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A NOTE ON SOME NEW FRACTIONAL RESULTS INVOLVING CONVEX FUNCTIONS

Z. DAHMANI

Abstract. In this paper, we establish some new integral inequalities for convex functions by using the Riemann-Liouville operator of non integer order. For our results some classical integral inequalities can be deduced as some special cases.

1. Introduction

The integral inequalities play a fundamental role in the theory of differential equations. Much significant development in this area has been established for the last two decades. For details we refer to [**10, 12, 14, 15**] and the references therein. Moreover, the study of fractional type inequalities is also of a great importance. For further information and applications we refer the reader to [**1, 13**]. Let us introduce now some results that have inspired our work. We begin by the paper of Ngo et al. [**11**], in which the authors proved that

$$\int_0^1 f^{\delta+1}(\tau)\mathrm{d}\tau \geq \int_0^1 \tau^{\delta} f(\tau)\mathrm{d}\tau \tag{1}$$

and

$$\int_0^1 f^{\delta+1}(\tau)\mathrm{d}\tau \geq \int_0^1 \tau f^{\delta}(\tau)\mathrm{d}\tau, \tag{2}$$

where $\delta > 0$ and f is a positive continuous function on $[0,1]$ such that

$$\int_x^1 f(\tau)\mathrm{d}\tau \geq \int_x^1 \tau\mathrm{d}\tau, x \in [0,1].$$

Then, in [**8**], W. J. Liu, G. S. Cheng and C. C. Li established the following result

$$\int_a^b f^{\alpha+\beta}(\tau)\mathrm{d}\tau \geq \int_a^b (\tau-a)^{\alpha} f^{\beta}(\tau)\mathrm{d}\tau, \tag{3}$$

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provided that $\alpha > 0$, $\beta > 0$ and f is a positive continuous function on $[a,b]$ satisfying

$$\int_x^b f^{\gamma}(\tau)\mathrm{d}\tau \geq \int_x^b (\tau - a)^{\gamma}\mathrm{d}\tau;\ \gamma := \min(1,\beta), x \in [a,b].$$

In [**9**], the following two theorems were proved.

Theorem 1.1. *Let f and h be two positive continuous functions on $[a,b]$ with $f \leq h$ on $[a,b]$ such that $\frac{f}{h}$ is decreasing and f is increasing. Assume that ϕ is a convex function ϕ; $\phi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(\tau)\mathrm{d}\tau}{\int_a^b h(\tau)\mathrm{d}\tau} \geq \frac{\int_a^b \phi(f(\tau))\mathrm{d}\tau}{\int_a^b \phi(h(\tau))\mathrm{d}\tau} \tag{4}$$

holds.

And

Theorem 1.2. *Let f, g and h be three positive continuous functions on $[a,b]$ with $f \leq h$ on $[a,b]$ such that $\frac{f}{h}$ is decreasing and f and g are increasing. Assume that ϕ is a convex function ϕ; $\phi(0) = 0$. Then the inequality*

$$\frac{\int_a^b f(\tau)\mathrm{d}\tau}{\int_a^b h(\tau)\mathrm{d}\tau} \geq \frac{\int_a^b \phi(f(\tau))g(\tau)\mathrm{d}\tau}{\int_a^b \phi(h(\tau))g(\tau)\mathrm{d}\tau} \tag{5}$$

holds.

Many researchers have given considerable attention to (1), (2) and (3) and a number of extensions, generalizations and variants have appeared in the literature, (e.g. [**2, 3, 4, 5, 7, 14**]).

The purpose of this paper is to generalize some classical integral inequalities of [**9**] using the Riemann-Liouville integral operator. For our results Theorem 1.1 and Theorem 1.2 can be deduced as some special cases.

2. Preliminaries

Let us introduce some definitions and properties concerning the Riemann-Liouville fractional integral operator.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$, is defined as

$$\begin{aligned} J^{\alpha}[f(t)] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau; \qquad \alpha > 0,\quad a < t \leq b, \\ J^{0}[f(t)] &= f(t), \end{aligned} \tag{6}$$

where $\Gamma(\alpha) := \int_0^{\infty} \mathrm{e}^{-u}\, u^{\alpha-1}\mathrm{d}u$.

For the convenience of establishing the results we give the semigroup property

$$J^{\alpha}J^{\beta}[f(t)] = J^{\alpha+\beta}[f(t)], \qquad \alpha \geq 0,\quad \beta \geq 0, \tag{7}$$

which implies the commutative property

$$J^\alpha J^\beta[f(t)] = J^\beta J^\alpha[f(t)]. \tag{8}$$

For more details one can consult [**6, 13**].

3. Main Results

Theorem 3.1. *Let f and h be two positive continuous functions on $[a,b]$ and $f \le h$ on $[a,b]$. If $\frac{f}{h}$ is decreasing and f is increasing on $[a,b]$, then for any convex function $\phi; \phi(0)=0$, the inequality*

$$\frac{J^\alpha[f(t)]}{J^\alpha[h(t)]} \ge \frac{J^\alpha[\phi(f(t))]}{J^\alpha[\phi(h(t))]}, \qquad a < t \le b, \quad \alpha > 0 \tag{9}$$

is valid.

Proof. The function ϕ is convex with $\phi(0)=0$. Then the function $\frac{\phi(x)}{x}$ is increasing. Since f is increasing, then $\frac{\phi(f(x))}{f(x)}$ is also increasing. This and the fact that $\frac{f(x)}{h(x)}$ is decreasing yield

$$\tfrac{\phi(f(\tau))}{f(\tau)}\tfrac{f(\rho)}{h(\rho)} + \tfrac{\phi(f(\rho))}{f(\rho)}\tfrac{f(\tau)}{h(\tau)} - \tfrac{\phi(f(\rho))}{f(\rho)}\tfrac{f(\rho)}{h(\rho)} - \tfrac{\phi(f(\tau))}{f(\tau)}\tfrac{f(\tau)}{h(\tau)} \ge 0 \tag{10}$$

for all $\tau, \rho \in [a,t]$, $a < t \le b$.

Hence, we can write

$$\begin{aligned} &\frac{\phi(f(\tau))}{f(\tau)} f(\rho)h(\tau) + \frac{\phi(f(\rho))}{f(\rho)} f(\tau)h(\rho) \\ &\qquad - \frac{\phi(f(\rho))}{f(\rho)} f(\rho)h(\tau) - \frac{\phi(f(\tau))}{f(\tau)} f(\tau)h(\rho) \ge 0 \end{aligned} \tag{11}$$

for all $\tau, \rho \in [a,t]$, $a < t \le b$.

Now, multiplying both sides of (11) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the resulting inequality with respect to τ over $[a,t], a < t \le b$, we get

$$\begin{aligned} &f(\rho)J^\alpha\Big[\frac{\phi(f(t))}{f(t)}h(t)\Big] + \frac{\phi(f(\rho))}{f(\rho)}h(\rho)J^\alpha[f(t)] \\ &\qquad - \frac{\phi(f(\rho))}{f(\rho)} f(\rho)J^\alpha[h(t)] - h(\rho)J^\alpha\left[\frac{\phi(f(t))}{f(t)}f(t)\right] \ge 0. \end{aligned} \tag{12}$$

With the same argument as before, we obtain

$$J^\alpha[f(t)]J^\alpha\left[\frac{\phi(f(t))}{f(t)}h(t)\right] - J^\alpha[h(t)]J^\alpha\left[\frac{\phi(f(t))}{f(t)}f(t)\right] \ge 0. \tag{13}$$

Since $f \le h$ on $[a,b]$, then using the fact that the function $\frac{\phi(x)}{x}$ is increasing, we can write

$$\frac{\phi(f(\tau))}{f(\tau)} \le \frac{\phi(h(\tau))}{h(\tau)}, \qquad \tau \in [a,t], \quad a < t \le b. \tag{14}$$

This implies that

$$\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tag{15}$$

where $\tau \in [a,t]$, $a < t \leq b$.

Integrating both sides of (15) with respect to τ over $[a,t], a < t \leq b$, yields

$$J^{\alpha}\left[\frac{\phi(f(t))}{f(t)}h(t)\right] \leq J^{\alpha}\left[\frac{\phi(h(t))}{h(t)}h(t)\right]. \tag{16}$$

Hence, thanks to (13) and (16), we obtain (9). □

Remark 3.2. Applying Theorem 3.1 for $\alpha = 1$, $t = b$, we obtain Theorem 1.1.

We further have the following theorem.

Theorem 3.3. *Let f and h be two positive continuous functions on $[a,b]$ and $f \leq h$ on $[a,b]$. If $\frac{f}{h}$ is decreasing and f is increasing on $[a,b]$, then for any convex function ϕ; $\phi(0) = 0$, we have*

$$\frac{J^{\alpha}[f(t)]J^{\omega}[\phi(h(t))] + J^{\omega}[f(t)]J^{\alpha}[\phi(h(t))]}{J^{\alpha}[h(t)]J^{\omega}[\phi(f(t))] + J^{\omega}[h(t)]J^{\alpha}[\phi(f(t))]} \geq 1, \tag{17}$$

where $\alpha > 0$, $\omega > 0$, $a < t \leq b$.

Proof. The relation (12) allows us to obtain

$$\begin{aligned} J^{\omega}[f(t)]J^{\alpha}\left[\frac{\phi(f(t))}{f(t)}h(t)\right]+J^{\omega}\left[\frac{\phi(f(t))}{f(t)}h(t)\right]J^{\alpha}[f(t)] \\ -J^{\omega}\left[\frac{\phi(f(t))}{f(t)}f(t)\right]J^{\alpha}[h(t)]-J^{\omega}[h(t)]J^{\alpha}\left[\frac{\phi(f(t))}{f(t)}f(t)\right]\geq 0. \end{aligned} \tag{18}$$

Since $f \leq h$ on $[a,b]$ and using the fact that the function $\frac{\phi(x)}{x}$ is increasing, then thanks to (14), we obtain

$$\frac{(t-\tau)^{\omega-1}}{\Gamma(\omega)}h(\tau)\frac{\phi(f(\tau))}{f(\tau)} \leq \frac{(t-\tau)^{\omega-1}}{\Gamma(\omega)}h(\tau)\frac{\phi(h(\tau))}{h(\tau)}, \tag{19}$$

where $\tau \in [a,t]$, $a < t \leq b$. And then,

$$J^{\omega}\left[\frac{\phi(f(t))}{f(t)}h(t)\right] \leq J^{\omega}\left[\frac{\phi(h(t))}{h(t)}h(t)\right]. \tag{20}$$

Hence, thanks to (16), (18) and (20), we get (17). □

Remark 3.4. (i) Applying Theorem 3.3 for $\alpha = \omega$, we obtain Theorem 3.1.
(ii) Applying Theorem 3.3 for $\alpha = \omega = 1$, $t = b$, we obtain Theorem 1.1.

Another result which generalizes Theorem 1.2 is described in the following theorem.

Theorem 3.5. *Let f, h and g be three positive continuous functions and $f \leq h$ on $[a,b]$. Suppose that $\frac{f}{h}$ is decreasing, f and g are increasing on $[a,b]$ and ϕ is a convex function, $\phi(0)=0$. Then, for any $\alpha > 0$, $a < t \leq b$, we have*

$$\frac{J^\alpha[f(t)]}{J^\alpha[h(t)]} \geq \frac{J^\alpha[\phi(f(t))g(t)]}{J^\alpha[\phi(h(t))g(t)]}. \tag{21}$$

Proof. Let $\tau, \rho \in [a,t], a < t \leq b$. We have

$$\begin{aligned} &\frac{\phi(f(\tau))g(\tau)}{f(\tau)} f(\rho)h(\tau) + \frac{\phi(f(\rho))g(\rho)}{f(\rho)} f(\tau)h(\rho) \\ &\qquad - \frac{\phi(f(\rho))g(\rho)}{f(\rho)} f(\rho)h(\tau) - \frac{\phi(f(\tau))g(\tau)}{f(\tau)} f(\tau)h(\rho) \geq 0. \end{aligned} \tag{22}$$

Hence we can write

$$\begin{aligned} &f(\rho)J^\alpha\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big] + \frac{\phi(f(\rho))g(\rho)}{f(\rho)}h(\rho)J^\alpha[f(t)] \\ &\qquad - \frac{\phi(f(\rho))g(\rho)}{f(\rho)}f(\rho)J^\alpha[h(t)] - h(\rho)J^\alpha\Big[\frac{\phi(f(t))g(t)}{f(t)}f(t)\Big] \geq 0. \end{aligned} \tag{23}$$

Therefore,

$$J^\alpha[f(t)]J^\alpha\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big] - J^\alpha[h(t)]J^\alpha\Big[\phi(f(t))g(t)\Big] \geq 0. \tag{24}$$

On the other hand, we have

$$\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\frac{\phi(f(\tau))g(\tau)}{f(\tau)} \leq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\frac{\phi(h(\tau))g(\tau)}{h(\tau)}, \tag{25}$$

where $\tau \in [a,t]$, $a < t \leq b$. Consequently,

$$J^\alpha\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big] \leq J^\alpha\Big[\phi(h(t))g(t)\Big], \tag{26}$$

and so,

$$J^\alpha[f(t)]J^\alpha\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big] \leq J^\alpha[f(t)]J^\alpha\Big[\phi(h(t))g(t)\Big]. \tag{27}$$

Hence, thanks to (24) and (27) we obtain (21). □

Remark 3.6. It is clear that Theorem 1.2 would follow as a special case of Theorem 3.5 when $\alpha = 1$ and $t = b$.

Another result which generalizes Theorem 3.5 is described in the following theorem.

Theorem 3.7. *Let f, h and g be three positive continuous functions and $f \leq h$ on $[a,b]$. Suppose that $\frac{f}{h}$ is decreasing, f and g are increasing on $[a,b]$ and ϕ is a convex function, $\phi(0)=0$. Then, for any $\alpha > 0, \omega > 0, a < t \leq b$, we have*

$$\frac{J^\alpha[f(t)]J^\omega[\phi(h(t))g(t)] + J^\omega[f(t)]J^\alpha[\phi(h(t))g(t)]}{J^\alpha[h(t)]J^\omega[\phi(f(t))g(t)] + J^\omega[h(t)]J^\alpha[\phi(f(t))g(t)]} \geq 1. \tag{28}$$

Proof. Using (23), we can write

$$
\begin{aligned}
&J^{\omega}[f(t)]J^{\alpha}\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big]+J^{\omega}\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big]J^{\alpha}[f(t)]\\
&-J^{\omega}\Big[\frac{\phi(f(t))g(t)}{f(t)}f(t)\Big]J^{\alpha}[h(t)]-J^{\omega}[h(t)]J^{\alpha}\Big[\frac{\phi(f(t))g(t)}{f(t)}f(t)\Big]\geq 0.
\end{aligned}
\tag{29}
$$

Then, using the fact that the function $\frac{\phi(x)g(x)}{x}$ is increasing and the hypothesis $f\leq h$ on $[a,b]$, we obtain

$$J^{k}\Big[\frac{\phi(f(t))g(t)}{f(t)}h(t)\Big]\leq J^{k}\Big[\frac{\phi(h(t))g(t)}{h(t)}h(t)\Big],\qquad k=\alpha,\omega. \tag{30}$$

Hence, thanks to (29) and (30), we get (28). □

Remark 3.8. It is clear that Theorem 3.5 would follow as a special case of Theorem 3.7 when $\alpha=\beta$.

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Z. Dahmani, Laboratory LPAM, FSEI, UAMB, University of Mostaganem, Algeria,
e-mail: `zzdahmani@yahoo.fr`