SUMMATION FORMULAE FOR THE LEGENDRE POLYNOMIALS

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ABSTRACT. In this paper, summation formulae for the 2-variable Legendre polynomials in terms of certain multi-variable special polynomials are derived. Several summation formulae for the classical Legendre polynomials are also obtained as applications. Further, Hermite-Legendre polynomials are introduced and summation formulae for these polynomials are also established.

1. INTRODUCTION AND PRELIMINARIES

We recall that the 2-variable Legendre polynomials (2VLeP) $R_n(x, y)$ are defined by the series [13]

$$(1.1) \qquad R_n(x, y) = (n!)^2 \sum_{k=0}^{n} \frac{(-1)^{n-k} y^k x^{n-k}}{(k!)^2 [(n-k)!]^2}$$

and specified by the following generating function

$$(1.2) \qquad C_0(-yt)\, C_0(xt) = \sum_{n=0}^{\infty} R_n(x, y) \frac{t^n}{(n!)^2}\ ,$$

where $C_0(x)$ denotes the 0th order Tricomi function. The nth order Tricomi functions $C_n(x)$ are defined as [20]

$$(1.3) \qquad C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}.$$

The 2VLeP $R_n(x, y)$ are linked to the classical Legendre polynomials $P_n(x)$ [1] by the following relation

$$(1.4) \qquad R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right) = P_n(x).$$

Further, we recall a second form of the 2-variable Legendre polynomials (2VLeP) $S_n(x, y)$ which are defined by the series [4, p. 158] (see also [9])

$$(1.5) \qquad S_n(x, y) = n! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-2k}}{(k!)^2(n-2k)!}$$

and specified by the following generating function

$$(1.6) \qquad \exp(yt)\, C_0(xt^2) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}.$$

Next, we recall that the higher-order Hermite polynomials, some times called the Kampé de Fériet or the Gould-Hopper polynomials (GHP) $H_n^{(m)}(x, y)$, are defined as [18, p. 58, (6.2)] (see also [3])

$$(1.7) \qquad g_n^m(x, y) = H_n^{(m)}(x, y) = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{y^k x^{n-mk}}{k!(n-mk)!},$$
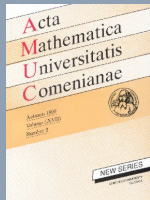
where m is a positive integer. These polynomials are specified by the generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!}. \tag{1.8}$$

In particular, we note that

$$H_n^{(1)}(x, y) = (x + y)^n, \tag{1.9}$$

$$H_n^{(2)}(x, y) = H_n(x, y), \tag{1.10}$$

where $H_n(x, y)$ denotes the 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) [2], defined by the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{1.11}$$

We note the following link between the 2VHKdFP $H_n(x, y)$ and the 2VLeP $S_n(x, y)$ [9, p. 613]

$$H_n(y, -D_x^{-1}) = S_n(x, y), \tag{1.12}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined in such a way that

$$D_x^{-n}\{f(x)\} = \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi) d\xi, \tag{1.13}$$

so that for $f(x) = 1$, we have

$$D_x^{-n}\{1\} = \frac{x^n}{n!}. \tag{1.14}$$

In view of equations (1.8) and (1.11), we note the following link

$$(1.15) \qquad H_n^{(2)}\left(x, -\frac{1}{2}\right) = H_n\left(x, -\frac{1}{2}\right) = He_n(x),$$

where $He_n(x)$ denotes the classical Hermite polynomials [1].

Also, we recall that the 2-variable generalized Laguerre polynomials (2VgLP) $_mL_n(x,y)$ are defined by the series [12, p. 213]

$$(1.16) \qquad {}_mL_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{x^r y^{n-mr}}{(r!)^2(n-mr)!}$$

and by the following generating function

$$(1.17) \qquad \exp(yt)\, C_0(-xt^m) = \sum_{n=0}^{\infty} {}_mL_n(x,y)\frac{t^n}{n!}.$$

We note the following link between the 2VgLP $_mL_n(x,y)$ and the GHP $H_n^{(m)}(x,y)$ [12, p. 213]

$$(1.18) \qquad {}_mL_n(x,y) = H_n^{(m)}(y, D_x^{-1}).$$

In particular, we note that

$$(1.19) \qquad {}_2L_n(-x,y) = S_n(x,y),$$

$$(1.20) \qquad {}_1L_n(-x,y) = L_n(x,y),$$

where $L_n(x,y)$ denotes the 2-variable Laguerre polynomials (2VLP) [14] (see also [16]), defined by means of the generating function

$$(1.21) \qquad \exp(yt)\, C_0(xt) = \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}.$$
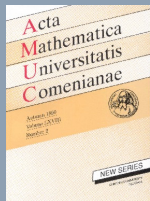
In terms of classical Laguerre polynomials $L_n(x)$ [1], it is easily seen from definition (1.16) and relation (1.20) that

$$(1.22) \qquad \qquad _1L_n(-x,1) = L_n(x,1) = L_n(x).$$

Again, we recall that the 2-variable generalized Laguerre type polynomials (2VgLtP) $_{[m]}L_n(x,y)$ are defined by the series [7, p. 603]

$$(1.23) \qquad \qquad _{[m]}L_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{y^k(-x)^{n-mk}}{k![(n-mk)!]^2}$$

and by the following generating function

$$(1.24) \qquad \qquad \exp(yt^m)\, C_0(xt) = \sum_{n=0}^{\infty} {}_{[m]}L_n(x,y)\frac{t^n}{n!}.$$

For $m=2$ and $x \to -x$, the polynomials $_{[m]}L_n(x,y)$ reduce to the 2-variable Hermite type polynomials (2VHtP) $G_n(x,y)$ [9], i.e., we have

$$(1.25) \qquad \qquad _{[2]}L_n(-x,y) = G_n(x,y).$$

In view of equations (1.21) and (1.24), we note the following link

$$(1.26) \qquad \qquad _{[1]}L_n(x,y) = L_n(x,y).$$

Further, we recall that the 3-variable Laguerre-Hermite polynomials (3VLHP) $_LH_n(x,y,z)$ are defined by the series [15, p. 241]

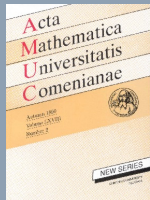$$(1.27) \qquad \qquad _LH_n(x,y,z) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{z^k L_{n-2k}(x,y)}{k!(n-2k)!}$$

and specified by the following generating function

$$\exp(yt + zt^2) \, C_0(xt) = \sum_{n=0}^{\infty} {}_L H_n(x, y, z) \frac{t^n}{n!}. \tag{1.28}$$

In particular, we note that

$${}_L H_n \left(x, y, -\frac{1}{2} \right) = {}_L H_n^*(x, y), \tag{1.29}$$

$${}_L H_n(x, 1, -1) = {}_L H_n(x), \tag{1.30}$$

where ${}_L H_n^*(x, y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) [16] and ${}_L H_n(x)$ denotes the Laguerre-Hermite polynomials (LHP) [17], respectively.

Furthermore, we recall that the 3-variable Hermite-Laguerre polynomials (3VHLP) ${}_H L_n(x, y, z)$ are defined by the series [11, p. 58]

$${}_H L_n(x, y, z) = n! \sum_{k=0}^{n} \frac{(-1)^k z^{n-k} H_k(x, y)}{(k!)^2 (n-k)!} \tag{1.31}$$

and by the following generating function

$$\exp(zt) \, {}_H C_0(x, y; t) = \sum_{n=0}^{\infty} {}_H L_n(x, y, z) \frac{t^n}{n!}, \tag{1.32}$$

where ${}_H C_0(x, y; t)$ denotes the Hermite-Tricomi functions defined by the following operational definition [11, p. 58]

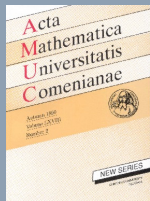$${}_H C_0(x, y; t) = \exp \left(y \frac{\partial^2}{\partial x^2} \right) \{ C_0(xt) \}. \tag{1.33}$$

The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums, involving special functions. Problems of this type arise, for example, in the computation of the higher-order moments of a distribution or in evaluation of transition matrix elements in quantum mechanics. In [5], Dattoli showed that the summation formulae of special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of Hermite polynomials of more than one variable.

In this paper, we derive the explicit summation formulae for the 2VLeP $R_n(x, y)$ in terms of the product of certain multi-variable special polynomials. Also, we derive the implicit summation formula for the 2VLeP $S_n(x, y)$. Summation formulae for the classical Legendre polynomials $P_n(x)$ are obtained as special cases of the summation formulae for the 2VLeP $R_n(x, y)$. Further, the Hermite-Legendre polynomials $_H R_n(x, y, z)$ are introduced and summation formulae for these polynomials are also obtained.

2. Summation formulae for the 2-variable Legendre polynomials

First, we prove the following explicit summation formula for the 2VLeP $R_n(x, y)$ by using generating functions (1.8), (1.21) and (1.24) of GHP $H_n^{(m)}(x, y)$, 2VLP $L_n(x, y)$ and 2VgLtP $_{[m]}L_n(x, y)$, respectively.

Theorem 2.1. *The following explicit summation formula for the 2VLeP $R_n(x,y)$ in terms of the product of GHP $H_n^{(m)}(x,y)$, 2VgLtP $_{[m]}L_n(x,y)$ and 2VLP $L_n(x,y)$ holds true*

(2.1)
$$R_n(x,z)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k^{(m)}(-y,-w) \, _{[m]}L_r(-z,w) \, L_{n-k-r}(x,y).$$

Proof. Consider the product of 2VLP generating function (1.21) and 2VgLtP generating function (1.24) in the following form

(2.2)
$$\exp(yt) \, C_0(xt) \, \exp(wt^m) \, C_0(zt)$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} L_n(x,y) \, _{[m]}L_r(z,w) \, \frac{t^{n+r}}{n!r!}.$$

Replacing n by $n-r$ in the r.h.s. of equation (2.2) and then using the lemma [20, p. 100]

(2.3)
$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A(r,n) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} A(r,n-r),$$

we find

(2.4)
$$C_0(xt) \, C_0(zt) \, \exp(yt+wt^m)$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \, _{[m]}L_r(z,w) \, L_{n-r}(x,y) \, \frac{t^n}{n!},$$
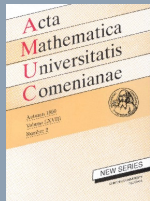
which on shifting the exponential to the r.h.s. and then using the generating function (1.8) in the r.h.s. becomes

$$
\begin{aligned}
C_0(xt)\ &C_0(zt) \\
(2.5) \qquad &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} H_k^{(m)}(-y,-w)\ _{[m]}L_r(z,w)\ L_{n-r}(x,y)\ \frac{t^{n+k}}{n!k!}.
\end{aligned}
$$

Again, replacing n by $n-k$ in the r.h.s. of equation (2.5), we get

$$
\begin{aligned}
C_0(zt)\ &C_0(xt) \\
(2.6) \qquad &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r} H_k^{(m)}(-y,-w)\ _{[m]}L_r(z,w)\ L_{n-k-r}(x,y)\ \frac{t^{n}}{n!}.
\end{aligned}
$$

Finally, using generating function (1.2) in the l.h.s. of equation (2.6) and then equating the coefficients of like powers of t in the resultant equation, we get assertion (2.1) of Theorem 2.1. $\qquad\square$

Remark 2.1. Taking $m = 2$ in assertion (2.1) of Theorem 2.1 and using relations (1.10) and (1.25), we deduce the following consequence of Theorem 2.1.

Corollary 2.1. *The following summation formula for the 2VLeP $R_n(x,y)$ involving product of 2VHKdFP $H_n(x,y)$, 2VHtP $G_n(x,y)$ and 2VLP $L_n(x,y)$ holds true*

$$
\begin{aligned}
R_n(x,z) \\
(2.7) \qquad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k}\binom{n-k}{r} H_k(-y,-w)\ G_r(z,w)\ L_{n-k-r}(x,y).
\end{aligned}
$$

Note. For $y = 1$, equation (2.7) yields to the following summation formula

(2.8)
$$R_n(x, z)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-1, -w) \; G_r(z, w) \; L_{n-k-r}(x).$$

Again, for $w = \frac{1}{2}$, equation (2.7) yields to the following summation formula

(2.9)
$$R_n(x, z)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} He_k(-y) \; G_r\left(z, \frac{1}{2}\right) \; L_{n-k-r}(x, y).$$

Remark 2.2. Taking $m = 1$ in assertion (2.1) of Theorem 2.1 and using relations (1.9) and (1.26), we deduce the following consequence of Theorem 2.1.

Corollary 2.2. *The following explicit summation formula for the 2VLeP $R_n(x, y)$ in terms of product of the 2VLP $L_n(x, y)$ holds true*

(2.10)
$$R_n(x, z)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} (-1)^k (y + w)^k \; L_r(-z, w) \; L_{n-k-r}(x, y).$$

Note. For $w = -y$, equation (2.10) yields to the following summation formula

(2.11)
$$R_n(x, z) = n! \sum_{r=0}^{n} \binom{n}{r} L_r(-z, -y) \; L_{n-r}(x, y),$$

which for $y = 1$ gives the following summation formula

$$(2.12) \qquad R_n(x, z) = n! \sum_{r=0}^{n} \binom{n}{r} L_r(-z, -1) \, L_{n-r}(x).$$

Again, for $y = w = 1$, equation (2.10) yields to the following summation formula:

$$(2.13) \qquad R_n(x, z) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} (-2)^k \, L_r(-z) \, L_{n-k-r}(x).$$

Remark 2.3. Using generating functions (1.11), (1.21) and (1.28) of 2VHKdFP $H_n(x, y)$, 2VLP $L_n(x, y)$ and 3VLHP $_L H_n(x, y, z)$ respectively and proceeding on the same lines of proof of Theorem 2.1, we get the following result.

Theorem 2.2. *The following explicit summation formula for the 2VLeP $R_n(x, y)$ in terms of the product of 2VHKdFP $H_n(x, y)$, 3VLHP $_L H_n(x, y, z)$ and 2VLP $L_n(x, y)$ holds true*

$$R_n(x, z)$$
$$(2.14)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-y-w, -v) \, _L H_r(-z, w, v) \, L_{n-k-r}(x, y).$$

Note. For $y = 1$ and $v = -\frac{1}{2}$, equation (2.14) yields to the following summation formula

$$R_n(x, z)$$
$$(2.15)$$
$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k \left(-1 - w, \frac{1}{2} \right) \, _L H_r^*(-z, w) \, L_{n-k-r}(x).$$
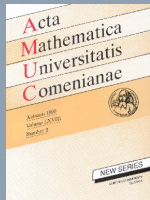
Again, for $y = 1$ and $v = \frac{1}{2}$, equation (2.14) yields to the following summation formula

$$
\begin{aligned}
&R_n(x, z) \\
(2.16) \quad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} He_k(-1-w) \; _L H_r\left(-z, w, \frac{1}{2}\right) \; L_{n-k-r}(x).
\end{aligned}
$$

Further, for $w = 1$ and $v = -1$, equation (2.14) yields to the following summation formula

$$
\begin{aligned}
&R_n(x, z) \\
(2.17) \quad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-y-1, 1) \; _L H_r(-z) \; L_{n-k-r}(x, y).
\end{aligned}
$$

Next, we prove the following result involving the 2VLeP $S_n(x, y)$.

Theorem 2.3. *The following implicit summation formula for the 2VLeP $S_n(x, y)$ holds true*

$$
(2.18) \quad S_{k+l}(y, w) = \sum_{n,r=0}^{k,l} \binom{k}{n} \binom{l}{r} (w - x)^{n+r} S_{k+l-n-r}(y, x).
$$

Proof. We start by a recently derived summation formula for the 2VHKdFP $H_n(x, y)$ [19, p. 1539]

$$
(2.19) \quad H_{k+l}(w, y) = \sum_{n,r=0}^{k,l} \binom{k}{n} \binom{l}{r} (w - x)^{n+r} H_{k+l-n-r}(x, y).
$$

Replacing y by $-D_y^{-1}$ in the above equation, we have

$$(2.20) \qquad H_{k+l}(w, -D_y^{-1}) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r}(w-x)^{n+r} H_{k+l-n-r}(x, -D_y^{-1}),$$

which using relation (1.12) gives assertion (2.18) of Theorem 2.3. $\qquad\square$

Alternate proof. Replacing y by D_y^{-1} in the following result [**19**, p. 1538]

$$(2.21) \qquad H_{k+l}^{(m)}(w, y) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r}(w-x)^{n+r} H_{k+l-n-r}^{(m)}(x, y)$$

and then using link (1.18), we get the following summation formula for 2VgLP $_m L_n(x, y)$

$$(2.22) \qquad {}_m L_{k+l}(y, w) = \sum_{n,r=0}^{k,l} \binom{k}{n}\binom{l}{r}(w-x)^{n+r} {}_m L_{k+l-n-r}(y, x).$$

Now, taking $m = 2$ and replacing y by $-y$ in equation (2.22) and using relation (1.19), we get assertion (2.18) of Theorem 2.3. $\qquad\square$

Remark 2.4. Taking $l = 0$ in assertion (2.18) of Theorem 2.3, we deduce the following consequence of Theorem 2.3

Corollary 2.3. *The following implicit summation formula for the 2VLeP $S_n(x,y)$ holds true*

$$(2.23) \qquad S_k(y, w) = \sum_{n=0}^{k} \binom{k}{n}(w-x)^n \, S_{k-n}(y, x).$$
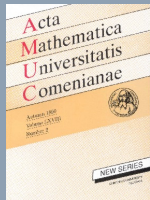
3. Applications

In this section, we derive the summation formulae for the classical Legendre polynomials $P_n(x)$ as applications of the results derived in the previous section.

I. Replacing x by $\frac{1-x}{2}$ and z by $\frac{1+x}{2}$ in equations (2.1), (2.7), (2.10) and (2.11) and using relation (1.4), we get the following explicit summation formulae for the classical Legendre polynomials $P_n(x)$

(3.1) $P_n(x)$

$$= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k^{(m)}(-y, -w) {}_{[m]}L_r \left(\frac{-1-x}{2}, w \right) L_{n-k-r} \left(\frac{1-x}{2}, y \right),$$

(3.2) $$P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-y, -w) G_r \left(\frac{1+x}{2}, w \right) L_{n-k-r} \left(\frac{1-x}{2}, y \right),$$

(3.3) $$P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} (-1)^k (y+w)^k L_r \left(\frac{-1-x}{2}, w \right) L_{n-k-r} \left(\frac{1-x}{2}, y \right),$$

(3.4) $$P_n(x) = n! \sum_{r=0}^{n} \binom{n}{r} L_r \left(\frac{-1-x}{2}, -y \right) L_{n-r} \left(\frac{1-x}{2}, y \right).$$

Next, taking $y = 1$ in equations (3.2) and (3.4) and using relation (1.22), we get

$$(3.5) \qquad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-1, -w) \, G_r\left(\frac{1+x}{2}, w\right) \, L_{n-k-r}\left(\frac{1-x}{2}\right),$$

$$(3.6) \qquad P_n(x) = n! \sum_{r=0}^{n} \binom{n}{r} L_r\left(\frac{-1-x}{2}, -1\right) \, L_{n-r}\left(\frac{1-x}{2}\right).$$

Further, taking $y = w = 1$ in equation (3.3) and using relation (1.22), we get

$$(3.7) \qquad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} (-2)^k \, L_r\left(\frac{-1-x}{2}\right) \, L_{n-k-r}\left(\frac{1-x}{2}\right).$$

Furthermore, taking $w = \frac{1}{2}$ in equation (3.2) and using relation (1.15), we get

$$(3.8) \qquad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} He_k(-y) \, G_r\left(\frac{1+x}{2}, \frac{1}{2}\right) \, L_{n-k-r}\left(\frac{1-x}{2}, y\right).$$

II. Replacing x by $\frac{1-x}{2}$ and z by $\frac{1+x}{2}$ in equation (2.14) and using relation (1.4), we get the following explicit summation formulae for the classical Legendre polynomials $P_n(x)$

$$(3.9) \qquad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r}$$
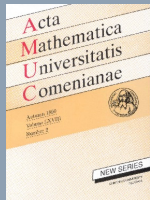$$\times \, H_k(-y-w, -v) \, _LH_r\left(\frac{-1-x}{2}, w, v\right) \, L_{n-k-r}\left(\frac{1-x}{2}, y\right).$$

Next, taking $y = 1$ and $v = -\frac{1}{2}$ in equation (3.9) and using relations (1.22) and (1.29), we get

$$
\begin{aligned}
(3.10) \quad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} \\
\times H_k\left(-1-w, \frac{1}{2}\right) \, {}_L H_r^*\left(\frac{-1-x}{2}, w\right) \, L_{n-k-r}\left(\frac{1-x}{2}\right).
\end{aligned}
$$

Further, taking $y = 1$ and $v = \frac{1}{2}$ in equation (3.9) and using relations (1.15) and (1.22), we get

$$
(3.11) \quad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} He_k(-1-w) \, {}_L H_r\left(\frac{-1-x}{2}, w, \frac{1}{2}\right) L_{n-k-r}\left(\frac{1-x}{2}\right).
$$

Furthermore, taking $w = 1$ and $v = -1$ in equation (3.9) and using relation (1.30), we get

$$
(3.12) \quad P_n(x) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-y-1, 1) \, {}_L H_r\left(\frac{-1-x}{2}\right) L_{n-k-r}\left(\frac{1-x}{2}, y\right).
$$

4. Concluding remarks

Operational methods can be used to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of functions. We recall that the 2VHKdFP $H_n(x, y)$ have the following operational definition

$$
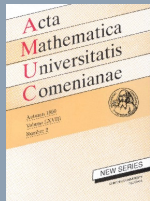(4.1) \qquad H_n(x, y) = \exp\left(y \frac{\partial^2}{\partial x^2}\right) \{x^n\}.
$$

Now, in view of definition (4.1) the 3VHLP $_H L_n(x, y, z)$ are specified by the following operational definition [11, p. 58]

$$(4.2) \qquad _H L_n(x, y, z) = \exp\left(y \frac{\partial^2}{\partial x^2}\right) \left\{L_n(x, z)\right\}.$$

In order to introduce Hermite-Legendre polynomials (HLeP) $_H R_n(x, y, z)$, we replace y by z in generating function (1.2) and then operate $\exp\left(y \frac{\partial^2}{\partial x^2}\right)$ on both sides of the resultant equation. Now, using operational definition (1.33) in the l.h.s. of the resultant equation, we get the following generating function of the HLeP $_H R_n(x, y, z)$

$$(4.3) \qquad C_0(-zt) \, _H C_0(x, y; t) = \sum_{n=0}^{\infty} \, _H R_n(x, y, z) \frac{t^n}{(n!)^2},$$

where $_H R_n(x, y, z)$ are defined as

$$(4.4) \qquad _H R_n(x, y, z) = \exp\left(y \frac{\partial^2}{\partial x^2}\right) \left\{R_n(x, z)\right\}.$$

It is worthy to note that the method adopted in this paper can be exploited to establish further consequences regarding other families of special polynomials. Here, we establish summation formulae for the HLeP $_H R_n(x, y, z)$. To this aim, we consider the product of generating functions (1.24) and (1.32) of the 2VgLtP and 3VHLP respectively, in the following form

$$\exp(zt) _H C_0(x, y; t) \, \exp(vt^m) \, C_0(wt)$$

$$(4.5) \qquad = \sum_{n,r=0}^{\infty} \, _H L_n(x, y, z) \, _{[m]} L_r(w, v) \frac{t^{n+r}}{n! r!}.$$

Now, following the same lines of proof of Theorem 2.1 and in view of generating function (4.3), we get the following summation formula for the HLeP $_H R_n(x, y, z)$

$$
\begin{aligned}
{}_H R_n &(x, y, w) \\
(4.6) \quad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k^{(m)}(-z, -v) \; {}_{[m]} L_r(-w, v) \; {}_H L_{n-k-r}(x, y, z),
\end{aligned}
$$

which for $m = 2$ gives the following summation formula

$$
\begin{aligned}
{}_H R_n &(x, y, w) \\
(4.7) \quad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} H_k(-z, -v) \; G_r(-w, v) \; {}_H L_{n-k-r}(x, y, z).
\end{aligned}
$$

Again, for $m = 1$, equation (4.6) yields to the following summation formula

$$
\begin{aligned}
{}_H R_n &(x, y, w) \\
(4.8) \quad &= n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r} (-z - v)^k \; L_r(-w, v) \; {}_H L_{n-k-r}(x, y, z).
\end{aligned}
$$

We remark that the summation formula (4.6) can also be obtained after replacing y by z, z by w and w by v in assertion (2.1) of Theorem 2.1 and operating $\exp\left(y \frac{\partial^2}{\partial x^2}\right)$ on the resultant equation and then using operational definitions (4.2) and (4.4).

Similarly, by considering the product of generating functions (1.28) and (1.32) of the 2VgLtP and 3VHLP, respectively, and following the same method, we get another summation formula for

the HLeP $_H R_n(x, y, z)$

$$_H R_n(x, y, w) = n! \sum_{k=0}^{n} \sum_{r=0}^{n-k} \binom{n}{k} \binom{n-k}{r}$$

(4.9)
$$\times H_k(-z - v, -u) \; _L H_r(-w, v, u) \; _H L_{n-k-r}(x, y, z),$$

which can also be obtained after replacing y by z, z by w, w by v and v by u in assertion (2.14) of Theorem 2.2 and operating $\exp\left(y \frac{\partial^2}{\partial x^2}\right)$ on the resultant equation and then using operational definitions (4.2) and (4.4).

Very recently Dattoli et al. [8] introduced a two-variable extension of the Legendre polynomials $P_n(x, y)$, defined by the generating function

(4.10)
$$\frac{1}{\sqrt{1 + xt + yt^2}} = \sum_{n=0}^{\infty} P_n(x, y) \; t^n.$$

To give another example of the method adopted in this paper, we derive a summation formula for the 2-variable Chebyshev polynomials (2VCP) $U_n(x, y)$ [6] in terms of the product of the polynomials $P_n(x, y)$. To this aim, we consider the product of generating function (4.10) in the following form

(4.11)
$$\frac{1}{(1 + xt + yt^2)} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} P_n(x, y) P_r(x, y) \; t^{n+r}.$$

Replacing x by $-2x$ in equation (4.11) and then replacing n by $n-r$ in the r.h.s. of the resultant equation, we find

(4.12)
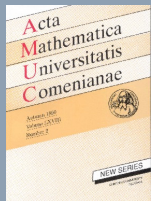$$\frac{1}{(1 - 2xt + yt^2)} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} P_{n-r}(-2x, y) P_r(-2x, y) \; t^n.$$

Now, using the generating function [**10**, p. 43] (see also [**6**])

$$(4.13) \qquad \frac{1}{(1 - 2xt + yt^2)} = \sum_{n=0}^{\infty} U_n(x, y) \ t^n$$

of the 2VCP $U_n(x, y)$ in the l.h.s. of equation (4.12), we get the following summation formula

$$(4.14) \qquad U_n(x, y) = \sum_{r=0}^{n} P_{n-r}(-2x, y) \ P_r(-2x, y).$$

The above examples prove the usefulness of the method adopted in this paper. Further, to bolster up the contention of using operational techniques, certain new families of special polynomials will be introduced in a forthcoming investigation.

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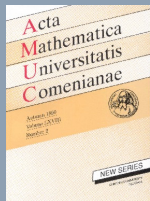
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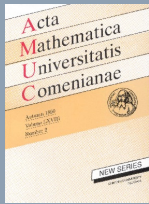
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