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ABOUT THE BIVARIATE OPERATORS OF KANTOROVICH TYPE

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Abstract. The aim of this paper is to study the convergence and approximation properties of the bivariate operators and GBS operators of Kantorovich type.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let us consider the operator $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} p_{m,k}(t) f(t) \mathrm{d}t, \tag{1.1}$$

for any $x \in [0,1]$, where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$. These operators were introduced in 1930 by L. V. Kantorovich in [**9**].

The purpose of this paper is to give a representation for the bivariate operators and GBS operators of Kantorovich type, to establish a convergence theorem for these operators. We also give an approximation theorem for these operators in terms of the first modulus of smoothness and of the mixed modulus of smoothness for bivariate functions.

In Section 2 we construct the bivariate operators of Kantorovich type. The method was inspired by the construction of Bernstein bivariate operators (see [**10**] or [**13**]). So, let the sets $\Delta_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x+y \leq 1\}$ and $\mathcal{F}(\Delta_2) = \{f | f : \Delta_2 \to \mathbb{R}\}$. For $m \in \mathbb{N}$, the operator $B_m : \mathcal{F}(\Delta_2) \to \mathcal{F}(\Delta_2)$

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defined for any function $f \in \mathcal{F}(\Delta_2)$ by

$$(1.3) \qquad (B_m f)(x,y) = \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y) f\left(\frac{k}{m},\frac{j}{m}\right)$$

for any $(x,y)\in\Delta_2$ is named the Bernstein bivariate operator, where

$$(1.4) \qquad p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j},$$

for any $k,j\ge 0$, $k+j\le m$ and any $(x,y)\in\Delta_2$. We also recall some properties for Bernstein bivariate operators.

In Section 3 we prove some identities and inequalities verified by the $(\mathcal{K}_m)_{m\ge1}$ operators.

In Section 4 we use Theorem 4.1 (a Shisha-Mond type theorem) for estimating the rate of convergence of the $(\mathcal{K}_m)_{m\ge1}$ operators in term of the ω_{total} modulus.

In Section 5 of our paper we give approximation theorems for B-continuous and B-differentiable functions by GBS operators, using the ω_{mixed} modulus.

2. The construct of the bivariate operators of Kantorovich type

For $m\in\mathbb{N}$, let the operator $\mathcal{K}_m : L_1([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ be defined for any function $f\in L_1([0,1]\times[0,1])$ by

$$(2.1) \qquad (\mathcal{K}_m f)(x,y) = (m+1)^2 \sum_{\substack{k,j=0\\ k+j\le m}} p_{m,k,j}(x,y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} f(s,t)\,\mathrm{d}s\mathrm{d}t$$

for any $(x,y)\in\Delta_2$. The operators defined above are called the bivariate operators of Kantorovich type. Clearly, the bivariate operators of Kantorovich type are linear and positive.

Let the functions $e_{ij} : \Delta_2 \to \mathbb{R}$, $e_{ij}(x,y) = x^i y^j$ for any $(x,y)\in\Delta_2$, where $i,j\in\mathbb{N}_0$.

Lemma 2.1 ([**11**]). *The operators $(B_m)_{m\ge1}$ verify for any $(x,y)\in\Delta_2$ the following equalities:*

$$(2.2) \qquad (B_m e_{00})(x,y) = 1,$$

$$(2.3) \qquad (B_m e_{10})(x,y) = x, \qquad (B_m e_{01})(x,y) = y,$$

$$(2.4) \qquad \left(B_m(\cdot - x)^2\right)(x,y) = \frac{x(1-x)}{m},$$

$$(2.5) \qquad \left(B_m(* - y)^2\right)(x,y) = \frac{y(1-y)}{m},$$

$$(2.6) \qquad \begin{aligned}\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) = {} & \frac{3(m-2)}{m^3}\, x^2y^2 - \frac{m-2}{m^3}\,(x^2y + xy^2)\\ & + \frac{m-1}{m^3}\, xy,\end{aligned}$$

$$
\begin{aligned}
&\left(B_m\ (\cdot - x)^4(* - y)^2\right)(x,y)\\
&= -\frac{5(3m^2-26m+24)}{m^5}\,x^4y^2 + \frac{6(3m^2-26m+24)}{m^5}\,x^3y^2\\
&\quad -\frac{6(m^2-7m+6)}{m^5}\,x^3y - \frac{3m^2-41m+42}{m^5}\,x^2y^2\\
&\quad +\frac{3m^2-26m+24}{m^5}\,x^4y + \frac{3m^2-17m+14}{m^5}\,x^2y\\
&\quad -\frac{m-2}{m^5}\,xy^2 + \frac{m-1}{m^5}\,xy
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
&\left(B_m\ (\cdot - x)^2(* - y)^4\right)(x,y)\\
&= -\frac{5(m^2-26m+24)}{m^5}\,x^2y^4 + \frac{6(3m^2-26m+24)}{m^5}\,x^2y^3\\
&\quad -\frac{6(m^2-7m+6)}{m^5}\,xy^3 - \frac{3m^2-41m+42}{m^5}\,x^2y^2\\
&\quad +\frac{3m^2-26m+24}{m^5}\,xy^4 + \frac{3m^2-17m+14}{m^5}\,xy^2\\
&\quad -\frac{m-2}{m^5}\,x^2y + \frac{m-1}{m^5}\,xy
\end{aligned} \tag{2.8}
$$

for any $m \in \mathbb{N}$, where " $\cdot$ " and " $*$ " stand for the first and the second variables.

Lemma 2.2 ([**11**])**.** *The operators $(B_m)_{m\geq 1}$ verify for any $(x,y) \in \Delta_2$ the following inequalities:*

$$(B_m(\cdot - x)^2)(x,y) \leq \frac{1}{4m}\,, \tag{2.9}$$

$$(B_m(* - y)^2)(x,y) \leq \frac{1}{4m} \qquad \text{for any } m \in \mathbb{N}, \tag{2.10}$$

$$\left(B_m(\cdot - x)^2(* - y)^2\right)(x,y) \leq \frac{9}{4m^2} \qquad \text{for any } m \in \mathbb{N},\ m \geq 2, \tag{2.11}$$

$$\left(B_m(\cdot - x)^4(* - y)^2\right)(x,y) \leq \frac{9}{m^3}\,, \tag{2.12}$$

$$\left(B_m(\cdot - x)^2(* - y)^4\right)(x,y) \leq \frac{9}{m^3} \qquad \text{for any } m \in \mathbb{N},\ m \geq 8. \tag{2.13}$$

3. Some relations verify by the $(\mathcal{K}_m)_{m\geq 1}$ operators

Lemma 3.1. *The operators $(\mathcal{K}_m)_{m\geq 1}$ verify for any $(x,y) \in \Delta_2$ the following equalities*

$$(\mathcal{K}_m e_{00})(x,y) = 1, \tag{3.1}$$

$$(\mathcal{K}_m e_{10})(x,y) = \frac{2mx+1}{2(m+1)}, \qquad (\mathcal{K}_m e_{01})(x,y) = \frac{2my+1}{2(m+1)}, \tag{3.2}$$

$$(\mathcal{K}_m(\cdot - x)^2)(x,y) = \frac{3(m-1)x(1-x)+1}{3(m+1)^2}, \tag{3.3}$$

$$(\mathcal{K}_m(* - y)^2)(x,y) = \frac{3(m-1)y(1-y)+1}{3(m+1)^2}, \tag{3.4}$$

$$\begin{aligned} &9(m+1)^4(\mathcal{K}_m(\cdot - x)^2(* - y)^2)(x,y) \\ &\quad = 9(3m^2-20m+1)x^2y^2 + 9(-m^2+10m-1)(x^2y+xy^2) \\ &\qquad + 9(m^2-6m+1)xy + 3(-m+1)(x^2+y^2) \\ &\qquad + 3(m-1)(x+y)+1, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &15(m+1)^6(\mathcal{K}_m(\cdot - x)^4(* - y)^2)(x,y) \\ &\quad = 15(-15m^3+295m^2-409m+1)x^4y^2 \\ &\qquad + 15(3m^3-87m^2+149m-1)x^4y \\ &\qquad + (18m^3-382m^2+558m-2)x^3y^2 \\ &\qquad + 15(-6m^3+142m^2-234m+2)x^3y \\ &\qquad + 15(-3m^3+122m^2-233m+2)x^2y^2 \\ &\qquad + 15(3m^3-60m^2+115m-2)x^2y \\ &\qquad + 15(-m^3-2m^2+28m-1)xy^2 + (5m^2-14m+1)xy \\ &\qquad + 5(3m^2-20m+1)x^4 + 5(-6m^2+40m+2)x^3 \\ &\qquad + 5(3m^2-25m+2)x^2 + 3(-m+1)y^2 \\ &\qquad + 5(4m-1)x + 3(m-1)y + 1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &15(m+1)^6(\mathcal{K}_m(\cdot - x)^2(* - y)^4)(x,y) \\ &\quad = 15(-15m^3+295m^2-409m+1)x^2y^4 \\ &\qquad + 15(3m^3-87m^2+149m-1)xy^4 \\ &\qquad + (18m^3-382m^2+558m-2)x^2y^3 \\ &\qquad + 15(-6m^3+142m^2-234m+2)xy^3 \\ &\qquad + 15(-3m^3+122m^2-233m+2)x^2y^2 \\ &\qquad + 15(3m^3-60m^2+115m-2)xy^2 \\ &\qquad + 15(-m^3-2m^2+28m-1)x^2y \\ &\qquad + (5m^2-14m+1)xy + 5(3m^2-20m+1)y^4 \\ &\qquad + 5(-6m^2+40m+2)y^3 + 5(3m^2-25m+2)y^2 \\ &\qquad + 3(-m+1)x^2 + 5(4m-1)y + 3(m-1)x + 1. \end{aligned} \tag{3.7}$$

Proof. In the following consideration we take the relations from Lemma 2.1 into account. We have

$$
\begin{aligned}
(\mathcal{K}_m e_{00})(x,y) &= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y) \int\limits_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int\limits_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} \mathrm{d}s\mathrm{d}t \\
&= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{1}{(m+1)^2} = (B_m e_{00})(x,y) = 1,
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{K}_m e_{10})(x,y) &= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y) \int\limits_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int\limits_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} s\mathrm{d}s\mathrm{d}t \\
&= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{2k+1}{2(m+1)^3} \\
&= \frac{1}{2(m+1)} \sum_{\substack{k,j=0\\k+j\le m}} (2k+1)p_{m,k,j}(x,y) \\
&= \frac{m}{m+1}(B_m e_{10})(x,y) + \frac{1}{2(m+1)}(B_m e_{00})(x,y) = \frac{2mx+1}{2(m+1)},
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{K}_m(\cdot - x)^2)(x,y) &= (\mathcal{K}_m e_{20})(x,y) - 2x(\mathcal{K}_m e_{10})(x,y) + x^2(\mathcal{K}_m e_{00})(x,y) \\
&= \frac{3(m-1)x(1-x)+1}{3(m+1)^2},
\end{aligned}
$$

because

$$
\begin{aligned}
(\mathcal{K}_m e_{20})(x,y) &= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y) \int\limits_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int\limits_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} s^2\mathrm{d}s\mathrm{d}t \\
&= (m+1)^2 \sum_{\substack{k,j=0\\k+j\le m}} p_{m,k,j}(x,y)\frac{3k^2+3k+1}{3(m+1)^4} \\
&= \frac{1}{3(m+1)^2} \sum_{\substack{k,j=0\\k+j\le m}} (3k^2+3k+1)p_{m,k,j}(x,y) \\
&= \frac{m^2}{(m+1)^2}(B_m e_{20})(x,y) + \frac{m}{(m+1)^2}(B_m e_{10})(x,y) \\
&\quad + \frac{1}{3(m+1)^2}(B_m e_{00})(x,y) \\
&= \frac{3(m-1)x(1-x)+1}{3(m+1)^2}.
\end{aligned}
$$

The other relations from the lemma can be obtained analogously. □

Lemma 3.2. *The operators* $(\mathcal{K}_m)_{m\geq 1}$ *verify for any* $(x,y)\in\Delta_2$ *the following estimations*

$$(\mathcal{K}_m(\cdot - x)^2)(x,y) \leq \frac{1}{m+1}, \tag{3.8}$$

$$(\mathcal{K}_m(* - y)^2)(x,y) \leq \frac{1}{m+1}, \tag{3.9}$$

$$(\mathcal{K}_m(\cdot - x)^2(* - y)^2)(x,y) \leq \frac{1}{(m+1)^2}, \tag{3.10}$$

for any $m\in\mathbb{N}$,

$$(\mathcal{K}_m(\cdot - x)^4(* - y)^2)(x,y) \leq \frac{1}{(m+1)^3}, \tag{3.11}$$

$$(\mathcal{K}_m(\cdot - x)^2(* - y)^4)(x,y) \leq \frac{1}{(m+1)^3}, \tag{3.12}$$

for any $m\in\mathbb{N}$, $m\geq 14$.

Proof. We use the relations $x,y\geq 0$, $x+y\leq 1$, $x(1-x)\leq 1/4$, $y(1-y)\leq 1/4$, $xy\leq 1/4$ and the results from Lemma 3.1 and we obtain

$$3(m+1)^2(\mathcal{K}_m(\cdot - x)^2)(x,y) = 3(m-1)x(1-x)+1 \leq 3(m+1),$$

for $m\in\mathbb{N}$, from where the relation (3.8) results. Further, we can write

$$\begin{aligned} &9(m+1)^4(\mathcal{K}_m(\cdot - x)^2(* - y)^2)(x,y)\\ &\quad= 9xy(3xy+1-x-y)m^2 + 9xy(-20xy+10x+10y-6)m\\ &\qquad+3(m-1)(x(1-x)+y(1-y))+1 \leq 9(m+1)^2, \end{aligned}$$

for any $m\in\mathbb{N}$, where we used the inequality $-20xy+10x+10y-6\leq 4$, for any $(x,y)\in\Delta_2$, from where the relation (3.10) results. Finally, we have analogously

$$15(m+1)^6(\mathcal{K}_m(\cdot - x)^4(* - y)^2)(x,y) \leq 15(m+1)^3,$$

where we can take any $m\in\mathbb{N}$, $m\geq 14$, from where the relation (3.11) results. □

4. Approximation and convergence theorems for the bivariate operators of Kantorovich type

Let $X,Y\subset\mathbb{R}$ be given intervals and the set $B(X\times Y)=\{f : X\times Y\to\mathbb{R} | f$ bounded on $X\times Y\}$. For $f\in B(X,Y)$, let the function $\omega_{\text{total}}(f;\cdot,*):[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ by

$$\begin{aligned}\omega_{\text{total}}(f;\delta_1,\delta_2) = \sup\{\,&|f(x,y)-f(x',y')| : (x,y),(x',y')\in X\times Y,\\ &|x-x'|\leq\delta_1, |y-y'|\leq\delta_2\,\}\end{aligned} \tag{4.1}$$

is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.

Theorem 4.1 ([**15**]). *Let $L : C(X \times Y) \to B(X \times Y)$ be a linear positive operator. For any $f \in C(X \times Y)$, any $(x, y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$
\begin{aligned}
|(Lf)(x,y) - f(x,y)| \le |(Le_{00})(x,y) - 1||f(x,y)| + \Big[(Le_{00})(x,y) \\
+ \delta_1^{-1}\sqrt{(Le_{00})(x,y)(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(Le_{00})(x,y)(L(* - y)^2)(x,y)} \\
+ \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{00})^2(x,y)(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{\text{total}}(f;\delta_1,\delta_2)
\end{aligned} \tag{4.2}
$$

holds.

For some further information on this measure of smoothness see, for example, [**5**], [**15**] or [**16**].

Theorem 4.2. *Let the function $f \in C([0,1]\times[0,1])$. Then, for any $(x, y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \ge 4$, we have*

$$
\begin{aligned}
&|(\mathcal{K}_m f)(x,y) - f(x,y)| \\
&\le \left(1 + \frac{1}{\delta_1\sqrt{m+1}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+1}}\right)\omega_{\text{total}}(f;\delta_1,\delta_2)
\end{aligned} \tag{4.3}
$$

for any $\delta_1, \delta_2 > 0$ and

$$
|(\mathcal{K}_m f)(x,y) - f(x,y)| \le 4\omega_{\text{total}}\left(f; \frac{1}{\sqrt{m+1}}, \frac{1}{\sqrt{m+1}}\right). \tag{4.4}
$$

Proof. We apply Theorem 4.1 and Lemma 3.2. For (4.4), we choose $\delta_1 = \delta_2 = \frac{1}{\sqrt{m+1}}$ in (4.3). □

Corollary 4.1. *If $f \in C([0,1] \times [0,1])$, then*

$$
\lim_{m\to\infty}(\mathcal{K}_m f)(x,y) = f(x,y) \tag{4.5}
$$

uniformly on Δ_2.

Proof. It results from (4.4). □

5. Approximation and convergence theorems for GBS operators of Kantorovich type

In the following consideration, let X and Y be compact real intervals. A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous (Bögel-continuous) function in $(x_0, y_0) \in X \times Y$ if

$$
\lim_{(x,y)\to(x_0,y_0)} \Delta f\left((x,y),(x_0,y_0)\right) = 0.
$$

Here $\Delta f\left((x,y),(x_0,y_0)\right) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable (Bögel-differentiable) function in $(x_0, y_0) \in X \times Y$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left((x,y),(x_0,y_0)\right)}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

The definition of B-continuity and B-differentiability were introduced by K. Bögel in the papers [**6**] and [**7**].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $K > 0$ such that

$$|\Delta f\left((x,y),(s,t)\right)| \leq K$$

for any $(x, y), (s, t) \in X \times Y$.

We shall use the function sets $B(X \times Y)$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f\ B\text{-bounded on } X \times Y\}$, $C_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f\ B\text{-continuous on } X \times Y\}$ and $D_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f\ B\text{-differentiable on } X \times Y\}$.

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, *) : [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined by

$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\left\{|\Delta f((x,y),(s,t))| : |x-s| \leq \delta_1, |y-t| \leq \delta_2\right\} \tag{5.1}$$

for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ is called the mixed modulus of smoothness.

For related topics, see [**1**], [**2**], [**3**] and [**4**].

Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator. The operator $UL : C_b(X \times Y) \to B(X \times Y)$ defined for any function $f \in C_b(X \times Y)$, any $(x, y) \in X \times Y$ by

$$(ULf)(x,y) = \left(L(f(\cdot,y) + f(x,*) - f(\cdot,*))\right)(x,y) \tag{5.2}$$

is called a *GBS* operator ("Generalized Boolean Sum" operator) associated with the operator L.

Let the functions $e_{ij} : X \times Y \to \mathbb{R}$, $e_{ij}(x, y) = x^i y^j$ for any $(x, y) \in X \times Y$, where $i, j \in \mathbb{N}_0$.

Theorem 5.1 ([**3**]). *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in C_b(X \times Y)$, any $(x, y) \in (X \times Y)$ and any $\delta_1, \delta_2 > 0$, we have*

$$\begin{aligned} |f(x,y) - (ULf)(x,y)| &\leq |f(x,y)|\,|1 - (Le_{00})(x,y)| \\ &+ \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)} \\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^2)(x,y)}\Big]\,\omega_{\text{mixed}}(f; \delta_1, \delta_2). \end{aligned} \tag{5.3}$$

In the following consideration, we need the following theorem for estimating the rate of the convergence of the B-differentiable functions.

Theorem 5.2 ([**12**]). *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x, y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have*

$$\begin{aligned} &|f(x,y) - (ULf)(x,y)| \\ &\quad \leq |f(x,y)||1-(Le_{00})(x,y)| + 3\|D_B f\|_\infty \sqrt{(L(\cdot - x)^2(* - y)^2)\,(x,y)} \\ &\quad + \Big[\sqrt{(L(\cdot - x)^2(* - y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot - x)^4(* - y)^2)(x,y)} \\ &\quad + \delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^4)(x,y)} \\ &\quad + \delta_1^{-1}\delta_2^{-1}\left(L(\cdot - x)^2(* - y)^2\right)(x,y)\Big]\omega_{\text{mixed}}(D_B f; \delta_1, \delta_2). \end{aligned} \tag{5.4}$$

If $m \in \mathbb{N}$ and $f \in C_b([0,1] \times [0,1])$, then the GBS operator associated with the $\mathcal{K}_m$ operator is defined by

$$\begin{aligned} (U\mathcal{K}_m f)(x,y) =&(m+1)^2 \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x,y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} [f(s,y) \\ &+ f(x,t) - f(s,t)]\mathrm{d}s\,\mathrm{d}t \qquad \text{for any } (x,y) \in \Delta_2. \end{aligned} \tag{5.5}$$

Theorem 5.3. *Let the function $f \in C_b([0,1]\times[0,1])$. Then for any $(x, y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \geq 4$, we have*

$$\begin{aligned} &|(U\mathcal{K}_m f)(x,y) - f(x,y)| \\ &\qquad \leq \left(1 + \frac{1}{\delta_1\sqrt{m+1}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+1}}\right)\omega_{\text{mixed}}(f; \delta_1, \delta_2) \end{aligned} \tag{5.6}$$

for any $\delta_1, \delta_2 > 0$ and

$$|(U\mathcal{K}_m f)(x,y) - f(x,y)| \leq 4\omega_{\text{mixed}}\left(f; \frac{1}{\sqrt{m+1}}, \frac{1}{\sqrt{m+1}}\right). \tag{5.7}$$

Proof. For the first inequality, we apply Theorem 5.1 and Lemma 3.2. The inequality (5.7) is obtained from (5.6) by choosing $\delta_1 = \delta_2 = \frac{1}{\sqrt{m+1}}$. □

Corollary 5.1. *If $f \in C_b([0,1] \times [0,1])$, then*

$$\lim_{m\to\infty} (U\mathcal{K}_m f)(x,y) = f(x,y) \qquad \textit{uniformly on } \Delta_2. \tag{5.8}$$

Proof. It results from (5.7). □

Theorem 5.4. *Let the function $f \in D_b([0,1] \times [0,1])$ with $D_B f \in B([0,1] \times [0,1])$. Then for any $(x, y) \in \Delta_2$, any $m \in \mathbb{N}$, $m \geq 14$, we have*

$$\begin{aligned} |(U\mathcal{K}_m f)(x,y) - f(x,y)| \leq& \frac{3}{m+1}\|D_B f\|_\infty \\ &+ \frac{1}{m+1}\left(1 + \frac{1}{\delta_1\sqrt{m+1}}\right)\left(1 + \frac{1}{\delta_2\sqrt{m+1}}\right)\omega_{\text{mixed}}(D_B f; \delta_1, \delta_2) \end{aligned} \tag{5.9}$$

for any $\delta_1, \delta_2 > 0$ *and*

$$|(U\mathcal{K}_m f)(x,y) - f(x,y)| \leq \frac{3}{m+1}\|D_B f\|_\infty +$$

$$+\frac{4}{m+1}\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m+1}}, \frac{1}{\sqrt{m+1}}\right). \tag{5.10}$$

Proof. It results from Theorem 5.2 and Lemma 3.2. □

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