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SOME GEOMETRIC PROPERTIES OF A GENERALIZED CESÀRO SEQUENCE SPACE

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ABSTRACT. In this paper we define the generalized Cesàro sequence space $\mathrm{Ces}_p(q)$ and exhibit some geometrical properties of the space. The results herein proved are exhibit analogous to those by Y. A. Cui [Southeast Asian Bull. Math. **24** (2000), 201–210] for the Cesàro sequence space Ces_p.

1. INTRODUCTION

Let $(X, \|.\|)$ be a real Banach space. By $B(X)$ and $S(X)$, we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X, χ_A represents a characteristics function of A.

A norm $\|.\|$ is called uniformly convex (UC) (cf. [**2**]) if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for $x, y \in S(X)$, $\|x - y\| > \varepsilon$ implies

$$\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta. \tag{1.0.1}$$

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence (x_n) in X admits a subsequence $\{z_n\}$ such that the sequence $\{t_k(z)\}$ is convergent in norm in X (see [**1**]), where

$$t_k(z) = \frac{1}{k}(z_1 + z_2 + \cdots + z_k).$$

Every Banach space X with the Banach-Saks property is reflexive, but the converse is not true (see [**4, 5**]). Kakutani [**6**] proved that any uniformly convex Banach space X has the Banach-Saks property. Moreover, he also proved that if X is a reflexive Banach space and $\theta \in (0, 2)$ such that for every sequence (x_n) in $S(X)$ weakly convergent to zero, there exist $n_1, n_2 \in \mathbb{N}$ satisfying the Banach-Saks property.

For a sequence $(x_n) \subset X$, we define

$$A(x_n) = \lim_{n\to\infty} \inf\{\|x_i + x_j\| : i, j \geq n, i \neq j\}.$$

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In [**6**], the following new geometric constant connected with the packing constant (see [**7**]) and with the Banach-Saks property was defined

$$C(X) = \sup\{A(x_n) : (x_n) \text{ is a weakly null sequence in } S(X)\}.$$

Recall that a sequence (x_n) is said to be an *ε-separated sequence* if, for some $\varepsilon > 0$,

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space X is said to satisfy *property* (β) if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geq \varepsilon$, there is an index k such that

$$\left\| \frac{x + x_k}{2} \right\| | \geq 1 - \delta, \qquad \text{for some } k \in \mathbb{N}.$$

In this paper, we define the generalized Cesàro sequence space as follows: let $p \in [1, \infty)$ and q be a bounded sequence of positive real numbers such that

$$Q_n = \sum_{k=0}^{n} q_k, \qquad (n \in \mathbb{N}),$$

$$\operatorname{Ces}_p(q) = \left\{ x = (x(i)) : \|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i)| \right)^p \right)^{1/p} < \infty \right\}.$$

If $q_i = 1$ for all $i \in \mathbb{N}$, then $\operatorname{Ces}_p(q)$ is reduced to Ces_p (cf. [**3, 8, 9**]).

Lemma 1.1. *Let $x, y \in \operatorname{Ces}_p(q)$. Then for any $\varepsilon > 0$ and $L > 0$, $\exists\ \delta > 0$ such that*

$$|\ \|x + y\|^p - \|x\|^p\ | < \varepsilon$$

whenever

$$\|x\|^p \leq L \qquad \textit{and} \qquad \|y\|^p \leq \delta.$$

Proof. For any fix $\varepsilon > 0$ and $L > 0$, take $\beta = 2^{-p} L^{-1} \varepsilon$ and $\delta = 2^{-p} \beta^{p-1} \varepsilon$. Then for any $x, y \in \operatorname{Ces}_p(q)$ with $\|x\|^p \leq L$ and $\|y\|^p \leq \delta$, we have

$$\begin{aligned}
\|x + y\|^p &= \sum_{n=1}^{\infty} \Big(\frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i) + q_i y(i)| \Big)^p \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{i=1}^{n} \left| (1-\beta) q_i x(i) + \beta \big(q_i x(i) + \frac{q_i y(i)}{\beta} \big) \right| \right)^p \\
&\leq \sum_{n=1}^{\infty} \left((1-\beta) \frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i)| + \beta \frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i) + \frac{q_i y(i)}{\beta}| \right)^p \\
&\leq (1-\beta) \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i)| \right)^p + \beta \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{i=1}^{n} |q_i x(i) + \frac{q_i y(i)}{\beta}| \right)^p
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_ix(i)|\right)^p + \frac{\beta}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|2q_ix(i)|\right)^p$$
$$+\frac{\beta}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|\frac{2q_iy(i)}{\beta}|\right)^p$$
$$\leq \|x\|^p + \varepsilon/2 + (2/\beta)^{p-1}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_iy(i)|\right)^p$$
$$\leq \|x\|^p + \varepsilon.$$

□

2. Main Results.

Theorem 2.1. *The space* $\mathrm{Ces}_p(q)$ *satisfies the property* (β).

Proof. Let $\mathrm{Ces}_p(q)$ not have property (β). Then there exists $\varepsilon_0 > 0$ such that, for any $\delta \in (0, \varepsilon_0/(1+2^{1+p}))$, there is a sequence $(x_n) \subset S(\mathrm{Ces}_p(q))$ with $\mathrm{sep}(x_n) > \varepsilon_0^{1/p}$ and an element $x_0 \in S(\mathrm{Ces}_p(q))$ such that

$$\left\|\frac{x_n + x_0}{2}\right\|^p > 1 - \delta \qquad \text{for any } n \in \mathbb{N}.$$

Fix $\delta \in (0, \varepsilon_0/(1 + 2^{1+p}))$. We want to show that

$$\lim_{j\to\infty}\sup_k \sum_{n=j+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_ix_k(i)|\right)^p \leq \frac{2^{1+p}\delta}{2^p - 1}. \tag{2.1.1}$$

Otherwise, without loss of generality, we can assume that there exists a sequence (j_k) such that $j_k \to \infty$ as $k \to \infty$ and

$$\sum_{n=j_k+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_ix_k(i)|\right)^p > \frac{2^{1+p}\delta}{2^p - 1} \qquad \text{for every } \ k \in \mathbb{N}. \tag{2.1.2}$$

Let $\delta > 0$ be a real number corresponding to $\varepsilon = \delta$ and $L = 1$ in Lemma 1. By absolute continuity of the norm of x_0, there exists a positive integer n_1 such that

$$\|x_0\chi_{n_1,n_1+1,n_1+2,\ldots}\|^p = \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_ix_0(i)|\right)^p < \delta_1.$$

Choose k so large that $j_k > n_1$. By the Lemma 1 and (2.1.2), we have

$$1 - \delta < \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\left|\frac{q_ix_k(i) + q_ix_0(i)}{2}\right|\right)^p$$
$$= \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\left|\frac{q_ix_k(i)+q_ix_0(i)}{2}\right|\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\left|\frac{q_ix_k(i) + q_ix_0(i)}{2}\right|\right)^p$$

$$\leq \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_0(i)|\right)^p + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p$$
$$+ \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}\left|\frac{q_i x_k(i)}{2}\right|\right)^p + \delta$$
$$\leq \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p + \frac{1}{2^p}\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p + \delta$$
$$\leq \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p - \frac{2^p-1}{2^p}\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p + \delta$$
$$< 1 - 2\delta + \delta = 1 - \delta.$$

That is a contradiction. Hence (2.1.1) must hold. Since

$$\left(\frac{1}{Q_{n_1}}\sum_{i=1}^{n_1}|q_i x_k(i)|\right)^p \leq \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i x_k(i)|\right)^p \leq 1,$$

we have $|q_i x_k(i)| \leq Q_{n_1}$ for $k \in \mathbb{N}$ and $i = 1, 2, \ldots, n_1$. Hence there is a subsequence (z_n) of (x_n) and a sequence (a_n) of real numbers such that

$$\lim_{k\to\infty} q_i z_k(i) = a_i, \qquad \text{for } \imath = 1, 2, \ldots, n_1.$$

Therefore,

$$\sum_{n=1}^{n_1}\left(\frac{1}{Q\,_n}\sum_{i=1}^{n}|q_i z_k(i) - q_i z_m(i)|\right)^p < \delta \text{ for n,m sufficiently large.}$$

Consequently,

$$\|z_k - z_m\|^p$$
$$= \sum_{n=1}^{\infty}(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_k(i) - q_i z_m(i)|)^p$$
$$= \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_k(i) - q_i z_m(i)|\right)^p + \sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_k(i) - q_i z_m(i)|\right)^p$$
$$\leq \sum_{n=1}^{n_1}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_k(i) - q_i z_m(i)|\right)^p$$
$$+ 2^p\left(\sum_{n=n_1+1}^{\infty}(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_k(i)|\right)^p + \left(\sum_{n=n_1+1}^{\infty}\left(\frac{1}{Q_n}\sum_{i=1}^{n}|q_i z_m(i)|\right)^p\right)$$
$$\leq \delta + 2^{p+1}\delta < \varepsilon_0,$$

i.e., $\mathrm{sep}(x_n) \leq \mathrm{sep}(z_n) < \varepsilon_0^{1/p}$. This is a contradiction. Therefore $\mathrm{Ces}_p(q)$ must satisfy the property (β). □

Theorem 2.2. $C(\mathrm{Ces}_p(q)) = 2^{1/p}$.

Proof. Let

$$K = \sup\{A(u_n) : u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)e_i \in S(\mathrm{Ces}_p(q)),$$

$$0 = i_0 < i_1 < i_2 < \ldots, u_n \overset{\omega}{\to} 0, .$$

Then $C(\mathrm{Ces}_p(q)) \geq K$. Moreover, for any $\varepsilon > 0$, there is a sequence $(x_n) \subset S(\mathrm{Ces}_p(q))$ with $x_n \overset{\omega}{\to} 0$ such that

$$A(x_n) + \varepsilon > C(\mathrm{Ces}_p(q)).$$

By the definition of $A(x_n)$, there exists a subsequence (y_n) of (x_n) such that

$$\|y_n + y_m\| + 2\varepsilon > C(\mathrm{Ces}_p(q)) \tag{2.2.1}$$

for any $n, m \in \mathbb{N}$ with $m \neq n$. Take $v_1 = y_1$. Then, by the absolute continuity of the norm of y_1, there exists $i_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=i_1+1}^{\infty} v_1(i)e_i \right\| < \varepsilon.$$

Putting $z_1 = \sum_{i=1}^{i_1} v_1(i)e_i$, we have

$$\|z_1 + y_m\| = \|y_1 + y_m - \sum_{i=i_1+1}^{\infty} v_1(i)e_i\| \geq \|y_1 + y_m\| - \varepsilon \qquad \text{for any } \ m > 1.$$

Hence by (2.2.1), we have

$$\|z_1 + y_m\| + 3\varepsilon > C(\mathrm{Ces}_p(q)) \qquad \text{for any } \ m > 1.$$

Since $y_n(i) \to 0$ for $i = 1, 2, \cdots$, there exists $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that

$$\left\| \sum_{i=1}^{i_1} y_n(i)e_i \right\| < \varepsilon \qquad \text{whenever } \ n \geq n_2.$$

Define $v_2 = y_{n_2}$. Then there is $i_2 > i_1$ such that

$$\left\| \sum_{i=i_2+1}^{\infty} v_2(i)e_i \right\| < \varepsilon.$$

Taking $z_2 = \sum_{i=i_1+1}^{i_2} v_2(i)e_i$, we obtain

$$\begin{aligned} \|z_1 + z_2\| &= \|y_1 - \sum_{i=i_1+1}^{\infty} v_1(i)e_i + y_{n_2} - \sum_{i=1}^{i_1} v_2(i)e_i - \sum_{i=i_2+1}^{\infty} v_2(i)e_i \\ &\geq \|y_1 + y_{n_2}\| - 3\varepsilon. \end{aligned}$$

Hence and by (2.2.1), we immediately obtain

$$\|z_1 + z_2\| + 5\varepsilon > C(\mathrm{Ces}_p(q)).$$

Suppose that increasing sequences $(i_j)_{j=1}^{k-1}$, $(n_j)_{j=1}^{k-1}$ of natural numbers and a sequence $(z_j)_{j=1}^{k-1}$ of elements of $\mathrm{Ces}_p(q)$ are already defined and

$$\|z_n + z_m\| + 6\varepsilon > C(\mathrm{Ces}_p(q)) \qquad \text{for } m, n \in \{1, 2, \ldots, k-1\}, \quad m \neq n.$$

Since $y_n(i) \to 0$ for $i = 1, 2, \cdots$, there exists a natural number $n_k > n_{k-1}$ such that

$$\left\| \sum_{i=1}^{i_{k-1}} y_n(i) e_i \right\| < \varepsilon$$

provided $n \geq n_k$. Put $v = y_{n_k}$. Then there is $i_k > i_{k-1}$ such that

$$\left\| \sum_{i=i_k+1}^{\infty} v_k(i) e_i \right\| < \varepsilon.$$

Defining $z_k = \sum_{i=i_{k-1}+1}^{i_k} v_k(i) e_i$, we obtain

$$\|z_j + z_k\| = \left\| y_{n_j} - \sum_{i=1}^{i_{j-1}} v_j(i) e_i - \sum_{i_{j+1}}^{\infty} v_j(i) e_i + y_{n_k} - \sum_{i=1}^{i_{k-1}} v_k(i) e_i - \sum_{i_{k+1}}^{\infty} v_k(i) e_i \right\|$$

$$\geq \|y_{n_j} + y_{n_k}\| - 4\varepsilon \qquad \text{for } j = 1, 2, \ldots, k-1.$$

Hence, by (2.2.1), we obtain

$$\|z_j + z_k\| + 6\varepsilon > C(\mathrm{Ces}_p(q)) \qquad \text{for } j = 1, 2, \cdots, k-1.$$

Using the induction principle, we can find a sequence (z_n) satisfying the following conditions:

(1) $z_n = \sum_{i=i_{n-1}+1} v_n(i) e_i$, where $0 = i_0 < i_1 < i_2 < \ldots$;

(2) $\|z_n + z_m\| + 6\varepsilon > C(\mathrm{Ces}_p(q))$ for $m, n, \in \mathbb{N}$, $m \neq n$;

(3) $\|z_n\| \leq 1$ for $n = 1, 2, \ldots$;

(4) $z_n \xrightarrow{\omega} 0$ as $n \to \infty$.

Define $u_n = z_n / \|z_n\|$ for each $n \in \mathbb{N}$. Then every $u_n \in S(\mathrm{Ces}_p(q))$ and

$$\|u_n + u_m\| = \left\| \frac{z_n}{\|z_n\|} + \frac{z_m}{\|z_m\|} \right\| \geq \|z_n + z_m\| \geq C(\mathrm{Ces}_p(q)) - 6\varepsilon$$

for any $m, n \in \mathbb{N}$, $m \neq n$. By the arbitrariness of ε, we have $C(\mathrm{Ces}_p(q)) = K$. Let $\varepsilon > 0$ be given. Take $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=i_{n_\varepsilon}+1}^{\infty} \left(\frac{a}{Q_k} \right)^p < \varepsilon,$$

where

$$a = \sum_{i=i_{n_\varepsilon-1}+1}^{i_{n_\varepsilon}} |q_i u_{n_\varepsilon}(i)|.$$

Hence for any $m > n_\varepsilon$, we have

$$\begin{aligned}
\|u_{n_\varepsilon} + u_m\|^p &= \sum_{i=i_{n_\varepsilon-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_{n_\varepsilon}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(a + \sum_{i=1}^{k}|q_i u_m(i)|\right)^p \\
&\geq \sum_{i=i_{n_\varepsilon-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_{n_\varepsilon}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(\sum_{i=1}^{k}|q_i u_m(i)|\right)^p \\
&= \sum_{i=i_{n_\varepsilon-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_{n_\varepsilon}(i)|\right)^p \\
&\quad - \sum_{k=i_{m-1}}^{\infty}\left(\frac{a}{Q_k}\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(\sum_{i=1}^{k}|q_i u_m(i)|\right)^p \\
&> 1 - \varepsilon + 1 = 2 - \varepsilon,
\end{aligned}$$

i. e. $A(u_n) \geq (2-\varepsilon)) \geq (2-\varepsilon)^{1/p}$.

On the other hand, for ε mentioned above, by Lemma 1.1, there exists $\delta > 0$ such that

$$|\|x+y\|^p - \|x\|^p| < \varepsilon$$

whenever $\|x\|^p \leq 1$ and $\|y\|^p < \delta$. Take $n_\delta \in \mathbb{N}$ such that

$$\sum_{k=i_{n_\delta+1}}^{\infty} (\frac{a}{Q_k})^p < \delta, \qquad \text{and} \qquad a = \sum_{i=i_{n_\delta-1}+1}^{i_{n_\delta}} |q_i u_{n_\delta}(i)|.$$

Hence for any $m > n_\delta$, we have

$$\begin{aligned}
&\|u_{n_\delta} + u_m\|^p \\
&= \sum_{i=i_{n_\delta-1}+1}^{i_{m-1}} \left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_{n_\delta}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(a + \sum_{i=1}^{k}|q_i u_m(i)|)\right)^p \\
&\leq \sum_{i=i_{n_\delta-1}+1}^{\infty} \left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_{n_\delta}(i)|\right)^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(a + \sum_{i=1}^{k}|q_i u_m(i)|)\right)^p \\
&= \|u_{n_\delta}(i)\|^p + \sum_{k=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}(a + \sum_{i=1}^{k}|q_i u_m(i)|)\right)^p \\
&\quad - \sum_{k1=i_{m-1}+1}^{\infty}\left(\frac{1}{Q_k}\sum_{i=1}^{k}|q_i u_m(i)|\right)^p + \|u_m\|^p \\
&< 2 + \varepsilon,
\end{aligned}$$

i. e. $A((u_n)) \leq (2+\varepsilon)^{1/p}$.

Since ε was arbitrary, we obtain $C(\mathrm{Ces}_p(q)) = 2^{1/p}$. □

Corollary 2.1. *The space* $\mathrm{Ces}_p(q))$ *satisfies the Banach-Saks property .*

Proof. The proof follows immediately from the above Theorem 1 of [**3**] and Theorem 2.2. □

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References

1. Banach S.and Saks S.; *Sur la convergence forte dans les champs L^p,* Studia Math. **2** (1930), 51–57.
2. Clarkson J. A., *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.
3. Cui Y. A. and Meng C. and Pluciennik, R., *Banach-Saks property and property (β) in Cesàro Sequence Space*, Southeast Asian Bull. Math. **24** (2000), 201–210.
4. Cui Y. A. and Hudzik H., *On the Banach-Saks and weak Banach-Saks properties of some Banach sequence spaces*, Acta Sci. Math. (Szeged), **65** (1999), 179–187.
5. Diestel J., *Sequences and series in Banach spaces*, Graduate Texts in Math. **92**, Springer--Verlag, 1984.
6. Kakutani S., *Weak convergence in Uniformly convex Banach spaces*, Tohoku Math. J. Math. **45** (1938), 188–193.
7. Kotmann C. A., *Packing and reflexivity in Banach Spaces*, Trans. Amer. Math. Soc. **150** (1970), 565–576.
8. Lee. P. Y., *Cesàro sequence Spaces*, Math cronical, New Zeland **13** (1984), 29–45.
9. Shue J. S., *Cesàro sequence spaces*, Tamkang J. Math. **1** (1970), 143–150.

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