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A NOTE ON THE INSTABILITY OF EVOLUTION PROCESSES

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Abstract. In this paper we obtain a Perron type characterization for the expansiveness of an evolution process in Banach spaces.

1. Introduction

The notion of exponential dichotomy was introduced by O. Perron [**24**] and it has an important role in the theory of dynamical systems as we can see in the literature.

The study of dichotomy for differential equations with bounded coefficients in infinite dimensional spaces was introduced by Daleckij and Krein [**6**], Massera and Schäffer [**12**] followed by the paper of W. A. Coppel [**5**] who approaches the finite dimensional case using proper methods for the case of Banach spaces. Recent results for the case of unbounded operators were obtained by Levitan and Zhikov [**11**], Neerven [**22**], Latushkin and Chicone [**3**].

Important results in this topic are the papers [**1**], [**2**], [**4**], [**7**] – [**10**], [**13**] – [**18**], [**20**], [**21**], [**23**], [**25**] – [**28**]. Following this line it must be mentioned the joint paper of N. van Minh, Räbiger and Schnaubelt [**19**] which offers a new characterization of the stability, instability and dichotomy of a dynamical system described by an evolution process using the so called evolution semigroup associated to the process $\Phi(t, t_0)$ which has the advantage that its generator verifies the Spectral Mapping Theorem. In case of "admissibility", the generator gives the restriction that the input space is equal to the output space and the associated evolution semigroup is a C_0-semigroup as in [**3**, Paragraph 3.3, p.73]. This paper establishes characterizations for the instability of an evolution family with the Perron method without using the associated evolution semigroup.

The paper gives a new proof for the result from the paper of V. Minh, Räbiger and Schnaubelt [**19**] for the instability, and even expansiveness of the evolutionary processes with a direct method using the test-functions and input-output spaces, the pair $(\mathcal{C}, \mathcal{C})$ where $\mathcal{C} = \{f \colon \mathbb{R}_+ \to X,\ f \text{ continuous and bounded on } \mathbb{R}_+\}$ and X is a Banach space.

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2. Preliminaries

Let X be a real or complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X and $\mathcal{C} = \{f : \mathbb{R}_+ \to X,\ f$ continuous and bounded on $\mathbb{R}_+\}$.

Definition 2.1. A family of bounded linear operators on X, $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is called an evolutionary process if

1) $\Phi(t,t) = I$ for every $t \geq 0$;
2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$ for all $t \geq s \geq t_0 \geq 0$;
3) $\Phi(\cdot,s)x$ is continuous on $[s,\infty)$ for all $s \geq 0$, $x \in X$;
   $\Phi(t,\cdot)x$ is continuous on $[0,t]$ for all $t \geq 0$, $x \in X$;
4) there exist M, $\omega > 0$ such that

$$\|\Phi(t,s)\| \leq M\,\mathrm{e}^{\omega(t-s)} \quad \text{for all} \quad t \geq s \geq 0.$$

Definition 2.2. The evolution process Φ is said to be exponentially instable if and only if there exist N, $\nu > 0$ such that

$$\|\Phi(t,t_0)x\| \geq N\,\mathrm{e}^{\nu(t-t_0)}\,\|x\|$$

for all $t \geq t_0 \geq 0$ and all $x \in X$.

Definition 2.3. The evolution process Φ is said to be exponentially expansive if Φ is exponentially instable and $\Phi(t,t_0)$ is invertible for all $t \geq t_0 \geq 0$.

Definition 2.4. The evolution process Φ satisfies the Perron condition for instability if and only if for every $f \in \mathcal{C}$, there exists an unique $x \in X$ such that

$$x_f(t) = \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)d\tau,$$

$x_f \in \mathcal{C}$.

Lemma 2.1. *If the process Φ satisfies the Perron condition for instability, then for every $f \in \mathcal{C}$, there exists an unique $u \in \mathcal{C}$ such that*

$$u(t) = \Phi(t,t_0)u(t_0) + \int_{t_0}^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau$$

for all $t \geq t_0 \geq 0$.

Proof. Let $f \in \mathcal{C}$ with $u = x_f$. We have

$$\begin{aligned} x_f(t) &= \Phi(t,0)x + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \\ &= \Phi(t,t_0)\Phi(t_0,0)x + \int_0^{t_0} \Phi(t,t_0)\Phi(t_0,\tau)f(\tau)\mathrm{d}\tau + \int_{t_0}^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \\ &= \Phi(t,t_0)x_f(t_0) + \int_{t_0}^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \end{aligned}$$

for all $t \geq t_0 \geq 0$.

Hence $u(t) = x_f(t)$ which is equivalent to

$$u(t) = \Phi(t, t_0)u(t_0) + \int_{t_0}^{t} \Phi(t, \tau)f(\tau)\mathrm{d}\tau$$

for all $t \geq t_0 \geq 0$ and $u \in \mathcal{C}$.

We suppose that there exists $v \in \mathcal{C}$ with

$$v(t) = \Phi(t, t_0)v(t_0) + \int_{t_0}^{t} \Phi(t, \tau)f(\tau)\mathrm{d}\tau$$

for all $t \geq t_0 \geq 0$.

Denoting by $w = u - v$ we have that

$$w(t) = \Phi(t, t_0)w(t_0) + \int_{t_0}^{t} \Phi(t, \tau)0\mathrm{d}\tau$$

for all $t \geq t_0 \geq 0$. Then we obtain

$$w(t) = \Phi(t, 0)w(0) + \int_{0}^{t} \Phi(t, \tau)0\mathrm{d}\tau.$$

Hence

$$0 = \Phi(t, 0)0 + \int_{0}^{t} \Phi(t, \tau)0\mathrm{d}\tau$$

for $t \geq 0$.

It results that $w(0) = 0$ and so $w(t) = 0$ for all $t \geq 0$, which is equivalent to $u(t) - v(t) = 0$. This means that $u(t) = v(t)$ for all $t \geq 0$.

So, for every $f \in \mathcal{C}$, there exists an unique $u \in \mathcal{C}$ such that

$$u(t) = \Phi(t, t_0)u(t_0) + \int_{t_0}^{t} \Phi(t, \tau)f(\tau)\mathrm{d}\tau$$

for all $t \geq t_0 \geq 0$. □

Lemma 2.2. *If the process Φ satisfies the Perron condition for instability and $x \neq 0$, it results that $\Phi(t, 0)x \neq 0$ for all $t \geq 0$.*

Proof. We suppose that there exists $t_0 > 0$ with $\Phi(t_0, 0)x = 0$. Then $\Phi(t, t_0)\Phi(t_0, 0)x = 0$ for all $t \geq t_0 \geq 0$, which is equivalent to $\Phi(t, 0)x = 0$ for all $t \geq t_0$, and in this way we obtain that $\Phi(\cdot, 0)x \in \mathcal{C}$. Then

$$\Phi(t, 0)x = \Phi(t, 0)x + \int_{0}^{t} \Phi(t, \tau)0\mathrm{d}\tau$$

and

$$0 = \Phi(t,0)0 + \int_0^t \Phi(t,\tau)0\mathrm{d}\tau$$

for all $t \geq 0$, which is equivalent to $x = 0$. This contradicts the hypothesis, so $\Phi(t,0)x \neq 0$ for all $t \geq 0$. □

Theorem 2.1. *If the process Φ satisfies the Perron condition for instability, then there exists $k > 0$ such that*

$$|||x_f||| \leq k|||f|||$$

for all $f \in \mathcal{C}$.

Proof. We define $\mathcal{U}\colon \mathcal{C} \to \mathcal{C}$, $\mathcal{U}f = x_f$. As $f_n \to f$ in $\mathcal{C}$ and $\mathcal{U}f_n \to g$ in $\mathcal{C}$, we show that $\mathcal{U}f = g$.

Since

$$\mathcal{U}f_n(t) = xf_n(t) = \Phi(t,0)x_n + \int_0^t \Phi(t,\tau)f_n(\tau)\mathrm{d}\tau$$

with $x_n = x_{f_n}(0)$ for $n \to \infty$, it results that

$$g(t) = \Phi(t,0)g(0) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau$$

and so $g(t) = x_f(t) = \mathcal{U}f(t)$. Thus $\mathcal{U}$ is bounded. From the Closed Graph Theorem it results that there exists $k > 0$ such that

$$|||x_f||| \leq k|||f|||$$

for all $f \in \mathcal{C}$. □

Theorem 2.2. *The process Φ satisfies the Perron condition for instability if and only if Φ is exponentially expansive.*

Proof. Necessity. Let $x \neq 0$, $\delta > 0$ and $\chi\colon \mathbb{R}_+ \to \mathbb{R}$ with

$$\chi(t) = \begin{cases} 1 & \text{if} \quad t \in [0,\delta], \\ 1+\delta-t & \text{if} \quad t \in (\delta,\delta+1], \\ 0 & \text{if} \quad t > \delta+1. \end{cases}$$

It results that $\chi \in \mathcal{C}$ and $|||\chi||| = 1$.

Let now $f\colon \mathbb{R}_+ \to X$,

$$f(t) = \chi(t)\frac{\Phi(t,0)x}{\|\Phi(t,0)x\|}.$$

It results that $f \in \mathcal{C}$ and $|||f||| = 1$.

We consider

$$y(t) = -\int_t^\infty \chi(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\Phi(t,0)x$$
$$= \Phi(t,0)(-\int_0^\infty \chi(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}x) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau = 0$$

for all $t > \delta + 1$.

It results that $y \in \mathcal{C}$ and $y = x_f$. Then

$$\|y(t)\| \le |||y||| \le k\,|||f||| = k.$$

We have that

$$\int_t^\infty \chi(\tau)\frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\|\Phi(t,0)x\| \le k$$

for all $t \ge 0$.

If $t \in [0,\delta]$, we have that

$$\int_t^\delta \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\|\Phi(t,0)x\| \le k$$

for all $\delta > 0$. For $\delta \to \infty$ we obtain that

(1) $$\int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\mathrm{d}\tau \le \frac{k}{\|\Phi(t,0)x\|}$$

for all $t \ge 0$.

We denote by

$$\psi(t) = \int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\mathrm{d}\tau$$

and from (1) it follows that

$$\psi(t) \le -k\dot{\psi}(t).$$

Hence

$$\psi(t)\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \psi(t_0) \le \frac{k}{\|\Phi(t_0,0)x\|},$$

which is equivalent to

$$\int_t^\infty \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \frac{k}{\|\Phi(t_0,0)x\|}$$

for all $t \ge t_0 \ge 0$. It follows that

(2) $$\int_t^{t+1} \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \le \frac{k}{\|\Phi(t_0,0)x\|}$$

for all $t \ge t_0 \ge 0$.

However

$$\|\Phi(\tau,0)x\| = \|\Phi(\tau,t)\Phi(t,0)x\| \leq M\,\mathrm{e}^{\omega}\,\|\Phi(t,0)x\|,$$

thus

$$\frac{1}{M\,\mathrm{e}^{\omega}\,\|\Phi(t,0)x\|} \leq \int_t^{t+1} \frac{\mathrm{d}\tau}{\|\Phi(\tau,0)x\|}.$$

From (2) it follows that

$$\frac{1}{M\,\mathrm{e}^{\omega}\,\|\Phi(t,0)x\|}\,\mathrm{e}^{\frac{1}{k}(t-t_0)} \leq \frac{k}{\|\Phi(t_0,0)x\|}$$

for all $t \geq t_0 \geq 0$, which means that

$$\frac{1}{M\,\mathrm{e}^{\omega}\,k}\,\mathrm{e}^{\frac{1}{k}(t-t_0)}\,\|\Phi(t_0,0)x\| \leq \|\Phi(t,0)x\|$$

for all $t \geq t_0 \geq 0$ and all $x \in X$. So there exist $N = \frac{1}{M\,\mathrm{e}^{\omega}\,k}$ and $\nu = \frac{1}{k}$ such that

$$\|\Phi(t,0)x\| \geq N\,\mathrm{e}^{\nu(t-t_0)}\,\|\Phi(t_0,0)x\|$$

for all $t \geq t_0 \geq 0$, and all $x \in X$.

We consider

$$\chi_1^{t_0}(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < t_0, \\ 4(t-t_0) & \text{if} \quad t_0 < t \leq t_0 + \frac{1}{2}, \\ 2 - 4(t - t_0 - \frac{1}{2}) & \text{if} \quad t_0 + \frac{1}{2} < t \leq t_0 + 1, \\ 0 & \text{if} \quad t > t_0 + 1. \end{cases}$$

It results that

$$\int_{t_0}^{t_0+1} \chi_1^{t_0}(\tau)\mathrm{d}\tau = 1.$$

We denote by

$$g(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < t_0, \\ \chi_1^{t_0}\Phi(t,t_0)z & \text{if} \quad t > t_0. \end{cases}$$

So $g(t) = \chi_1^{t_0}\Phi(t,t_0)z$ for all $z \in X$. Therefore $g \in \mathcal{C}$ with

$$|||g||| \leq 2M\,\mathrm{e}^{\omega}\,\|z\|$$

and

$$z(t) = -\int_t^{\infty} \chi_1^{t_0}(\tau)\mathrm{d}\tau\Phi(t,t_0)z$$

with $z\colon [t_0,\infty) \to X$. Then

$$\begin{aligned} z(t) &= -\int_s^{\infty} \chi_1^{t_0}(\tau)\mathrm{d}\tau\Phi(t,s)\Phi(s,t_0)z + \int_s^{t} \chi_1^{t_0}(\tau)\mathrm{d}\tau\Phi(t,s)\Phi(s,t_0)z \\ &= \Phi(t,s)z(s) + \int_s^t \Phi(t,\tau)g(\tau)\mathrm{d}\tau \end{aligned}$$

for all $t \geq s \geq 0$.

But $z(t) = 0$ for all $t \geq t_0 + 1$ and $g \in \mathcal{C}$. It results that there exists an unique $x_g \in \mathcal{C}$ and

$$x_g(t) = \Phi(t,s)x_g(s) + \int_s^t \Phi(t,\tau)g(\tau)\mathrm{d}\tau$$

for all $t \geq s \geq 0$. Hence $x_g(t) = z(t)$ for all $t \geq t_0$. Therefore

$$x_g(t_0) = z(t_0) = -\int_{t_0}^{t_0+1} \chi_1^{t_0}(z)\mathrm{d}z = -z.$$

But

$$x_g(t_0) = \Phi(t_0,0)x_g(0) + \int_0^{t_0} \Phi(t_0,\tau)g(\tau) = \Phi(t_0,0)x_g(0).$$

So it results that $\Phi(t_0,0)(-x_g(0)) = z$. In this way we obtain that for all $z \in X$, there exists an unique $-x_g(0) \in X$ with $\Phi(t_0,0)(-x_g(0)) = z$, so $\Phi(t_0,0)x = x$ for all $t_0 \geq 0$.

Let $t \geq t_0 \geq 0$ and $z \in X$. Then there exists $u \in X$ with $\Phi(t_0,0)u = z$ and

$$\|\Phi(t,0)u\| \geq N\,\mathrm{e}^{\nu(t-t_0)}\,\|\Phi(t_0,0)u\|$$

which is equivalent to

$$\|\Phi(t,t_0)z\| \geq N\,\mathrm{e}^{\nu(t-t_0)}\,\|z\|$$

for all $t \geq t_0 \geq 0$ and all $z \in X$. Thus Φ is exponentially instable.

Let $w \in X$. Then there exists $u \in X$ with $\Phi(t,0)u = w = \Phi(t,t_0)\Phi(t_0,0)u$. So for $w \in X$ there exists $v = \Phi(t_0,0)u \in X$ such that $\Phi(t,t_0)v = w$.

It results that $\Phi(t,t_0)$ is surjective.

As $\Phi(t,t_0)$ is injective from Definition 2.2, it follows that $\Phi(t,t_0)$ is invertible, hence Φ is exponentially expansive.

Sufficiency. Let $f \in \mathcal{C}$ and

$$y(t) = -\int_t^{\infty} \Phi^{-1}(\tau,t)f(\tau)\mathrm{d}\tau.$$

Then

$$\|y(t)\| \leq \int_t^{\infty} \frac{1}{N}\,\mathrm{e}^{-\nu(\tau-t)}\,\|f(\tau)\|\,\mathrm{d}\tau \leq \frac{1}{N}|||f|||$$

for all $t \geq 0$.

It results that $y \in \mathcal{C}$ and $y(0) = -\int_0^\infty \Phi^{-1}(\tau,0)f(\tau)\mathrm{d}\tau$. So

$$\begin{aligned}\Phi(t,0)y(0) &= -\int_0^t \Phi(t,0)\Phi^{-1}(\tau,0)f(\tau)\mathrm{d}\tau - \int_t^\infty \Phi(t,0)\Phi^{-1}(\tau,0)f(\tau)\mathrm{d}\tau \\ &= -\int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau - \int_t^\infty \Phi(t,0)(\Phi(\tau,t)\Phi(t,0))^{-1}f(\tau)\mathrm{d}\tau \\ &= -\int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau - \int_t^\infty \Phi^{-1}(\tau,t)f(\tau)\mathrm{d}\tau.\end{aligned}$$

It results that

$$\Phi(t,0)y(0) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau = -\int_t^\infty \Phi^{-1}(\tau,t)f(\tau)\mathrm{d}\tau,$$

which is equivalent to

$$y(t) = \Phi(t,0)y(0) + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau. \tag{3}$$

But there exists $z \in X$ with

$$y(t) = \Phi(t,0)z + \int_0^t \Phi(t,\tau)f(\tau)\mathrm{d}\tau \tag{4}$$

By decreasing the relations (3) and (4), we obtain that

$$0 = \Phi(t,0)(y(0) - z),$$

hence

$$y(0) = z.$$

It results in this way that the evolution process Φ satisfies the Perron condition for instability and the proof is complete. □

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