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ON THE RICCATI DIFFERENTIAL POLYNOMIALS

A. AYAD

Abstract. In this paper we present some properties of the Riccatti differential polynomial associated with a homogeneous linear ordinary differential equation. We give a complete description of the differential Newton polygons of their derivatives and its evaluations.

1. Introduction

Many problems from quantum physics, optimal filtering and control can be modelized by Riccatti differential equations. Grigoriev [**2**] (see also [**3**]) has used a differential version of Newton polygons to compute formal power series solutions of Riccatti differential equations and consequently to factorize linear ordinary differential equations.

This paper will describe some properties of the Riccatti differential polynomials associated with homogeneous linear ordinary differential equations. First, we introduce them in Section 1. Second, we compute their derivatives in Section 2. Section 3 describes the Newton polygons of the Riccatti differential polynomials and their derivatives. Newton polygons of different evaluations of the Riccatti differential polynomials are given in Section 4.

Let K be a field and $\overline{K}$ be an algebraic closure of K. Let $S(y) = 0$ be a homogeneous linear ordinary differential equation of order n with coefficients in $K[x]$. This equation can be written in the form

$$S(y) = s_n y^{(n)} + \cdots + s_1 y' + s_0 y$$

where $s_i \in K[x]$ for all $0 \le i \le n$ and $s_n \neq 0$. Let $y_0, \ldots, y_n$ be new variables algebraically independent over $K(x)$. Let $(r_i)_{i \ge 0}$ be the following sequence of differential polynomials

$$r_0 = 1, \qquad r_1 = y_0, \qquad r_2 = y_0^2 + y_1,$$
$$r_3 = y_0^3 + 3y_0y_1 + y_2, \qquad r_{i+1} = y_0 r_i + Dr_i, \qquad \text{for all } i \ge 1,$$

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where $Dy_i = y_{i+1}$ for any $0 \le i \le n-1$. For all $i \ge 1$, $r_i \in \mathbb{Z}[y_0, \dots, y_{i-1}]$ has total degree equal to i w.r.t. $y_0, \dots, y_{i-1}$ and the only term of r_i of degree i is y_0^i.

Definition 1.1. The non-linear differential polynomial

$$R = s_n r_n + \cdots + s_1 r_1 + s_0 r_0 \in K[x][y_0, \dots, y_n]$$

is called the Riccatti differential polynomial associated with $S(y) = 0$. The equation $R(y) := R(y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx}) = 0$ is called the Riccatti differential equation associated with $S(y) = 0$.

Remark 1. The Riccatti differential equations defined in Definition 1.1 are a generalization of the well-known first order Riccatti differential equations. Namely, for $n = 2$, i.e., $S(y) = s_2 y'' + s_1 y' + s_0 y$, the Riccatti differential equation associated with $S(y) = 0$ is the following first order Riccatti differential equation

$$R(y) = s_2 y' + s_2 y^2 + s_1 y + s_0 = 0.$$

Lemma 1.2. *Let R be the Riccatti differential polynomial associated with $S(y) = 0$. y is a solution of $S(y) = 0$ if and only if $\frac{y'}{y}$ is a solution of $R(y) = 0$.*

Proof. See page 12 of [**2**] (See also [**4**]). □

Another way to compute the Riccatti differential polynomial associated with $S(y) = 0$ is by considering the change of variable $z = \frac{y'}{y}$, i.e., $y' = zy$, one computes the successive derivatives of y and we put them in the equation $S(y) = 0$ to get a non-linear differential equation $R(z) = 0$ which satisfies the property of Lemma 1.2.

2. Partial derivatives of the Riccatti differential polynomial

For each $i \ge 0$ and $k \ge 0$, the k-th derivative of r_i is the differential polynomial defined by

$$r_i^{(0)} := r_i, \qquad r_i^{(1)} := r_i' := \frac{\partial r_i}{\partial y_0}$$
$$r_i^{(k+1)} := (r_i^{(k)})' = \frac{\partial^{k+1} r_i}{\partial y_0^{k+1}}.$$

Lemma 2.1. *For all $i \ge 1$, we have $r_i' = i r_{i-1}$. Thus for all $k \ge 0$, $r_i^{(k)} = (i)_k r_{i-k}$, where $(i)_0 := 1$ and $(i)_k := i(i-1)\cdots(i-k+1)$.*

Proof. We prove the first item by induction on i. For $i = 1$, we have $r_1' = 1 = 1 \cdot r_0$. Suppose that this property holds for a certain i and prove it for $i+1$. Namely,

$$\begin{aligned} r_{i+1}' &= (y_0 r_i + D r_i)' = y_0 r_i' + r_i + D r_i' \\ &= i y_0 r_{i-1} + r_i + D(i r_{i-1}) = i(y_0 r_{i-1} + D r_{i-1}) + r_i \\ &= i r_i + r_i = (i+1) r_i. \end{aligned}$$

The second item can be easily deduced from the the first item by induction on k. □

Definition 2.2. Let R be the Riccati differential polynomial associated with $S(y)=0$. For any $k\geq 0$, the k-th derivative of R is defined by

$$R^{(k)}:=\frac{\partial^k R}{\partial y_0^k}=\sum_{0\leq i\leq n} s_i r_i^{(k)}.$$

Lemma 2.3. *For all $k\geq 0$, we have*

$$R^{(k)}=\sum_{0\leq i\leq n-k}(i+k)_k s_{i+k} r_i.$$

Proof. For all $i<k$, we have $r_i^{(k)}=0$, because $\deg_{y_0}(r_i)=i$. Then by Lemma 2.1, we get

$$\begin{aligned}R^{(k)}&=\sum_{k\leq i\leq n} s_i r_i^{(k)}\\&=\sum_{k\leq i\leq n} s_i (i)_k r_{i-k}\\&=\sum_{0\leq j\leq n-k}(j+k)_k s_{j+k} r_j,\end{aligned}$$

where the last equality is done by the change $j=i-k$. □

Corollary 2.4. For all $k\geq 0$, the k-th derivative of R is the Riccati differential polynomial of the following linear ordinary differential equation of order $n-k$

$$S^{(k)}(y):=\sum_{0\leq i\leq n-k}(i+k)_k s_{i+k} y^{(i)}.$$

Proof. By Definition 1.1 and Lemma 2.3. □

Remark 2. If $s_i\in K$ for all $0\leq i\leq n$, then $S^{(k)}$ is the k-th derivative of S w.r.t. x.

3. Newton polygon of the Riccati differential polynomial and its derivatives

Definition 3.1. Let $F(y_0,\ldots,y_n)=\sum_{i\in\mathbb{Q},\alpha\in A} c_{i,\alpha}x^i y_0^{\alpha_0}\cdots y_n^{\alpha_n}$ be a multivariate polynomial in $y_0,\ldots,y_n$ with coefficients $c_{i,\alpha}\in K$, where $\alpha=(\alpha_0,\ldots,\alpha_n)$ belongs to a finite subset A of $\mathbb{N}^{n+1}$. For every couple $(i,\alpha)\in\mathbb{Q}\times A$ such that $c_{i,\alpha}\neq 0$, we mark the point

$$P_{i,\alpha}:=(i-\alpha_1-2\alpha_2-\cdots-n\alpha_n,\alpha_0+\alpha_1+\cdots+\alpha_n)\in\mathbb{Q}\times\mathbb{N},$$

and we denote by $P(F)$ the set of all the points $P_{i,\alpha}$. The convex hull of these points and the point $(+\infty,0)$ in the plane $\mathbb{R}^2$ is denoted by $\mathcal{N}(F)$ and is called the Newton polygon of the differential equation $F(y)=0$ in the neighborhood of $x=0$. If $\deg_{y_0,\ldots,y_n}(F)=m$, then $\mathcal{N}(F)$ is located between the two horizontal lines $y=0$ and $y=m$.

- For any $(a,b)\in\mathbb{Q}^2\setminus\{(0,0)\}$, we define the set

$$N(F,a,b):=\{(u,v)\in P(F),\ \forall(u',v')\in P(F),\quad au'+bv'\geq au+bv\}.$$

- A point $P_{i,\alpha} \in P(F)$ is a vertex of the Newton polygon $\mathcal{N}(F)$ if there exists $(a,b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $N(F,a,b) = \{P_{i,\alpha}\}$. We remark that $\mathcal{N}(F)$ has a finite number of vertices. We denote by $V(F)$ the set of all vertices p of $\mathcal{N}(F)$ for which $a > 0$ and $b \geq 0$. By the inclination of a line we mean the negative inverse of its geometric slope. If $p \in V(F)$ and $N(F,a,b) = \{p\}$ for a certain $(a,b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, then the fraction $\mu = \frac{b}{a} \in \mathbb{Q}$ is the inclination of a straight line which intersects $\mathcal{N}(F)$ exactly in the vertex p.
- A pair of different vertices $e = (P_{i,\alpha}, P_{i',\alpha'})$ forms an edge of $\mathcal{N}(F)$ if there exists $(a,b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $e \subset N(F,a,b)$. We denote by $E(F)$ the set of all the edges e of $\mathcal{N}(F)$ for which $a > 0$ and $b \geq 0$. It is easy to prove that if $e \in E(F)$, then there exists a unique pair $(a(e), b(e)) \in \mathbb{Z}^2$ such that $a(e) > 0,\ b(e) \geq 0$ are relatively prime and $e \subset N(F, a(e), b(e))$. If $e \in E(F)$, we can prove that the fraction $\mu_e = \frac{b(e)}{a(e)} \in \mathbb{Q}$ is the inclination of the straight line passing through the edge e.
- For each edge $e \in E(F)$, we define the univariate polynomial (in a new variable Z)
$$H_{(F,e)}(Z) = \sum_{P_{i,\alpha} \in N(F,a(e),b(e))} c_{i,\alpha} Z^{\alpha_0+\alpha_1+\cdots+\alpha_n} (\mu_e)_1^{\alpha_1} \cdots (\mu_e)_n^{\alpha_n} \in K[Z],$$
where $(\mu_e)_k := \mu_e(\mu_e - 1)\cdots(\mu_e - k + 1)$ for any positive integer k. We call $H_{(F,e)}(Z)$ the *characteristic* polynomial of F associated with the edge $e \in E(F)$. Its degree is at most $m = \deg_{y_0,\ldots,y_n}(F)$.
- For each vertex $p = (u,v) \in V(F)$, let $\mu_1 < \mu_2$ be the inclinations of the adjacent edges at p in $\mathcal{N}(F)$. It is easy to prove that for all rational numbers $\mu = \frac{b}{a},\ a \in \mathbb{N}^*,\ b \in \mathbb{N}$ such that $N(F,a,b) = \{p\}$, we have $\mu_1 < \mu < \mu_2$. We associate with p the polynomial
$$h_{(F,p)}(\mu) = \sum_{P_{i,\alpha} = p} c_{i,\alpha} (\mu)_1^{\alpha_1} \cdots (\mu)_n^{\alpha_n} \in K[\mu],$$
which is called the *indicial* polynomial of F associated with the vertex p (here μ is considered as an indeterminate). Let $H_{(F,p)}(Z) = Z^v h_{(F,p)}(\mu)$ defined as above for edges $e \in E(F)$.
- Let $p = (u,v) \in V(F)$ and e be the edge of $\mathcal{N}(F)$ descending from p, then $h_{(F,p)}(\mu_e)$ is the coefficient of the monomial Z^v in the expansion of the *characteristic* polynomial of F associated with e.

Let R be the *Riccatti* differential polynomial associated with $S(y) = 0$. We will describe the Newton polygons of R and polynomial derivatives. For every $0 \leq i \leq n$, we mark the points $(\deg(s_i), i)$ and $(\mathrm{ord}(s_i), i)$ in the plane $\mathbb{R}^2$, where $\mathrm{ord}(s_i)$ is the order of multiplicity of 0 as a root of the polynomial s_i. Let $\mathcal{N}$ be the convex hull of these points and the two points $(\min_{0 \leq i \leq n}\{\mathrm{ord}(s_i) - i + 1\}, 1)$ and $(+\infty, 0)$.

Lemma 3.2. *$\mathcal{N}$ is the Newton polygon of R, i.e., $\mathcal{N}(R) = \mathcal{N}$.*

Proof. For all $0 \leq i \leq n$, $\deg_{y_0,\ldots,y_{i-1}}(r_i) = i$ and the only term of r_i of degree i is y_0^i, then $lc(s_i)x^{\deg(s_i)}y_0^i$ is a term of R and $\mathcal{N} \subset \mathcal{N}(R)$. For any other term of

$s_i r_i$ in the form $bx^j y_0^{\alpha_0} \cdots y_{i-1}^{\alpha_{i-1}}$, where $b \in K$, $j < \deg(s_i)$ and $\alpha_0 + \cdots + \alpha_{i-1} < i$, the corresponding point $(j - \alpha_1 - \cdots - (i-1)\alpha_{i-1}, \alpha_0 + \cdots + \alpha_{i-1})$ is in the interior of $\mathcal{N}$. Thus $\mathcal{N}(R) \subset \mathcal{N}$. □

Lemma 3.3. *For any edge e of $\mathcal{N}(R)$, the characteristic polynomial of R associated with e is a non-zero polynomial. For any vertex p of $\mathcal{N}(R)$, the indicial polynomial of R associated with p is a non-zero constant. Moreover, if the ordinate of p is i_0, then $h_{(R,p)}(\mu) = lc(s_{i_0}) \neq 0$, where $lc(s_{i_0}) \in K$ is the leading coefficient of s_{i_0}.*

Proof. By Lemma 3.2, each edge $e \in E(R)$ joints two vertices $(\deg(s_{i_1}), i_1)$ and $(\deg(s_{i_2}), i_2)$ of $\mathcal{N}(R)$. Moreover, the set $N(R, a(e), b(e))$ contains these two points. Then

$$0 \neq H_{(R,e)}(Z) = lc(s_{i_1})Z^{i_1} + lc(s_{i_2})Z^{i_2} + t,$$

where t is a sum of terms of degree different from i_1 and i_2. For any vertex $p \in V(R)$ of ordinate i_0, $lc(s_{i_0})x^{\deg(s_{i_0})}y_0^{i_0}$ is the only term of R with the corresponding point p. Then

$$h_{(R,p)}(\mu) = lc(s_{i_0}) \neq 0.$$

□

Let $0 \leq k \leq n$ and $R^{(k)}$ be the k-th derivative of R which is the k-th partial derivative of R w.r.t. y_0 (Definition 2.2). Then by [**1**, Section 2], the Newton polygon of $R^{(k)}$ is the translation of that of R defined by the point $(0, -k)$, i.e., $\mathcal{N}(R^{(k)}) = \mathcal{N}(R) + \{(0, -k)\}$. The vertices of $\mathcal{N}(R^{(k)})$ are among the points $(\deg(s_{i+k}), i)$ for $0 \leq i \leq n - k$ by Lemma 2.3. Then for each edge e_k of $\mathcal{N}(R^{(k)})$, there are two possibilities:

- e_k is parallel to a certain edge e of $\mathcal{N}(R)$, i.e., e_k is the translation of e by the point $\{(0, -k)\}$.
- The upper vertex of e_k is the translation of the upper vertex of a certain edge e of $\mathcal{N}(R)$ and the lower vertex of e_k is the translation of a certain point $(\deg(s_{i_0}), i_0)$ of $\mathcal{N}(R)$ which does not belong to e.

In both possibilities we say that the edge $e \in E(R)$ is associated with the edge $e_k \in E(R^{(k)})$.

Lemma 3.4. *Let $e_k \in E(R^{(k)})$ be parallel to an edge $e \in E(R)$. Then the characteristic polynomial of $R^{(k)}$ associated with e_k is the k-th derivative of that of R associated with e, i.e.,*

$$H_{(R^{(k)}, e_k)}(Z) = H^{(k)}_{(R,e)}(Z).$$

Proof. The edges e_k and e have the same inclination $\mu_e = \mu_{e_k}$ and $N(R^{(k)}, a(e_k), b(e_k)) = N(R, a(e), b(e)) + \{(0, -k)\}$. Then

$$\begin{aligned} H_{(R^{(k)}, e_k)}(Z) &= \sum_{(\deg(s_{i+k}), i) \in N(R^{(k)}, a(e_k), b(e_k))} (i+k)_k lc(s_{i+k}) Z^i \\ &= \sum_{(\deg(s_j), j) \in N(R, a(e), b(e))} (j)_k lc(s_j) Z^{j-k} = H^{(k)}_{(R,e)}(Z). \end{aligned}$$

□

4. Newton polygons of evaluations of the Riccatti differential polynomial

Let $0 \leq c \in \overline{K}$, $\mu \in \mathbb{Q}$ and $R_1(y) = R(y + cx^\mu)$ be the differential polynomial obtained from R by replacing y_k by $c(\mu)_k x^{\mu-k} + y_k$ for all $0 \leq k \leq n$. We will describe the Newton polygon of R_1 for different values of c and μ.

Lemma 4.1. *R_1 is the Riccatti differential polynomial of the following linear ordinary differential equation of order less than or equal to n:*

$$S_1(y) := \sum_{0\leq i\leq n} \frac{1}{i!} R^{(i)}(cx^\mu) y^{(i)}.$$

Proof. It is equivalent to prove the following analogy of Taylor formula

$$R_1 = \sum_{0\leq i\leq n} \frac{1}{i!} R^{(i)}(cx^\mu) r_i$$

which is done in [**2**, Lemma 2.1]. □

Then the vertices of $\mathcal{N}(R_1)$ are among the points $(\deg(R^{(i)}(cx^\mu), i)$ for $0 \leq i \leq n$. Thus the Newton polygon of R_1 is given by [**2**, Lemma 2.2]

Lemma 4.2. *If μ is the inclination of an edge e of $\mathcal{N}(R)$, then the edges of $\mathcal{N}(R_1)$ situated above e are the same as in $\mathcal{N}(R)$. Moreover, if c is a root of $H_{(R,e)}$ of multiplicity $\eta > 1$ then, $\mathcal{N}(R_1)$ contains an edge e_1 parallel to e originating from the same upper vertex as e where the ordinate of the lower vertex of e_1 equals to η. If $\eta = \deg H_{(R,e)}$, then $\mathcal{N}(R_1)$ contains an edge with inclination less than μ originating from the same upper vertex as e.*

Remark 3. If we evaluate R on cx^μ we get

$$R(cx^\mu) = \sum_{0\leq i\leq n} s_i \times (c^i x^{i\mu} + t),$$

where t is a sum of terms of degree strictly less than $i\mu$. Then

$$lc(R(cx^\mu)) = \sum_{i\in B} lc(s_i)c^i = \sum_{(\deg(s_i),i)\in e} lc(s_i)c^i = H_{(R,e)}(c),$$

where

$$\begin{aligned} B &:= \{0 \leq i \leq n;\ \deg(s_i) + i\mu = \max_{0\leq j\leq n} (\deg(s_j) + j\mu;\ s_j \neq 0)\} \\ &= \{0 \leq i \leq n;\ (\deg(s_i), i) \in e \text{ and } s_i \neq 0\}. \end{aligned}$$

Lemma 4.3. *Let μ be the inclination of an edge e of $\mathcal{N}(R)$ and c be a root of $H_{(R,e)}$ of multiplicity $\eta > 1$. Then*

$$H_{(R_1,e_1)}(Z) = H_{(R,e)}(Z + c)$$

where e_1 is the edge of $\mathcal{N}(R_1)$ given by Lemma 4.2. In addition, if e' is an edge of $\mathcal{N}(R_1)$ situated above e (which is also an edge of $\mathcal{N}(R)$ by Lemma 4.2) then $H_{(R_1,e)}(Z) = H_{(R,e)}(Z)$.

Proof. We have

$$\begin{aligned} H_{(R,e)}(Z+c) &= \sum_{\eta\le k\le n} \frac{1}{k!} H^{(k)}_{(R,e)}(c) Z^k \\ &= \sum_{\eta\le k\le n} \frac{1}{k!} H_{(R^{(k)},e)}(c) Z^k \\ &= \sum_{\eta\le k\le n} \frac{1}{k!} lc(R^{(k)}(cx^\mu)) Z^k \\ &= H_{(R_1,e_1)}(Z) \end{aligned}$$

where the first equality is just the Taylor formula taking into account that c is a root of $H_{(R,e)}$ of multiplicity $\eta > 1$. The second equality holds by Lemma 3.4, the third one by Remark 3, the fourth one by Lemma 4.1 and by the definition of the *characteristic* polynomial. □

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A. Ayad, Lebanese university, Faculty of sciences, Section 1, Hadath, Beirut, Lebanon and Section 5, Nabatieh, South Lebanon, *e-mail*: `ayadali99100@hotmail.com`,
`http://ali.ayad.free.fr`