ON (k,l)-RADIUS OF RANDOM GRAPHS

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Abstract. We introduce the concept of (k,l)-radius of a graph and prove that for any fixed pair k,l the (k,l)-radius is equal to $2\binom{k}{2} - \binom{l}{2}$ for almost all graphs. Since for $k = 2$ and $l = 0$ the (k,l)-radius is equal to the diameter, our result is a generalization of the known fact that almost all graphs have diameter two.

All graphs in this note are finite, undirected and simple. As usual, by *distance between two vertices* in a graph we mean the minimum length of a path connecting them. Then the *diameter* is the maximum distance between two vertices. The *transmission* of the graph, also called a *distance of the graph*, is defined as the sum of distances between all pairs of vertices (for general properties of the distance see [4]). The concepts of diameter and distance were generalized by Goddard, Swart and Swart in [3] by introducing the k-*diameter* as follows. The distance of k vertices $d_k(v_1, v_2, \ldots, v_k)$ is the sum of distances between all pairs of vertices from $\{v_1, v_2, \ldots, v_k\}$. The k-*diameter* is the maximum distance of a set of k vertices. Hence the 2-*diameter* is the usual diameter and if n is the order of the graph, the n-*diameter* is the distance of the graph.

In this note we use the definition of distance of a set of k vertices to define (k,l)-*eccentricity* and (k,l)-*radius*. We study (k,l)-*radius* of random graphs and determine the value of this parameter for almost all graphs in a probability space. We also discus the relationship between the (k,l)-*radius* and the k-*diameter* of a graph.

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Let S be a set of l vertices, $0 \leq l \leq k$. We define (k,l)-*eccentricity* of S, $e_{k,l}(S)$, as the maximum distance of k vertices $u_1, u_2, \ldots, u_k$, such that $S \subseteq \{u_1, u_2, \ldots, u_k\}$. In symbols,

$$e_{k,l}(S) = \max_T \{d_k(T), |T| = k, S \subseteq T \subseteq V(G)\}.$$

The (k,l)-*radius*, $rad_{k,l}(G)$, is the minimum (k,l)-eccentricity of a set of l vertices in G, that is

$$\mathrm{rad}_{k,l}(G) = \min_S(e_{k,l}(S)) = \min_S(\max_{S \subseteq T \subseteq V(G)} d_k(T))$$

where $|S| = l$, $|T| = k$.

We recall that the eccentricity $e(v)$ of a vertex v is the maximum distance to another vertex, the radius $\mathrm{rad}(G)$ is the minimum eccentricity, whereas the diameter $\mathrm{diam}(G)$ is the maximum eccentricity. From the definition of (k,l)-*radius* it follows that $\mathrm{rad}_{2,1}(G)$ is the usual radius and $\mathrm{rad}_{k,0}(G)$ is the k-*diameter*.

Now, consider the probability space in the following sense. Let p be a real number, $0 < p < 1$, and let n be an integer. By $G(n,p)$ we denote a class of labelled random graphs on n vertices, in which the probability of an edge equals p. More precisely, for every $u, v \in V(G)$, we have $P[uv \in E(G)] = p$. Hence $G(n,p)$ is a probability space the elements of which are the $2^{\binom{n}{2}}$ differently labelled graphs. We say that almost all graphs have property A if

$$\lim_{n \to \infty} P[G \in G(n,p) \text{ has property} A] = 1.$$

The space of random graphs is one of the random structures studied in connection with the 0-1 law. This law states that for many properties. The probability that a random structure satisfies the property is guaranteed to approach either 0 or 1. The 0-1 law for graphs was proved by Glebskij [2] and later on by Fagin [1]. Fagin's method is based on considering the following properties.

Let r and s be nonnegative integers. By $A_{r,s}$ we denote the property that for any disjoint sets of vertices X and Y, such that $|X| = r$ and $|Y| = s$, there exists a vertex z, $z \notin X \cup Y$ such that z is adjacent to every vertex of X and to no vertex of Y.

The following statements are well-known and their proofs can be found in the excellent survey by Winkler [5].

Theorem 1. [5] *For any fixed nonnegative integers r and s and a real number p, $0 < p < 1$ we have*

$$\lim_{n\to\infty} P[G \in G(n,p) \text{ has property } A_{r,s}] = 1.$$

Theorem 2. [5] *Let be $T = \{A_{r_1,s_1}, A_{r_2,s_2}, \ldots, A_{r_k,s_k}\}$ for some $k \geq 0$. Then almost all graphs have all the properties of T.*

From the fact that almost every graph has property $A_{2,0}$ (the distance of every pair of vertices is at most 2) and $A_{0,1}$ (the graph is not complete) we have:

Corollary 3. *For any fixed real p, $0 < p < 1$, almost all graphs are connected and have diameter 2.*

Now we are in a position to prove the main statement of this note.

Theorem 4. *Let k, l be nonnegative integers, $l \leq k$. For any fixed real p, $0 < p < 1$, almost all graphs G have*

$$\mathrm{rad}_{k,l}(G) = 2\binom{k}{2} - \binom{l}{2}.$$

Proof. Let L be a set of l vertices in a graph $G \in G(n,p)$. Let p_n denote the probability $P[G \in G(n,p)$ has diameter 2]. Then with the same probability p_n it holds

$$e_{k,l}(L) \leq d_l(L) + 2l(k-l) + 2\binom{k-l}{2}. \tag{1}$$

By Corollary 3 $\lim_{n\to\infty} p_n = 1$, so that (1) holds for almost all graphs $G \in G(n,p)$. Now we prove that for almost all graphs

$$e_{k,l}(L) \geq d_l(L) + 2l(k-l) + 2\binom{k-l}{2}. \tag{2}$$

To do this, it sufficies to prove that for almost all graphs there exist $k-l$ vertices from $V(G)\setminus L$ that are mutually nonadjacent and that are adjacent to no vertex of L. Let $T = \{A_{0,l}, A_{0,l+1}, \dots, A_{0,k-1}\}$. By Theorem 2, $\lim_{n\to\infty} P[G \in G(n,p)$ has all properties of $T] = 1$, i.e. almost all graphs G have all properties of T.

1. Let $L_l = L$. Property $A_{0,l}$ says that there exists a vertex $z_{l+1} \in V(G)\setminus L$ that is adjacent to no vertex of L.
2. For $i = l+1, l+2, \dots, k-1$ we define L_i inductively by $L_i = L_{i-1} \cup z_i$. Then property $A_{0,i}$ impies that there exists a vertex z_{i+1} that is adjacent to no vertex of L_i.

Hence, we have $k-l$ vertices $z_{l+1}, z_{l+2}, \dots, z_k$ that are mutually nonadjacent and are adjacent to no vertex of L_l, which proves (2). Thus, from (1) and (2) we have that for almost all graphs

$$e_{k,l}(L) = d_l(L) + 2l(k-l) + 2\binom{k-l}{2}. \tag{3}$$

Since $\mathrm{rad}_{k,l}(G) = \min_L e_{k,l}(L)$, the radius is minimal whenever $d_l(L)$ is minimal, (see (3)). We show that in almost all graphs $G \in G(n,p)$ there exists a set L' of l vertices, such that $d_l(L') = \binom{l}{2}$. In other words, we show that there is a set L' of l mutually adjacent vertices. Let $T' = \{A_{1,0}, A_{2,0}, \dots, A_{l-1,0}\}$. Then almost all graphs have all properties of T', since by Theorem 2 $\lim_{n\to\infty} P[G \in G(n,p)$ has all properties of $T'] = 1$.

1. Let L'_1 be a set containing a single vertex of G, say $L'_1 = \{z'_1\}$. Then $|L'_1| = 1$ and $A_{1,0}$ says that there exists a vertex z'_2 that is adjacent to z'_1 .
2. For $i = 2, 3, \dots l-1$ let L'_i be a set of vertices, such that $L'_i = L'_{i-1} \cup z'_i$. Then $|L'_i| = i$ and $A_{i,0}$ implies that there exists a vertex z'_{i+1} that is adjacent to all vertices of L'_i.

In this way we obtain a set $L' = L'_l$ of l vertices that are mutually adjacent, so that $d_l(L') = \binom{l}{2}$. Since $d_l(L)$ cannot be less then $\binom{l}{2}$ for any set of l vertices, we have

$$\operatorname{rad}_{k,l} = \binom{l}{2} + 2l(k-l) + 2\binom{k-l}{2} = 2\binom{k}{2} - \binom{l}{2}$$

for almost all graphs $G \in G(n,p)$, as required. □

Setting $l = 0$ in Theorem 2 we obtain:

Corollary 5. *For any $k \geq 0$ and for almost all graphs G we have*

$$\operatorname{diam}_k(G) = k(k-1).$$

It is obvious that Corollary 5 generalizes Corollary 3. Further, setting $k = 2$ and $l = 1$ we obtain:

Corollary 6. *For almost all graphs G we have* $\operatorname{rad}(G) = 2$.

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