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DUNFORD-PETTIS SETS IN BANACH LATTICES

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Abstract. We study the class of Dunford-Pettis sets in Banach lattices. In particular, we establish some sufficient conditions for which a Dunford-Pettis set is relatively weakly compact (resp. relatively compact).

1. Introduction and notation

Let us recall from [**2**] that a norm bounded subset A of a Banach space X is said to be a *Dunford-Pettis set* whenever every weakly compact operator from X to an arbitrary Banach space Y carries A to a norm relatively compact set of Y. This is equivalent to saying that A is a Dunford-Pettis set if and only if every weakly null sequence (f_n) of X' converges uniformly to zero on the set A, that is, $\sup_{x\in\mathbf{A}}|f_n(x)| \to 0$ (see [**7**, Theorem 1]).

It is well known that the class of Dunford-Pettis sets contains strictly that of relatively compact sets, that is, every relatively compact set is a Dunford-Pettis set. But a Dunford-Pettis set is not necessarily relatively compact. In fact, the closed unit ball B_{c_0} is a Dunford-Pettis set in c_0 (because $(c_0)' = \ell^1$ has the Schur property), but it is not relatively compact. However, if X is a reflexive Banach space, the class of Dunford-Pettis sets and that of relatively compact sets in X coincide. Also, we will prove that if E is a discrete KB-space, these two classes coincide (Corollary 3.10).

On the other hand, we note that a Dunford-Pettis set is not necessarily relatively weakly compact. In fact, the closed unit ball B_{c_0} is a Dunford-Pettis set in c_0, but it is not relatively weakly compact. And conversely a relatively weakly compact set is not necessarily Dunford-Pettis. In fact, the closed unit ball B_{ℓ^2} is a relatively weakly compact set in ℓ^2, but it is not a Dunford-Pettis set in ℓ^2.

However, we will establish that if E is a dual KB-space, then each Dunford-Pettis set of E is relatively weakly compact (see Corollary 3.4). And conversely, if X is a Banach space with the Dunford-Pettis property, then each relatively weakly compact subset of X is a Dunford-Pettis set (see Proposition 2.3).

The aim of this paper is to study the class of Dunford-Pettis sets in Banach lattices. Also, we give some consequences. As an example we will give some

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equivalent conditions for $T(A)$ to be a Dunford-Pettis set where A is a norm bounded solid subset of E and T is an operator from a Banach lattice E into a Banach space X (see Theorem 2.12).

To do this, we need to introduce a new class of operators, that we call order Dunford-Pettis operators. An operator T from a Banach lattice E into a Banach space X is called order Dunford-Pettis if it carries each order bounded subset of E onto a Dunford-Pettis set of X. For example, the identity operator of the Banach lattice c_0 is order Dunford-Pettis.

On the other hand, there exist operators which are not order Dunford-Pettis. In fact, the natural embedding $J\colon L^{\infty}[0,1] \to L^{2}[0,1]$ fails to be order Dunford--Pettis (if not, that is, if $J\colon L^{\infty}[0,1] \to L^{2}[0,1]$ is an order Dunford-Pettis operator, it follows from Theorem 2.7 that $i_{L^2[0,1]} \circ J = J$ is an AM-compact operator, but J fails to be AM-compact (see [**8**, Example on p. 222])).

Let us recall from [**8**] that an operator T from a Banach lattice E into a Banach space X is said to be AM-compact if it carries each order bounded subset of E onto a relatively compact set of X.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \le |y|$, we have $\|x\| \le \|y\|$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

An operator $T\colon E \to F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \ge 0$ in F whenever $x \ge 0$ in E. If $T\colon E \to F$ is a positive operator between two Banach lattices, then its adjoint $T'\colon F' \to E'$, defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$, is also positive. We refer the reader to [**2**] for unexplained terminologies on Banach lattice theory and positive operators.

2. Main results

The following result gives some characterizations of Dunford-Pettis sets in a Banach space.

Proposition 2.1 ([**7**]). *Let X be a Banach space and let A be a norm bounded set in X. The following statements are equivalent:*

1. *A is Dunford-Pettis set.*
2. *For each sequence (x_n) in A, $f_n(x_n) \to 0$ for every weakly null sequence (f_n) of X'.*

Proposition 2.2. *Let X be a Banach space and let (x_n) be a norm bounded sequence in X. The following statements are equivalent:*

1. *The subset $\{x_n,\ n \in \mathbb{N}\}$ is a Dunford-Pettis set.*

2. *$f_k(y_k) \to 0$ for each sequence (y_k) of $\{x_n,\ n \in \mathbb{N}\}$ and for every weakly null sequence (f_k) of X'.*
3. *$f_n(x_n) \to 0$ for every weakly null sequence (f_n) of X'.*

A Banach space X has the Dunford-Pettis property if every continuous weakly compact operator T from X into another Banach space Y transforms weakly compact sets in X into norm-compact sets in Y. This is equivalent to the saying that for any weakly convergent sequences (x_n) of X and (f_n) of X', the sequence $(f_n(x_n))$ converges.

Proposition 2.3 ([**6**])**.** *Let X be a Banach space. Then the following statements are equivalent:*

1. *X has the Dunford-Pettis property.*
2. *For every weakly null sequence (x_n) in X, the subset $\{x_n,\ n \in \mathbb{N}\}$ is a Dunford-Pettis set.*
3. *Every relatively weakly compact subset of X is a Dunford-Pettis set.*

Remark 1. If the Banach space X does not have the Dunford-Pettis property, then there exists a weakly null sequence (x_n) in X such that $\{x_n,\ n \in \mathbb{N}\}$ is not a Dunford-Pettis set.

Let us recall that a Banach lattice E has the weak Dunford-Pettis property if every weakly compact operator T defined on E (and taking their values in a Banach space X) is almost Dunford-Pettis, that is, the sequence $(\|T(x_n)\|)$ converges to 0 in X for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E. This is equivalent to the saying that for any weakly null sequence (x_n) consisting of pairwise disjoint elements in E and for any weakly null sequence (f_n) of X', $f_n(x_n) \to 0$.

Proposition 2.4. *Let E be a Banach lattice. Then the following statements are equivalent:*

1. *X has the weak Dunford-Pettis property.*
2. *For every disjoint weakly null sequence (x_n) E^+, the subset $\{x_n,\ n \in \mathbb{N}\}$ is a Dunford-Pettis set.*

Remark 2. If the Banach lattice E does not have the weak Dunford-Pettis property, then there exists a disjoint weakly null sequence (x_n) in E^+ such that $\{x_n,\ n \in \mathbb{N}\}$ is not a Dunford-Pettis set.

Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be order weakly compact if for each $x \in E^+$, the set $T([0,x])$ is relatively weakly compact in X.

The following result gives some examples of Dunford-Pettis sets in a Banach lattice.

Theorem 2.5. *Let E be a Banach lattice. Then for every order bounded disjoint sequence (x_n) in E, the subset $\{x_n,\ n \in \mathbb{N}\}$ is a Dunford-Pettis set.*

Proof. Let (x_n) be an order bounded disjoint sequence in E. To prove that $\{x_n,\ n \in \mathbb{N}\}$ is a Dunford-Pettis set it suffices to show that $f_n(x_n) \to 0$ for every weakly null sequence (f_n) of X' (see Proposition 2.2).

For that, let (f_n) be a weakly null sequence in E'. Consider the operator $S\colon E \to c_0$ defined by $S(x) = (f_n(x))_{n=0}^{\infty}$ for each $x \in E$. Then S is weakly compact ([**2**, Theorem 5.26]), and so S is order weakly compact. Hence by [**2**, Theorem 5.57] $\|S(z_i)\|_\infty = \|(f_n(z_i))_{n=0}^{\infty}\|_\infty \to 0$ for every order bounded disjoint sequence (z_i) in E. Finally, $|f_n(x_n)| \le \|(f_i(x_n))_{i=0}^{\infty}\|_\infty \to 0$, so the proof is finished. □

Remark 3. In ℓ^∞, the closed unit ball $B_{\ell^\infty} = [-\mathrm{e}, \mathrm{e}]$ is not a Dunford-Pettis set. Hence there exists a sequence (x_n) in $[-\mathrm{e}, \mathrm{e}]$ such that (x_n) is not Dunford-Pettis (Proposition 2.1).

Let E be a Banach lattice and E' its topological dual. The absolute weak topology $|\sigma|(E, E')$ is the locally convex solid topology on E generated by the family of lattice seminorms $\{P_f : f \in E'\}$ where $P_f(x) = |f|(|x|)$ for each $x \in E$. For more information about locally convex solid topologies, we refer the reader to the book of Aliprantis and Burkinshaw [**1**].

Other examples of Dunford-Pettis sets in a Banach lattice, are given by the following Theorem.

Theorem 2.6. *Let E be a Banach lattice and let A be an order bounded set of E. If A is $|\sigma|(E, E')$-totally bounded, then A is a Dunford-Pettis set.*

Proof. Let (f_n) be a weakly null sequence in E', let $x \in E^+$ such that $|y| \le x$ for every $y \in A$. Fix $\varepsilon > 0$. By [**2**, Theorem 4.37] there exists $f \in (E')^+$ such that $(|f_n| - f)^+(x) < \frac{\varepsilon}{4}$ for each n.

Since A is $|\sigma|(E, E')$-totally bounded, there exists a finite set $\{x_1, \ldots, x_k\} \subset A$ such that for each $z \in A$, we have $f(|z - x_i|) < \frac{\varepsilon}{4}$ for at least one $1 \le i \le k$. Since $f_n \to 0$ weakly, there exists N with $|f_n(x_i)| < \frac{\varepsilon}{4}$ for each $i = 1, \ldots, k$ and all $n \ge N$.

Now, let $z \in A$. Choose $1 \le i \le k$ with $f(|z - x_i|) < \frac{\varepsilon}{4}$ and note that $|z - x_i| \le 2x$ holds. In particular, for $n \ge N$, we have

$$\begin{aligned}
|f_n(z)| &\le |f_n(z - x_i)| + |f_n(x_i)| \\
&\le |f_n|(|z - x_i|) + \frac{\varepsilon}{4} \\
&\le (|f_n| - f)^+(|z - x_i|) + f(|z - x_i|) + \frac{\varepsilon}{4} \\
&\le 2(|f_n| - f)^+(x) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&\le \varepsilon.
\end{aligned}$$

This implies $\sup_{z \in \mathbf{A}} |f_n(z)| \to 0$, and then A is a Dunford-Pettis set. □

The next result characterizes the class of order Dunford-Pettis operators.

Theorem 2.7. *For an operator T from a Banach lattice E into a Banach space X, the following statements are equivalent:*

1. *$T\colon E \to X$ is an order Dunford-Pettis operator.*
2. *If S is a weakly compact operator from X into an arbitrary Banach space Z, then $S \circ T$ is an AM-compact operator.*
3. *For every weakly null sequence (f_n) of X', $|T'(f_n)| \to 0$ for the topology $\sigma(E', E)$.*

Proof. (1) ⇒ (2) Let S be a weakly compact operator from X into an arbitrary Banach space Z. It follows from (1) that for each $x \in E^+$, $T([-x,x])$ is a Dunford-Pettis set, and hence $S(T([-x,x]))$ is a norm relatively compact subset of Z. This proves that $S \circ T$ is AM-compact.

(2) ⇒ (3) Let (f_n) be a weakly null sequence of X'. Consider the operator $S\colon X \to c_0$ defined by

$$S(x) = (f_n(x))_{n=0}^{\infty} \qquad \text{for each} \quad x \in X.$$

Then S is weakly compact ([**2**, Theorem 5.26]). But according to our hypothesis, $S(T([-x,x]))$ is a norm relatively compact subset of c_0 for each $x \in E^+$. From this it follows that $|T'(f_n)|(x) = \sup_{y\in[-x,x]} |T'(f_n)(y)| = \sup_{z\in T([-x,x])} |f_n(z)| \to 0$ for each $x \in E^+$ (see [**2**, Exercise 14 in Section 3.2]).

(3) ⇒ (1) For each $x \in E^+$, $\sup_{y\in T([-x,x])} |f_n(y)| = |T'(f_n)|(x) \to 0$ for every weakly null sequence (f_n) of X'. This shows that $T([-x,x])$ is a Dunford-Pettis set for each $x \in E^+$. ☐

As a consequence of Theorem 2.6 and Theorem 2.7, we obtain the following corollaries

Corollary 2.8. *Let $T\colon E \to F$ be a regular operator between two Banach lattices such that $T([-x,x])$ is $|\sigma|(F,F')$-totally bounded for each $x \in E^+$. If $f_n \to 0$ for $\sigma(F', F'')$, then $|T'(f_n)| \to 0$ for $\sigma(E', E)$.*

Proof. By Theorem 2.6, the subset $T([-x,x])$ is a Dunford-Pettis set for each $x \in E^+$ and the conclusion follows from Theorem 2.7. ☐

We note that each AM-compact operator from a Banach lattice E into a Banach space F is order Dunford-Pettis. However an order Dunford-Pettis operator is not necessarily AM-compact. In fact, the identity operator of the Banach lattice $L^1[0,1]$ is order Dunford-Pettis but it is not AM-compact.

Corollary 2.9. *Let E and F be two Banach lattices such that F is reflexive. Then the class of order Dunford-Pettis operators from E into F coincide with that of AM-compact operators from E into F.*

Proof. It suffices to show that if $T\colon E \to F$ is an order Dunford-Pettis operator, then T is AM-compact. In fact, since F is reflexive, its identity operator $\mathrm{Id}_F\colon F \to F$ is weakly compact. Hence Theorem 2.7 implies $\mathrm{Id}_F \circ T = T$ is an AM-compact operator. ☐

Corollary 2.10. *Let E be a Banach lattice. Then the following statements are equivalent:*

1. *The identity operator $\mathrm{Id}_E\colon E \to E$ is order Dunford-Pettis.*

2. *For each* $x \in E^+$, $[-x, x]$ *is a Dunford-Pettis set.*
3. *For every weakly null sequence* (f_n) *of* E', *we have* $|f_n| \to 0$ *for* $\sigma(E', E)$.
4. *Every weakly compact operator from* E *into an arbitrary Banach space is* AM*-compact.*

Lemma 2.11. *Let* E *be a Banach lattice, let* A *be a norm bounded subset of* E^+, $(x_n) \subset A$, $(f_n) \subset (E')^+$ *and* $\varepsilon > 0$. *If* $f_n(x) \to 0$ *for every* $x \in A$, *then there exists a subsequence* (x_{n_k}, f_{n_k}) *of* (x_n, f_n) *such that*

$$f_{n_k}\left(\sum_{j=1}^{k-1} x_{n_j}\right) < \frac{\varepsilon}{2^{2k+2}} \qquad \text{for } k \geq 2.$$

Proof. Put $x_{n_1} = x_1$ and $f_{n_1} = f_1$. Since $f_n(x_{n_1}) = f_n(x_1) \to 0$, there exists $n_2 > n_1 = 1$ such that $f_{n_2}(x_{n_1}) < \frac{\varepsilon}{2^{4+2}}$. Now, assume constructed $(x_{n_k})_{k=1}^p$, $(f_{n_k})_{k=1}^p$ such that $f_{n_k}(\sum_{j=1}^{k-1} x_{n_j}) < \frac{\varepsilon}{2^{2k+2}}$ for all $k \in \{2, \ldots, p\}$. As $f_n(\sum_{k=1}^p x_{n_k}) = \sum_{k=1}^p f_n(x_{n_k}) \to 0$, there exists $n_{p+1} > n_p$ such that

$$f_{n_{p+1}}\left(\sum_{k=1}^{p} x_{n_k}\right) < \frac{\varepsilon}{2^{2(p+1)+2}}.$$

This completes the proof. □

The next main result gives some equivalent conditions for $T(A)$ to be a Dunford-Pettis set where A is a norm bounded solid subset of E and T is an operator from a Banach lattice E into a Banach space X.

Theorem 2.12. *Let* T *be an operator from a Banach lattice* E *into a Banach space* X *and let* A *be a norm bounded solid subset of* E. *Then the following statements are equivalent:*

1. $T(A)$ *is a Dunford-Pettis set.*
2. *The subsets* $T([-x, x])$ *and* $\{T(x_n),\ n \in \mathbb{N}\}$ *are Dunford-Pettis for each* $x \in A^+ = A \cap E^+$ *and for each disjoint sequence* (x_n) *in* A^+.
3. *For every weakly null sequence* (f_n)*of* E', *we have* $|T'(f_n)|(x) \to 0$ *for all* $x \in A^+$ *and* $f_n(T(x_n)) \to 0$ *for each disjoint sequence* (x_n) *in* A^+.

Proof. 1. ⇒ 2. Obvious.

2. ⇒ 3. Obvious.

3. ⇒ 1. To prove that $T(A)$ is a Dunford-Pettis set, it is sufficient to show that $\sup_{x \in \mathbf{A}} |T'(f_n)(x)| \to 0$ for every weakly null sequence (f_n) of X'. Assuming this to be false, let (f_n) be such a sequence satisfying $\sup_{x \in \mathbf{A}} |T'(f_n)(x)| > \varepsilon > 0$ for some $\varepsilon > 0$ and all n. For every n there exists y_n in A^+ such that $|T'(f_n)|(y_n) > \varepsilon$. Since $T([-y, y])$ is a Dunford-Pettis set for each $y \in A^+$, then $|T'(f_n)|(y) \to 0$ for every $y \in A^+$, and hence by Lemma 2.11, we may assume (by passing to a subsequence, if necessary) that

$$|T'(f_n)|\left(\sum_{i=1}^{n-1} y_i\right) < \frac{\varepsilon}{2^{2n+2}} \qquad \text{for } n \geq 2.$$

For $n \geq 2$, let

$$x_n = \left(y_n - 4^n \sum_{i=1}^{n-1} y_i - 2^{-n} \sum_{i=1}^{\infty} 2^{-i} y_i \right)^+ .$$

Note that $\sum_{i=1}^{\infty} 2^{-i} y_i$ exists since E is a Banach space. Now, the disjointness of (x_n) follows from

$$\begin{aligned} &x_n \leq (y_n - 4^n y_m)^+ \quad \text{and} \\ &x_m \leq (y_m - 4^{-n} y_n)^+ = 4^{-n}(4^n y_m - y_n)^+ = 4^{-n}(y_n - 4^n y_m)^- \quad \text{for } m < n. \end{aligned}$$

Also, since $0 \leq x_n \leq y_n$ for every n and (y_n) in A^+, then $(x_n) \subset A^+$.

On the other hand, the inequality

$$\begin{aligned} |T'(f_n)|(x_n) &\geq |T'(f_n)| \left(y_n - 4^n \sum_{i=1}^{n-1} y_i - 2^{-n} \sum_{i=1}^{\infty} 2^{-i} y_i \right) \\ &\geq \varepsilon - \frac{\varepsilon}{4} - 2^{-n} |f_n \circ T| \left(\sum_{i=1}^{\infty} 2^{-i} y_i \right) \end{aligned}$$

shows that $|T'(f_n)|(x_n) > \frac{\varepsilon}{2}$ for n sufficiently large (because $2^{-n}|T'(f_n)| \cdot (\sum_{i=1}^{\infty} 2^{-i} y_i) \to 0$).

In view of $|T'(f_n)|(x_n) = \sup\{|f_n(T(z))| : |z| \leq x_n\}$, for each n sufficiently large there exists some $|z_n| \leq x_n$ with $|f_n(T(z_n))| > \frac{\varepsilon}{2}$. Since (z_n^+) and (z_n^-) are both norm bounded disjoint sequences in A^+, it follows from our hypothesis that

$$\begin{aligned} \frac{\varepsilon}{2} &< |f_n(T(z_n))| \\ &\leq |f_n(T(z_n^+))| + |f_n(T(z_n^-))| \to 0 \end{aligned}$$

which is impossible. This proves that $T(A)$ is a Dunford-Pettis set. □

A relationship between a solid Dunford-Pettis set and its disjoint sequences is included in the next result.

Corollary 2.13. *Let E be a Banach lattice and let A be a norm bounded solid subset of E. The following statements are equivalent*

1. *A is a Dunford-Pettis set.*
2. *The subsets $[-x, x]$ and $\{x_n, n \in \mathbb{N}\}$ are Dunford-Pettis, for each $x \in A^+$ and for each disjoint sequence (x_n) in A^+.*
3. *For every weakly null sequence (f_n) of E', we have $|f_n|(x) \to 0$ for all $x \in A^+$ and $f_n(x_n) \to 0$ for each disjoint sequence (x_n) in A^+.*

Remark 4. Let T be an operator from a Banach space X into a Banach space Y. By the equality $\sup_{y \in T(B_X)} |f_n(y)| = \|T'(f_n)\|_{X'}$ for every weakly null sequence (f_n) in Y', it follows easily that $T(B_X)$ is a Dunford-Pettis set in Y if and only if T' is a Dunford-Pettis operator, where B_X is the closed unit ball of X.

The next result characterizes the adjoint of Dunford-Pettis operators from a Banach lattice into a Banach space.

Corollary 2.14. *For an operator T from a Banach lattice E into a Banach space X, the following statements are equivalent:*

1. *The adjoint $T'\colon X' \to E'$ is Dunford-Pettis.*
2. *$T(B_E)$ is a Dunford-Pettis set.*
3. *$T\colon E \to X$ is order Dunford-Pettis and $\{T(x_n) : n \in \mathbb{N}\}$ is a Dunford-Pettis set for each disjoint sequence (x_n) in B_E^+.*
4. *$|T'(f_n)| \to 0$ for $\sigma(E', E)$ and $f_n(T(x_n)) \to 0$ for every weakly null sequence (f_n) of E' and for each disjoint sequence (x_n) in B_E^+.*

Proof. 1. ⇔ 2. See Remark 4.
2. ⇔ 3. ⇔ 4. See Theorem 2.12. □

A Banach lattice E has the Schur property if each weakly null sequence in E converges to zero in norm.

Corollary 2.15. *Let E be a Banach lattice. Then the following statements are equivalent:*

1. *E' has the Schur property.*
2. *B_E is a Dunford-Pettis set.*
3. *$|f_n| \to 0$ for $\sigma(E', E)$ and $f_n(x_n) \to 0$ for every weakly null sequence (f_n) of E' and for each disjoint sequence (x_n) in B_E^+.*

3. DUNFORD-PETTIS SETS WHICH ARE RELATIVELY WEAKLY COMPACT (RESP. RELATIVELY COMPACT)

Let us recall from [**5**] that a norm bounded subset K of the topological dual X' and of a Banach space X is called an (L) set in X' whenever every weakly null sequence (x_n) of X converges uniformly to zero on the set K, that is, $\sup_{f\in K} |f(x_n)| \to 0$.

As examples, the closed unit ball B_{ℓ^∞} is an (L) set in ℓ^∞, but the closed unit ball B_{ℓ^1} is not an (L) set in ℓ^1. On the other hand, every Dunford-Pettis set in X' is an (L) set, but an (L) set is not necessarily Dunford-Pettis. In fact in ℓ^∞, the closed unit ball B_{ℓ^∞} is an (L) set, but it is not Dunford-Pettis.

Let us recall from [**8**] that a non-empty bounded subset A of a Banach lattice E is said to be L-weakly compact if $\|x_n\| \to 0$ for every disjoint sequence (x_n) contained in the solid hull of A. Every L-weakly compact set is relatively weakly compact ([**8**, Proposition 3.6.5]). In ℓ^∞ the closed unit ball $B_{\ell^\infty} = [-e, e]$ is an (L) set, but it is not relatively weakly compact, and then it is not L-weakly compact.

In the following we use this notion to give a characterization of the order continuity of the dual norm.

Theorem 3.1. *Let E be a Banach lattice. The following statements are equivalent:*

1. *The norm of E' is order continuous.*
2. *Any (L) set in E' is L-weakly compact.*
3. *Any (L) set in E' is relatively weakly compact.*
4. *Each Dunford-Pettis operator from E to any Banach space X is weakly compact.*

Proof. 1. ⇒ 2. Let K be an (L) set in E' and for each $x \in E$, let

$$\rho_K(x) = \sup\{|x'|(|x|) : x' \in K\} = \sup\{x'(z) : x' \in K \ \text{ and } \ |z| \le |x|\}.$$

Since K is norm bounded, $\rho_K(x) \in \mathbb{R}$ holds for each $x \in E$, and clearly ρ_K is a lattice seminorm on E.

On the other hand, if (x_n) is a disjoint sequence of B_E where B_E is the closed unit ball of E, then $\rho_K(x_n) \to 0$ holds. To see this, let $\varepsilon > 0$. For each n choose $x'_n \in K$ and $|z_n| \le |x_n|$ with $\rho_K(x_n) < \varepsilon + x'_n(z_n)$. Since the norm of E' is order continuous and as (z_n) is a disjoint sequence of B_E (because $|z_n| \le |x_n|$ and (x_n) is disjoint), it follows from [**8**, Theorem 2.4.14] that $z_n \to 0$ weakly. Hence the definition of an (L) set in E' proves that $x'_n(z_n) \to 0$, and so $\limsup \rho_K(x_n) < \varepsilon$ holds for all $\varepsilon > 0$. Therefore, $\lim \rho_K(x_n) \to 0$. Finally, by [**8**, Proposition 3.6.3], we have K is L-weakly compact.

2. ⇒ 3. Follows from of [**8**, Proposition 3.6.5].

3. ⇒ 4. Let $T\colon E \to X$ be a Dunford-Pettis operator. Then $T'(B_{X'})$ is an (L) set in E' where $B_{X'}$ is the closed unit ball of X'. Hence, 3. proves that $T'(B_{X'})$ is relatively weakly compact, and then T' (and T) is weakly compact.

4. ⇒ 1. See [**2**, Theorem 5.102]. ☐

A Dunford-Pettis set in E' is not necessarily relatively weakly compact. In fact, let $i\colon c_0 \to \ell^\infty$ be the canonical injection of c_0 into ℓ^∞. Then $i(B_{c_0})$ is a Dunford--Pettis set in ℓ^∞ (B_{c_0} is a Dunford-Pettis set in c_0), but it is not relatively weakly compact ($i\colon c_0 \to \ell^\infty$ is not weakly compact).

Corollary 3.2. *Let E be a Banach lattice such that the norm of E' is order continuous. Then any Dunford-Pettis set in E' is relatively weakly compact.*

Proof. Let K be a Dunford-Pettis set in E'. By the definition of Dunford-Pettis set, K is an (L) set in E'. Theorem 3.1 concludes the proof. ☐

A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space. It is clear that each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-space. In fact, the Banach lattice c_0 has an order continuous norm, but it is not a KB-space. However, for each Banach lattice E, its topological dual E' is a KB-space if and only if its norm is order continuous.

Let us recal that a Banach lattice E is called a dual Banach lattice if $E = G'$ for some Banach lattice G. A Banach lattice E is called a dual KB-space if E is a dual Banach lattice and E is a KB-space.

As a consequence of Theorem 3.1, we obtain the following corollaries.

Corollary 3.3. *Let E be a dual Banach lattice. The following statements are equivalent:*

1. *E is a KB-space.*
2. *Any (L) set in E is L-weakly compact.*
3. *Any (L) set in E is relatively weakly compact.*

Corollary 3.4. *Let E be a dual KB-space. Then any Dunford-Pettis set in E is relatively weakly compact.*

Proof. Follows from Corollary 3.2. □

In [**3**] we introduced and used the class of Banach lattices which satisfy the AM-compactness property. A Banach lattice E is said to have the AM-compactness property if E satisfies the four equivalent assertions of Corollary 2.10. For example, the Banach lattice $L^2[0,1]$ does not have the AM-compactness property, but l^1 has the AM-compactness property.

Theorem 3.5. *Let E be a Banach lattice with the AM-compactness property such that the norm of E' is order continuous. Then for each Banach space X every Dunford-Pettis operator $T\colon E \to X$ is compact.*

Proof. Let $T\colon E \to X$ be a Dunford-Pettis operator. Since the norm of E' is order continuous, it follows from [**8**, Theorem 3.7.10] that T is M-weakly compact (and then T is weakly compact). As E has the AM-compactness property, T is AM-compact. The rest of the proof follows from [**8**, Theorem 3.7.4]. □

Corollary 3.6. *Let E be a Banach lattice with the AM-compactness property such that the norm of E' is order continuous. Then any Dunford-Pettis set in E' is relatively compact (and then the class of Dunford-Pettis sets and that of relatively compact sets in E' coincide).*

Proof. By Theorem 3.5, any Dunford-Pettis operator from E to any Banach space X is compact. We conclude from [**5**, Theorem 1 and Corollary 1] that any Dunford-Pettis set in E' is relatively compact. □

Next, recall from [**3**] the following sufficient conditions guaranteeing that a Banach lattice has the AM-compactness property.

Theorem 3.7 ([**3**]). *Let E be a Banach lattice. Then E has the AM-compactness property if one of the following assertions is valid:*

1. *The norm of E is order continuous and E has the Dunford-Pettis property.*
2. *The topological dual E' is discrete.*
3. *The lattice operations in E' are weakly sequentially continuous.*
4. *The lattice operations in E' are weak * sequentially continuous.*

Let us recall from [**8**] that an operator $T\colon E \to X$ from a Banach lattice to a Banach space is said to be M-weakly compact if $\|T(x_n)\| \to 0$ for every norm bounded disjoint sequence (x_n) in E.

Let us, the lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 in the weak* topology $\sigma(E',E)$ whenever the sequence (f_n) converges to 0 in $\sigma(E',E)$.

A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements.

As a consequence of Theorem 3.5 and Theorem 3.7 we obtain a generalization and another proof of [**4**, Theorem 2.2].

Theorem 3.8. *Let E be a Banach lattice. Then each Dunford-Pettis operator from E to any Banach space X is compact if one of the following assertions is valid:*

1. *The topological dual E' is discrete and its norm is order continuous.*
2. *The norm of E' is order continuous and the lattice operations in E' are weak* sequentially continuous.*
3. *The norms of E and of E' are order continuous.*

Proof. 1. If E' is discrete, then it follows from Theorem 3.7 that the Banach lattice E has the AM-compactness property. Since the norm of E' is order continuous, the result follows from Theorem 3.5.

2. If the lattice operations in E' are weak* sequentially continuous, then it follows from Theorem 3.7 that the Banach lattice E has the AM-compactness property. Since the norm of E' is order continuous, the result follows from Theorem 3.5.

3. is exactly [**8**, Theorem 3.7.11(3)]. ☐

Corollary 3.9. *Let E be a Banach lattice. Then any Dunford-Pettis set in E' is relatively compact if one of the following assertions is valid:*

1. *The topological dual E' is discrete and its norm is order continuous.*
2. *The norm of E' is order continuous and the lattice operations in E' are weak* sequentially continuous.*
3. *The norms of E and of E' are order continuous.*

Corollary 3.10. *Let E be a discrete KB-space. Then any Dunford-Pettis set in E is relatively compact (and then the class of Dunford-Pettis sets and that of relatively compact sets in E coincide).*

Proof. Since each discrete KB-space is a dual (see [**8**, Exercise 5.4.E2]), it is sufficient to use 1. of Corollary 3.9. ☐

Corollary 3.11. *For an operator T from a Banach space X into a discrete KB-space F, the following statements are equivalent:*

1. *$T\colon X \to F$ is compact.*
2. *The adjoint $T'\colon F' \to X'$ is Dunford-Pettis.*

Proof. Since F is discrete KB-space, then $T\colon X \to F$ is compact if and only if $T(B_X)$ is relatively compact if and only if $T(B_X)$ is a Dunford-Pettis set in F (Theorem 3.10) if and only if T' is a Dunford-Pettis operator (Remark 4). ☐

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