) + l(bd - ae) = \alpha_6, \end{aligned}$$

what completes the proof. □

Proposition 3. *Let $F(X,Y,Z)$, $G(X,Y,Z)$, $H(X,Y,Z)$ denote the polynomials in the polynomial ring $P = k[X,Y,Z]$ (k algebraic closed) such that the ideal $Q = (F,G,H) \cdot P$ is $(X,Y,Z) \cdot P$-primary, $R = k[X,Y,Z]_{(X,Y,Z)}$. Let the surface W in k^3 given by: $F(X,Y,Z) = 0$ has rational parametrization*

$$\begin{aligned} X &= u_1(s,t) \\ Y &= u_2(s,t) \\ Z &= u_3(s,t) \end{aligned}$$

such that the module $k[u_1,u_2,u_3]$ is finite over $k[s,t]$ and $u_i(s,t)$ are polynomial in $k[s,t]$. Assume in addition that surface W is birational isomorph to the plane $(t \cdot m \cdot k(u_1,u_2,u_3) = k(s,t))$. Then

$$e_0(Q,R) = e_0((G(u_1,u_2,u_3), H(u_1,u_2,u_3)).k[s,t]_{(s,t)}, k[s,t]_{(s,t)})$$

Proof. Let's construct following homomorphism of polynomial rings

$$\begin{aligned} \phi : k[X,Y,Z] &\longrightarrow k[s,t] \\ X &\longrightarrow u_1(s,t) \\ Y &\longrightarrow u_2(s,t) \\ Z &\longrightarrow u_3(s,t) \end{aligned}$$

The kernel of ϕ is the ideal $(F).k[X,Y,Z]$, so there is a monomorphism

$$k[X,Y,Z]/(F)\cdot k[X,Y,Z]. \cong k[u_1,u_2,u_3] \hookrightarrow k[s,t]$$

and within also the local monomorphism of local rings

$$R/(F)\cdot R \;\cong k[u_1,u_2,u_3]_{(u_1,u_2,u_3)} \hookrightarrow k[s,t]_{(s,t)}$$

Let's apply the additive formula for multiplicity [**3**, Chapter 14]. By assumptions the module $k[u_1,u_2,u_3]_{(u_1,u_2,u_3)}$ is finite over $[s,t]_{(s,t)}$ and $[k(u_1,u_2,u_3):k(s,t)] = 1$. So by Theorem 14.7 of cited book is

$$e_0(Q\cdot R/(F)\cdot R, R/(F)\cdot R) = e_0((G(u_1,u_2,u_3),H(u_1,u_2,u_3))\cdot k[s,t]_{(s,t)}, k[s,t]_{(s,t)})$$

As the ideal Q in R is a parameter ideal, we have

$$e_0(Q,R) = e_0(Q\cdot R/(F)\cdot R, R/(F)\cdot R)$$

and thereby is the proof complete. □

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E. Boďa, KAGDM, Faculty of Mathematics, Physics and Informatics Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, *e-mail*: `bodae@fmph.uniba.sk`

I. Šuňal, KAGDM, Faculty of Mathematics, Physics and Informatics Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, *e-mail*: `igor.sunal@fmph.uniba.sk`