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PRE-IMAGE ENTROPY FOR MAPS ON NONCOMPACT TOPOLOGICAL SPACES

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Abstract. We propose a new definition of pre-image entropy for continuous maps on noncompact topological spaces, investigate fundamental properties of the new pre-image entropy, and compare the new pre-image entropy with the existing ones. The defined pre-image entropy generates that of Cheng and Newhouse. Yet, it holds various basic properties of Cheng and Newhouse's pre-image entropy, for example, the pre-image entropy of a subsystem is bounded by that of the original system, topologically conjugated systems have the same pre-image entropy, the pre-image entropy of the induced hyperspace system is larger than or equal to that of the original system, and in particular this new pre-image entropy coincides with Cheng and Newhouse's pre-image entropy for compact systems.

1. Introduction

The concepts of entropy are useful for studying topological and measure-theoretic structures of dynamical systems, that is, topological entropy (see [**1, 3, 4**]) and measure-theoretic entropy (see [**8, 13**]). For instance, two conjugate systems have the same entropy and thus entropy is a numerical invariant of the class of conjugated dynamical systems. The theory of expansive dynamical systems has been closely related to the theory of topological entropy [**6, 12, 19**]. Entropy and chaos are closely related, for example, a continuous map of interval is chaotic if and only if it has a positive topological entropy [**2**].

In [**10**], Hurley introduced several other entropy-like invariants for noninvertible maps. One of these, which Nitecki and Przytycki [**16**] called pre-image branch entropy (retaining Hurley's notation), distinguishes points according to the branches of the inverse map. Cheng and Newhouse [**7**] further extended the concept of topological entropy of a continuous map and gave the concept of pre-image entropy for compact dynamical systems. Several important pre-image entropy invariants, such as pointwise pre-image, pointwise branch entropy, partial pre-image entropy, and bundle-like pre-image entropy, etc., have been introduced and their relationships with topological entropy have been established. Zhang, Zhu and He [**22**] extended

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and studied some entropy-like invariants for the non-autonomous discrete dynamical systems given by a sequence of continuous self-maps of a compact topological space as mentioned above.

This paper investigates a more general definition of pre-image entropy for continuous maps defined on noncompact topological spaces and explore the properties of such pre-image entropy. This definition generalizes that of Cheng and Newhouse's. Moreover, we have proved that the pre-image entropy defined in this paper holds most properties of the pre-image entropy under Cheng and Newhouse's definition, for example, for compact systems, this new pre-image entropy coincides with the pre-image entropy defined by Cheng and Newhouse's, the defined pre-image entropy (over noncompact topological spaces) either retains the fundamental properties of pre-image entropy (over compact topological spaces) or has similar properties, the pre-image entropy of a subsystem is bounded by that of the original system, topologically conjugated systems have the same pre-image entropy, the pre-image entropy of an autohomeomorphism from R onto itself is 0, and the pre-image entropy of the induced hyperspace map is at least that of the original mapping.

2. The new definition of pre-image entropy and its general properties

Let (X, d) be an arbitrary metric space and $f\colon X \to X$ be a continuous mapping. Then the pair (X, f) is said to be a topological dynamical system. If X is compact, (X, f) is called a compact dynamical system. Let $\mathbb{N}$ denote the set of all positive integers and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Definition 2.1 ([**7**])**.** Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous map and let $\varepsilon > 0$ and $n \in \mathbb{N}$. Then

$$h_{\mathrm{pre}}(f) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \sup \frac{1}{n} \log \sup_{x\in X, k\geq n} r(n, \varepsilon, f^{-k}(x), f)$$

is called the pre-image entropy of f, where $r(n, \varepsilon, f^{-k}(x), f)$ is the maximal cardinality of (n, ε)-separated subsets of $f^{-k}(x)$.

Note that if f is a homeomorphism, then $h_{\mathrm{pre}}(f) = 0$. When X needs to be explicitly mentioned, we write $h_{\mathrm{pre}}(f, X)$ instead of $h_{\mathrm{pre}}(f)$.

Now, we begin our process to introduce our new definition of pre-image entropy. Let (X, f) be a topological dynamical system, where X is an arbitrary topological space with metric d and f is a continuous self-map of the metric space (X, d). Let $n \in \mathbb{N}$. Define the metric $d_{f,n}$ on X by

$$d_{f,n}(x, y) = \max_{0\leq j<n} d(f^j(x), f^j(y)).$$

A set $E \subseteq X$ is an $(n, \varepsilon)-$separated set if for any $x \neq y$ in E, one has $d_{f,n}(x, y) > \varepsilon$. Given a subset $K \subseteq X$, we define the quantity $r(n, \varepsilon, K, f)$ to be the maximal cardinality of (n, ε, K, f)-separated subset of K. A subset $E \subseteq K$ is an (n, ε, K)-spanning set if for every $x \in K$, there is a $y \in E$ such that $d_{f,n}(x, y) \leq \varepsilon$.

Let $s(n,\varepsilon,K,f)$ be the minimal cardinality of any (n,ε,K,f)-spanning set. Uniform continuity of f^j for $0 \le j < n$, guarantees that $r(n,\varepsilon,K,f)$ and $s(n,\varepsilon,K,f)$ are both finite for all $n,\varepsilon > 0$. It is a standard that for any subset $K \subseteq X$,

$$s(n,\varepsilon,K,f) \le r(n,\varepsilon,K,f) \le s\left(n,\frac{\varepsilon}{2},K,f\right). \tag{2.1}$$

Next, using techniques as in Bowen [**5**], we have the following.

If $n_1, n_2, l \in \mathbb{N}$ with $l \ge n_1$, then

$$\begin{aligned} r(n_1+n_2,\varepsilon,f^{-l}(K),f) &\le s\left(n_1,\frac{\varepsilon}{2},f^{-l}(K),f\right) s\left(n_2,\frac{\varepsilon}{2},f^{-l+n_1}(K),f\right) \\ &\le r\left(n_1,\frac{\varepsilon}{2},f^{-l}(K),f\right) r\left(n_2,\frac{\varepsilon}{2},f^{-l+n_1}(K),f\right). \end{aligned} \tag{2.2}$$

By $K(X,f)$, denote the set of all f-invariant nonempty compact subsets of X, that is, $K(X,f) = \{F \subseteq X : F \ne \emptyset, F \text{ is compact and } f(F) \subseteq F\}$. If X is compact, it follows from $f(X) \subseteq X$ that $K(X,f) \ne \emptyset$. However, when X is noncompact, $K(X,f)$ could be empty. The translation $f\colon \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x+1$ is such an example. Another example is $f\colon (0,\infty) \to (0,\infty)$, where $f(x) = 2x$ and $(0,\infty)$ has the subspace topology of $\mathbb{R}$.

Definition 2.2. Let (X,f) be a topological dynamical system, where (X,d) is a metric space and let $\varepsilon > 0$ and $n \in \mathbb{N}$. For $F \in K(X,f)$,

$$h^*_{\text{pre}}(f|_F,F) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup \frac{1}{n}\log \sup_{x\in F, k\ge n} r(n,\varepsilon,(f|_F)^{-k}(x),f|_F)$$

is called the pre-image entropy of f on F, where $f|_F\colon F \to F$ is the induced map of f, that is, for any $x \in F$, $f|_F(x) = f(x)$.

Remark 1. By Definition 2.2, if $F \in K(X,f)$ and $x \in F$, then $f(F) \subseteq F$ and $(f|_F)^{-k}(x) \subseteq F$. Furthermore, we have $f|_F\colon F \to F$ is a uniformly continuous mapping. Hence, $r(n,\varepsilon,(f|_F)^{-k}(x),f|_F)$ is finite for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Moreover, by Definition 2.1, we have $h^*_{\text{pre}}(f|_F,F) = h_{\text{pre}}(f|_F)$.

Theorem 2.1. *Let (X,f) be a topological dynamical system where (X,d) is a metric space. For $F_1, F_2 \in K(X,f)$ with $F_1 \subseteq F_2$, the inequality $h^*_{\text{pre}}(f|_{F_1},F_1) \le h^*_{\text{pre}}(f|_{F_2},F_2)$ holds.*

Proof. Let $\varepsilon > 0$ and n, $k \in \mathbb{N}$ with $k \ge n$, and let $x \in F_1$ and $E \subseteq (f|_{F_1})^{-k}(x)$ be an $(n,\varepsilon,f|_{F_1})^{-k}(x),f|_{F_1})$-separated subset with the maximal cardinality. Let $\text{card}(E) = m$, that is, $r(n,\varepsilon,(f|_{F_1})^{-k}(x),f|_{F_1}) = m$. Since $x \in F_1$ and $F_1 \subseteq F_2$, then $x \in F_2$ and $(f|_{F_1})^{-k}(x) \subseteq F_1$. Furthermore, we have $(f|_{F_1})^{-k}(x) \subseteq (f|_{F_2})^{-k}(x)$ and $(f|_{F_1})^{-k}(x) \subseteq F_2$. Hence, E is an $(n,\varepsilon,f|_{F_2})^{-k}(x),f|_{F_2})$-separated subset of $(f|_{F_2})^{-k}(x)$. Therefore, $r(n,\varepsilon,(f|_{F_2})^{-k}(x)) \ge m$, that is,

$$r(n,\varepsilon,(f|_{F_1})^{-k}(x),f|_{F_1}) \le r(n,\varepsilon,(f|_{F_2})^{-k}(x),f|_{F_2}).$$

Furthermore, we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in F_1,k\geq n} r(n,\varepsilon,(f|_{F_1})^{-k}(x),f|_{F_1})\\&\leq\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in F_2,k\geq n} r(n,\varepsilon,(f|_{F_2})^{-k}(x),f|_{F_2}).\end{aligned}$$

Therefore, $h^*_{\text{pre}}(f|_{F_1},F_1)\leq h^*_{\text{pre}}(f|_{F_2},F_2)$. □

Definition 2.3. Let (X,f) be a topological dynamical system, where (X,d) is a metric space. When $K(X,f)\neq\emptyset$, define

$$h^*_{\text{pre}}(f)=\sup_{F\in K(X,f)}\{h^*_{\text{pre}}(f|_F,F)\},$$

where the supremum is taken over F of $K(X,f)$. When $K(X,f)=\emptyset$, define $h^*_{\text{pre}}(f)=0$. $h^*_{\text{pre}}(f)$ is called the pre-image entropy of f.

Proposition 2.1. *$h^*_{\text{pre}}(f)$ is independent of the choice of metric on X.*

Proof. We only prove that $h^*_{\text{pre}}(f|_F,F)$ is independent of the choice of metric on X for every $F\in K(X,f)$. Let d_1 and d_2 be two metrics on X. Then, by compactness of F and $f|_F\colon F\to F$, for every $\varepsilon>0$, there is $\delta>0$ such that for all $x,y\in F$, if $d_1(x,y)<\delta$, then $d_2(x,y)<\varepsilon$. It follows that $r(n,\varepsilon,(f|_F)^{-k}(x),f|_F,d_2)\leq r(n,\delta,(f|_F)^{-k}(x),f|_F,d_1)$ for all $x\in F$, $\varepsilon>0$ and for every $n\in\mathbb{N}$ with $k\geq n$. This shows that $h^*_{\text{pre}}(f|_F,F,\varepsilon,d_2)\leq h^*_{\text{pre}}(f|_F,F,\delta,d_1)$. Letting $\delta\to0$, $h^*_{\text{pre}}(f|_F,F,\varepsilon,d_2)\leq h^*_{\text{pre}}(f|_F,F,d_1)$ holds. Now, letting $\varepsilon\to0$, $h^*_{\text{pre}}(f|_F,F,d_2)\leq h^*_{\text{pre}}(f|_F,F,d_1)$ hols. Interchanging d_1 and d_2, it gives the opposite inequality, proving that $h^*_{\text{pre}}(f|_F,F,d_1)=h^*_{\text{pre}}(f|_F,F,d_2)$. □

The next theorem indicates the concept of pre-image entropy $h^*_{\text{pre}}(f)$ defined above, generating that of Cheng and Newhouse [**7**], that is, $h^*_{\text{pre}}(f)$ coincides with $h_{\text{pre}}(f)$ when X is compact. Recall that $h_{\text{pre}}(f)$ is defined for compact dynamical systems only while in the preceding section, $h^*_{\text{pre}}(f)$ is defined for arbitrary topological spaces.

Theorem 2.2. *Let (X,f) be a compact topological dynamical system, where (X,d) is a metric space. Then $h^*_{\text{pre}}(f)=h_{\text{pre}}(f,X)$.*

Proof. Since X is compact and $f(X)\subseteq X$, we have $X\in K(X,f)$ implying $K(X,f)\neq\emptyset$. Thus from Definition 2.3, $h^*_{\text{pre}}(f)=\sup\limits_{F\in K(X,f)}\{h^*_{\text{pre}}(f|_F,F)\}$. By Theorem 2.1, for any $F\in K(X,f)$, it holds $h^*_{\text{pre}}(f|_F,F)\leq h^*_{\text{pre}}(f,X)$, that is, the supremum is achieved when $F=X$. Recall the definitions of $h^*_{\text{pre}}(f,X)$ and $h_{\text{pre}}(f,X)$, that is,

$$\begin{aligned}h^*_{\text{pre}}(f,X)=h^*_{\text{pre}}(f|_X,X)&=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X,k\geq n} r(n,\varepsilon,(f|_X)^{-k}(x),f|_X)\\&=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log\sup_{x\in X,k\geq n} r(n,\varepsilon,f^{-k}(x),f)\end{aligned}$$

and

$$h_{\text{pre}}(f, X) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log \sup_{x \in X, k \geq n} r(n, \varepsilon, f^{-k}(x), f).$$

Hence, we have $h^*_{\text{pre}}(f, X) = h_{\text{pre}}(f, X)$. So, from the previous proved equality $h^*_{\text{pre}}(f) = h^*_{\text{pre}}(f, X)$, we conclude $h^*_{\text{pre}}(f) = h_{\text{pre}}(f, X)$. □

From Definition 2.3, $h^*_{\text{pre}}(f)$ may be $+\infty$. The following example is given.

Example 2.1. Let $(\sum_{\mathbb{Z}_+}, \sigma)$ be one-sided infinite symbolic dynamics, $\sum_{\mathbb{Z}_+} = \{x = (x_n)_{n=0}^{\infty} : x_n \in \mathbb{Z}_+ \text{ for every } n\}$, $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$. Then $h^*_{\text{pre}}(\sigma)$ is $+\infty$.

Considering $\mathbb{Z}_+$ as a discrete space and putting product topology on $\sum_{\mathbb{Z}_+}$, an admissible metric ρ on the space $\sum_{\mathbb{Z}_+}$ is defined by

$$\rho(x, y) = \sum_{n=0}^{\infty} \frac{d(x_n, y_n)}{2^n},$$

where

$$d(x_n, y_n) = \begin{cases} 0 & \text{if } x_n = y_n, \\ 1 & \text{if } x_n \neq y_n \end{cases}$$

for $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots) \in \sum_{\mathbb{Z}_+}$. Then $\sum_{\mathbb{Z}_+}$ is a noncompact metric space.

Let $p \in \mathbb{N}$ and $\sum_p = \{x = (x_n)_{n=0}^{\infty} : x_n \in \{0, 1, \ldots, p-1\} \text{ for every } n\}$. Then we have $\sum_p \subseteq \sum_{\mathbb{Z}_+}$. By Robinson [**18**] and Zhou [**23**], $\sum_p$ is a compact space and $\sigma(\sum_p) \subseteq \sum_p$. Hence $\sum_p \in K(\sum_{\mathbb{Z}_+}, \sigma)$. Furthermore, we have $h^*_{\text{pre}}(\sigma) \geq h^*_{\text{pre}}(\sigma|_{\sum_p}, \sum_p)$ from Definition 2.3.

By Nitecki [**17**] and Cheng-Newhouse [**7**], $h_{\text{pre}}(\sigma|_{\sum_p}) = \log p$. By Definition 2.3, we have $h^*_{\text{pre}}(\sigma|_{\sum_p}, \sum_p) = h_{\text{pre}}(\sigma|_{\sum_p})$. Hence, $h^*_{\text{pre}}(\sigma) \geq \log p$. Since p is an arbitrary positive integer, it implies $h^*_{\text{pre}}(\sigma) = +\infty$.

3. Fundamental properties and main results of the pre-image entropy

Proposition 3.1. *Let (X, d) be a metric space and* id *be the identity map from X onto itself. Then for the dynamical system (X, id), we have $h^*_{\text{pre}}(\text{id}) = 0$.*

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. For any $F \in K(X, \text{id})$ and $x \in F$, $k \geq n$, we have $r(n, \varepsilon, (\text{id}|_F)^{-k}(x), \text{id}|_F) = r(n, \varepsilon, \{x\}, \text{id}|_F)$. Hence, $r(n, \varepsilon, (\text{id}|_F)^{-k}(x), \text{id}|_F) \leq 1$. Then

$$h^*_{\text{pre}}(\text{id}|_F, F) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log \sup_{x \in F, k \geq n} r(n, \varepsilon, (\text{id}|_F)^{-k}(x), \text{id}|_F) = 0.$$

It follows from Definitions 2.3 that $h^*_{\text{pre}}(\text{id}) = \sup_{F \in K(X, f)} \{h^*_{\text{pre}}(\text{id}|_F, F)\} = 0$. □

Let (X, f) and (Y, g) be two topological dynamical systems. Then, (X, f) is an extension of (Y, g), or (Y, g) is a factor of (X, f) if there exists a surjective continuous map $\pi\colon X \to Y$ (called a factor map) such that $\pi \circ f(x) = g \circ \pi(x)$ for every $x \in X$. If further, π is a homeomorphism, then (X, f) and (Y, g) are said to be topologically conjugate and the homeomorphism π is called a conjugate map.

Let (X, f) and (Y, g) be two topological dynamical systems, where (X, d_1) and (Y, d_2) are metric spaces. For the product space $X\times Y$, define a map $f\times g\colon X\times Y \to X\times Y$ by $(f\times g)(x, y) = (f(x), g(y))$. This map $f\times g$ is continuous and $(X\times Y, f\times g)$ forms a topological dynamical system. Given $X \times Y$, the metric

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}.$$

Theorem 3.1. ([**7**, Theorem 2.1]) *Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous self-maps of the compact metric spaces X, Y, respectively. Then*

(1) *(power rule) for any $m \in \mathbb{N}$, we have $h_{\text{pre}}(f^m) = m \cdot h_{\text{pre}}(f)$;*

(2) *(product rule) $h_{\text{pre}}(f \times g) = h_{\text{pre}}(f) + h_{\text{pre}}(g)$;*

(3) *(topological invariance) if f is topologically conjugate to g, then $h_{\text{pre}}(f) = h_{\text{pre}}(g)$.*

Let $F_x \in K(X, f)$ and $F_y \in K(Y, g)$. By Definition 2.1, we have $h^*_{\text{pre}}(f|_{F_x}, F_x) = h_{\text{pre}}(f|_{F_x})$ and $h^*_{\text{pre}}(g|_{F_y}, F_y) = h_{\text{pre}}(g|_{F_y})$. Furthermore, we have the following corollary.

Corollary 3.1. *Let $f\colon X \to X$ and $g\colon Y \to Y$ be continuous self-maps of the metric spaces X, Y, respectively. Let $F_x \in K(X, f)$ and $F_y \in K(Y, g)$. Then*

(1) *(power rule) for any $m \in \mathbb{N}$, we have $h^*_{\text{pre}}(f^m|_{F_x}, F_x) = m \cdot h^*_{\text{pre}}(f|_{F_x}, F_x)$;*

(2) *(product rule) $h^*_{\text{pre}}(f \times g|_{F_x\times F_y}, F_x \times F_y) = h^*_{\text{pre}}(f|_{F_x}, F_x) + h^*_{\text{pre}}(g|_{F_y}, F_y)$;*

(3) *(topological invariance) if f is topologically conjugate to g and π is their conjugate map, then $h^*_{\text{pre}}(f|_{F_x}, F_x) = h^*_{\text{pre}}(g|_{\pi(F_x)}, \pi(F_x))$.*

Proposition 3.2. *For any $m \in \mathbb{N}$, $h^*_{\text{pre}}(f^m) \geq m \cdot h^*_{\text{pre}}(f)$. When $K(X, f) = K(X, f^m)$, $h^*_{\text{pre}}(f^m) = m \cdot h^*_{\text{pre}}(f)$.*

Proof. If $K(X, f) = \emptyset$, then $h^*_{\text{pre}}(f) = 0$, thus $h^*_{\text{pre}}(f^m) \geq m \cdot h^*_{\text{pre}}(f)$. If $K(X, f)\neq\emptyset$, then $K(X, f)\subseteq K(X, f^m)$. For any $F\in K(X, f)$, thus $F\in K(X, f^m)$. By Corollary 3.1 (1), $h^*_{\text{pre}}(f^m|_F, F) = h^*_{\text{pre}}((f|_F)^m, F) = m \cdot h^*_{\text{pre}}(f|_F, F)$. Then

$$\begin{aligned} h^*_{\text{pre}}(f^m) &= \sup_{L\in K(X,f^m)} \{h^*_{\text{pre}}(f^m|_L, L)\} \\ &\geq \sup_{F\in K(X,f)} \{h^*_{\text{pre}}(f^m|_F, F)\} = m \cdot \sup_{F\in K(X,f)} \{h^*_{\text{pre}}(f|_F, F)\} \\ &= m \cdot h^*_{\text{pre}}(f). \end{aligned}$$

Hence, $h^*_{\text{pre}}(f^m) \geq m \cdot h^*_{\text{pre}}(f)$. Next, we show that when $K(X, f) = K(X, f^m)$, the equality holds, that is, $h^*_{\text{pre}}(f^m) = m \cdot h^*_{\text{pre}}(f)$. Consider two cases.

Case 1. $K(X, f) = K(X, f^m) = \emptyset$. By applying Definition 2.3, we have $h^*_{\text{pre}}(f^m) = m \cdot h^*_{\text{pre}}(f) = 0$.

Case 2. $K(X,f) = K(X,f^m) \neq \emptyset$. For any $F \in K(X,f) = K(X,f^m)$, we have $h^*_{\rm pre}(f^m|_F, F) = h^*_{\rm pre}((f|_F)^m, F) = m \cdot h^*_{\rm pre}(f|_F, F)$. Then

$$\begin{aligned} h^*_{\rm pre}(f^m) &= \sup_{F\in K(X,f^m)} \{h^*_{\rm pre}(f^m|_F, F)\} = \sup_{F\in K(X,f)} \{h^*_{\rm pre}(f^m|_F, F)\} \\ &= m \cdot \sup_{F\in K(X,f)} \{h^*_{\rm pre}(f|_F, F)\} = m \cdot h^*_{\rm pre}(f). \end{aligned}$$

□

Lemma 3.1. ([**14**]) *Let (X,f) and (Y,g) be two topological dynamical systems. Let $P_x\colon X\times Y \to X$ and $P_y\colon X\times Y\to Y$ be the projections on X and Y, respectively. If $F\in K(X\times Y, f\times g)$, then $P_x(F)\in K(X,f)$, $P_y(F)\in K(Y,g)$ and $F\subseteq P_x(F)\times P_y(F)$.*

Proposition 3.3. *Let (X,f) and (Y,g) be two topological dynamical systems, where X and Y are two metric spaces. If $K(X\times Y, f\times g)\neq\emptyset$, then $h^*_{\rm pre}(f\times g) = h^*_{\rm pre}(f) + h^*_{\rm pre}(g)$.*

Proof. Recall the projections $P_x\colon X\times Y\to X$ and $P_y\colon X\times Y\to Y$. Since $K(X\times Y, f\times g)\neq\emptyset$, then for any $F\in K(X\times Y, f\times g)$, by Lemma 3.1, $P_x(F)\in K(X,f)$, $P_y(F)\in K(Y,g)$ and $F\subseteq P_x(F)\times P_y(F)$. Denote $P_x(F)$ by F_x and $P_y(F)$ by F_y. By Theorem 2.1, $h^*_{\rm pre}(f\times g|_F, F)\le h^*_{\rm pre}(f\times g|_{F_x\times F_y}, F_x\times F_y)$. From Corollary 3.1 (2), we have $h^*_{\rm pre}(f\times g|_{F_x\times F_y}, F_x\times F_y) = h^*_{\rm pre}(f|_{F_x}, F_x) + h^*_{\rm pre}(g|_{F_y}, F_y)$. Then

$$\begin{aligned} h^*_{\rm pre}(f\times g) &= \sup\{h^*_{\rm pre}(f\times g|_F, F) : F\in K(X\times Y, f\times g)\} \\ &\le \sup\{h^*_{\rm pre}(f\times g|_{F_x\times F_y}, F_x\times F_y) : \ F_x\in K(X,f) \text{ and } F_y\in K(Y,g)\} \\ &\le \sup\{h^*_{\rm pre}(f|_{F_x}, F_x) : F_x\in K(X,f)\} \\ &\quad + \sup\{h^*_{\rm pre}(f|_{F_y}, F_y) : F_y\in K(Y,g)\} \\ &= h^*_{\rm pre}(f) + h^*_{\rm pre}(g). \end{aligned}$$

We prove the converse inequality. Let $F_x\in K(X,f)$ and $F_y\in K(Y,g)$. By Corollary 3.1 (2), $h^*_{\rm pre}(f\times g|_{F_x\times F_y}, F_x\times F_y) = h^*_{\rm pre}(f|_{F_x}, F_x) + h^*_{\rm pre}(g|_{F_y}, F_y)$. Then

$$\begin{aligned} h^*_{\rm pre}(f\times g) &= \sup\{h^*_{\rm pre}(f\times g|_F, F) : F\in K(X\times Y, f\times g)\} \\ &\ge \sup\{h^*_{\rm pre}(f\times g|_{F_x\times F_y}, F_x\times F_y) : \ F_x\in K(X,f) \text{ and } F_y\in K(Y,g)\} \\ &= \sup\{h^*_{\rm pre}(f|_{F_x}, F_x) \\ &\quad + h^*_{\rm pre}(g|_{F_y}, F_y) : \ F_x\in K(X,f) \text{ and } F_y\in K(Y,g)\} \\ &= \sup\{h^*_{\rm pre}(f|_{F_x}, F_x) : F_x\in K(X,f)\} \\ &\quad + \sup\{h^*_{\rm pre}(f|_{F_y}, F_y) : F_y\in K(Y,g)\} \\ &= h^*_{\rm pre}(f) + h^*_{\rm pre}(g). \end{aligned}$$

□

Definition 3.1. Let (X, f) be a topological dynamical system. If $\Lambda \subseteq X$ and $f(\Lambda) \subseteq \Lambda$, then $(\Lambda, f|_\Lambda)$ is said to be a topological subsystem of (X, f), or simply a subsystem of (X, f).

Remark 2. In above definition, Λ is not necessarily compact or closed. In the literature of dynamics, many authors assume subsystems to be compact or closed.

Theorem 3.2. *Let $(\Lambda, f|_\Lambda)$ be a subsystem of (X, f), where X is a metric space. Then $h^*_{\text{pre}}(f|_\Lambda) \le h^*_{\text{pre}}(f)$.*

Proof. If $K(\Lambda, f|_\Lambda) = \emptyset$, it follows from Definition 2.3 that $h^*_{\text{pre}}(f|_\Lambda) = 0$, thus $h^*_{\text{pre}}(f|_\Lambda) \le h^*_{\text{pre}}(f)$. If $K(\Lambda, f|_\Lambda) \ne \emptyset$, then $K(\Lambda, f|_\Lambda) \subseteq K(X, f)$. For any $F \in K(\Lambda, f|_\Lambda)$, we have $h^*_{\text{pre}}((f|_\Lambda)|_F, F) = h^*_{\text{pre}}(f|_F, F)$. Hence,

$$\begin{aligned} h^*_{\text{pre}}(f|_\Lambda) &= \sup_{F \in K(\Lambda, f|_\Lambda)} h^*_{\text{pre}}((f|_\Lambda)|_F, F) = \sup_{F \in K(\Lambda, f|_\Lambda)} h^*_{\text{pre}}(f|_F, F) \\ &\le \sup_{F \in K(X,f)} h^*_{\text{pre}}(f|_F, F) = h^*_{\text{pre}}(f). \end{aligned}$$

□

Theorem 3.3. *Let (X, f) and (Y, g) be two topological dynamical systems, where X, Y are two metric spaces. If (X, f) and (Y, g) are topologically conjugate, that is, there exists a continuous map $\pi \colon X \to Y$ satisfying $\pi \circ f = g \circ \pi$, then $h^*_{\text{pre}}(f) = h^*_{\text{pre}}(g)$.*

Proof. Consider two cases.

Case 1. $K(X, f) = \emptyset$. We claim $K(Y, g) = \emptyset$. If not, assume $K(Y, g) \ne \emptyset$. Then there exists $F \in K(Y, g) \ne \emptyset$ satisfying $g(F) \subseteq F$. As $\pi \colon X \to Y$ is a conjugate map, that is, $\pi \circ f = g \circ \pi$, the inverse π^{-1} is a conjugate map from (Y, g) and (X, f), that is, $\pi^{-1} \circ g = f \circ \pi^{-1}$. Note that $\pi^{-1}(F)$ is a nonempty compact subset of X and $f(\pi^{-1}(F)) = \pi^{-1}(g(F)) \subseteq \pi^{-1}(F)$. Hence, $\pi^{-1}(F) \in K(X, f)$, which contradicts $K(X, f) = \emptyset$. Therefore, $K(X, f) = \emptyset$ implies $K(Y, g) = \emptyset$. Similarly, we can prove that $K(Y, g) = \emptyset$ implies $K(X, f) = \emptyset$. So we have proved that $K(X, f) = \emptyset$ if and only if $K(Y, g) = \emptyset$, and thus by Definition 2.3, $h^*_{\text{pre}}(f) = h^*_{\text{pre}}(g)$.

Case 2. $K(X, f) \ne \emptyset$. We prove that for every $F \in K(X, f)$, $2^\pi \colon K(X, f) \to K(Y, g)$, $2^\pi(F) = \pi(F)$ is a one-to-one correspondence between $K(X, f)$ and $K(Y, g)$. Recall $\pi \colon X \to Y$ is a conjugate map, that is, $\pi \circ f = g \circ \pi$. Since $2^\pi(F) = \pi(F)$ and $g(\pi(F)) = \pi(f(F)) \subseteq \pi(F)$, so we have $\pi(F) \in K(Y, g)$. Hence, 2^π is well definite. Furthermore, for any $F_1, F_2 \in K(X, f)$ and $F_1 \ne F_2$, we have $2^\pi(F_1) = \pi(F_1)$, $2^\pi(F_2) = \pi(F_2)$ and $\pi(F_1) \ne \pi(F_2)$, thus $2^\pi(F_1) \ne 2^\pi(F_2)$. Moreover, for any $F \in K(Y, g)$, we have $\pi^{-1}(F) \in K(X, f)$ and $2^\pi(\pi^{-1}(F)) = \pi(\pi^{-1}(F)) = F$. Therefore, $2^\pi \colon K(X, f) \to K(Y, g)$ is bijective. We consider $F \in K(X, f)$, then $\pi \colon F \to \pi(F)$ is a conjugate map, that is, $\pi \circ f|_F = g|_{\pi(F)} \circ \pi$. By Corollary 3.1 (3), we have $h_{\text{pre}}(f|_F, F) = h_{\text{pre}}(g|_{\pi(F)}, \pi(F))$. Furthermore, $h^*_{\text{pre}}(f|_F, F) = h^*_{\text{pre}}(g|_{\pi(F)}, \pi(F))$. Hence,

$$h^*_{\text{pre}}(f) = \sup_{F \in K(X,f)} h^*_{\text{pre}}(f|_F, F) = \sup_{F \in K(X,f)} h^*_{\text{pre}}(g|_{\pi(F)}, \pi(F)).$$

Since $2^\pi \colon K(X,f) \to K(Y,g)$ is a one-to-one correspondence, we have

$$\sup_{F\in K(X,f)} h^*_{\mathrm{pre}}(g|_{\pi(F)}, \pi(F)) = \sup_{F'\in K(Y,g)} h^*_{\mathrm{pre}}(g|_{F'}, F') = h^*_{\mathrm{pre}}(g).$$

Therefore, $h^*_{\mathrm{pre}}(f) = h^*_{\mathrm{pre}}(g)$. □

4. Pre-image entropies of locally compact spaces and induced hyperspaces

Let R denote the one-dimensional Euclidean space and X denote a (noncompact) locally compact metrizable space, if not indicated otherwise. From Kelley's result [**11**], the Alexandroff compactification (that is, one-point compactification) $\omega X = X \cup \{\omega\}$ of X is also metrizable.

Definition 4.1 ([**14**])**.** Let $f \colon X \to X$ be a continuous map.

(1) If there exists an $a \in X$ such that for every sequence x_n of points of X, $\lim_{n\to\infty} f(x_n) = a$ holds whenever x_n does not have any convergent subsequence in X, then f is said to be convergent to a at infinity.
(2) If for every sequence x_n of points of X, x_n does not have any convergent subsequence in X, $f(x_n)$ does not have any convergent subsequence, then f is said to be convergent to infinity at the infinity.
(3) If (1) or (2) hold, f is said to be convergent at the infinity.

Theorem 4.1 ([**21**])**.** *A continuous map $f \colon X \to X$ is convergent at the infinity if and only if f can be extended to a continuous map $\bar{f}$ on the Alexandroff compactification ωX.*

Theorem 4.2. *Let (X, f) be a dynamical system. If f can be extended to a continuous map on the Alexandroff compactification ωX, that is, f is convergent at the infinity and $\bar{f}(\omega) = a$ or $\bar{f}(\omega) = \omega$ (refer to Definition* 4.1*), then $h^*_{\mathrm{pre}}(f) \le h^*_{\mathrm{pre}}(\bar{f})$.*

Proof. By the assumption, $(\omega X, \bar{f})$ is a topological dynamical system and (X, f) is a subsystem of $(\omega X, \bar{f})$ (by a clear embedding). Hence, from Theorem 3.2, $h^*_{\mathrm{pre}}(f) \le h^*_{\mathrm{pre}}(\bar{f})$. □

Example 4.1. Let $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$, $x \in \mathbb{R}$. Then $h^*_{\mathrm{pre}}(f) = 0$.

From assumption, the only invariant compact subset of f is $\{0\}$, that is, $K(\mathbb{R}, f) = \{\{0\}\}$. Denote $F = \{0\}$. We prove $h^*_{\mathrm{pre}}(f|_F, F) = 0$. In fact, $f \colon F \to F$ is a homeomorphism from compact space F onto itself. Hence, $h_{\mathrm{pre}}(f|_F) = 0$. As $h^*_{\mathrm{pre}}(f|_F, F) = h_{\mathrm{pre}}(f|_F)$, which implies $h^*_{\mathrm{pre}}(f|_F, F) = 0$. Therefore, by Definition 2.3, we have $h^*_{\mathrm{pre}}(f) = 0$.

If $\mathbb{R}$ is replaced by $(0, \infty)$ which is equipped with the subspace topology of $\mathbb{R}$, $K((0,\infty), f) = \emptyset$. It follows from Definition 2.3 that $h^*_{\mathrm{pre}}(f) = 0$.

Theorem 4.3. *If $f \colon \mathbb{R} \to \mathbb{R}$ is an autohomeomorphism, then $h^*_{\mathrm{pre}}(f) = 0$.*

Proof. Let x_n be a sequence of points of $\mathbb{R}$ that does not have any convergent subsequence in $\mathbb{R}$. As f is a homeomorphism, the sequence $f(x_n)$ does not have any convergent subsequence in $\mathbb{R}$ neither. By Theorem 4.1, f can be extended to a continuous map $\bar{f}\colon \omega\mathbb{R} \to \omega\mathbb{R}$ and $\bar{f}(\omega) = \omega$. Clearly, $\bar{f}$ is also an autohomeomorphism. On the other hand, ωR is homeomorphic to the unit circle S^1. Let $\pi\colon \omega\mathbb{R} \to S^1$ be such a homeomorphism. Define $g\colon S^1 \to S^1$ by $g = \pi \circ \bar{f} \circ \pi^{-1}$. Then, g is a homeomorphism and π gives the conjugace between $(\omega\mathbb{R}, \bar{f})$ and (S^1, g). Hence, it follows from Theorem 3.3 that $h^*_{\text{pre}}(\bar{f}) = h^*_{\text{pre}}(g)$. Now, from the result given in Walters book [**20**], $h(g) = 0$, where $h(g)$ denotes topological entropy of g. By [**7**], $h^*_{\text{pre}}(g) \leq h(g)$, which implies $h^*_{\text{pre}}(g) = 0$. Hence, $h^*_{\text{pre}}(\bar{f}) = 0$. From Theorem 4.2, $h^*_{\text{pre}}(f) \leq h^*_{\text{pre}}(\bar{f})$. Therefore, $h^*_{\text{pre}}(f) = 0$. □

We investigate the pre-image entropy relation between a topological dynamical system and its induced hyperspace topological dynamical system. The hyperspace is employed with the Vietoris topology. Notice that if X is a noncompact metric space, the Vietoris topology is non-metrizable [**15**].

The Vietoris topology on 2^X, the family of all nonempty closed subsets of X, is generated by the base

$$\upsilon(U_1, U_2, \cdots, U_n) = \left\{ F \in 2^X : F \subseteq \bigcup_{i=1}^{n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for all } i \leq n \right\},$$

where $U_1, U_2, \cdots, U_n$ are open subsets of X [**9**].

Let (X, f) be a topological dynamical system, where $f\colon X \to X$ is a closed mapping. The hyperspace map $2^f\colon 2^X \to 2^X$ is induced by f as follows: for every $F \in 2^X$, $2^f(F) = f(F)$. When f is a closed and continuous map, 2^f is well defined and it is continuous [**11, 15**], thus ensuring that $(2^X, 2^f)$ forms a topological dynamical system, i.e., the induced hyperspace topological dynamical system of (X, f).

By Michael's results [**15**], we have the following facts.

Fact 1: If X is compact, then 2^X is compact.

Fact 2: If X is compact and Hausdorff, then 2^X is compact and Hausdorff.

Fact 3: $\pi\colon X \to 2^X$ defined by $\pi(x) = \{x\}$ for $x \in X$, is continuous. If X is compact and Hausdorff, then π is homeomorphic embedding and (X, f) and $(\pi(X), 2^f)$ are topologically conjugate.

Theorem 4.4. [**14**] *Let (X, f) be a topological dynamical system, where X is Hausdorff and f is a closed map. If $F \in K(X, f)$, then $2^F \in K(2^X, 2^f)$. Hence, $(2^F, 2^f)$ is a subsystem of $(2^X, 2^f)$.*

Theorem 4.5. *Let (X, f) be a topological dynamical system, where X is Hausdorff and f is a closed map. Then $h^*_{\text{pre}}(2^f) \geq h^*_{\text{pre}}(f)$.*

Proof. Case 1. $K(X, f) = \emptyset$. By Definition 2.3, we have $h^*_{\text{pre}}(f) = 0$. Hence, $h^*_{\text{pre}}(2^f) \geq h^*_{\text{pre}}(f)$.

Case 2. $K(X, f) \neq \emptyset$. For $F \in K(X, f)$, it follows from Theorem 4.4 that $2^F \in K(2^X, 2^f)$. Define $\pi\colon F \to 2^F$ by $\pi(x) = \{x\}$, $x \in F$. From Fact 3 in the preceding paragraph of Theorem 4.4, (F, f) and $(\pi(F), 2^f)$ are topologically conjugate. From Cheng and Newhouse's result [**7**], $h_{\rm pre}(f|_F, F) = h_{\rm pre}(2^f|_{\pi(F)}, \pi(F))$. By Remark 1, $h^*_{\rm pre}(f, F) = h_{\rm pre}(f|_F, F)$ and $h^*_{\rm pre}(2^f, \pi(F)) = h_{\rm pre}(2^f|_{\pi(F)}, \pi(F))$, which imply $h^*_{\rm pre}(f, F) = h^*_{\rm pre}(2^f, \pi(F))$. Again, by the Fact 3, $\pi(F)$ is a compact subset of 2^X. On the other hand, from $2^f(\pi(F)) = \pi(f(F))$ and $f(F) \subseteq F$, we have $2^f(\pi(F)) = \pi(f(F)) \subseteq \pi(F)$, thus $\pi(F) \in K(2^X, 2^f)$. Furthermore, it follows from Definition 2.3 that $h^*_{\rm pre}(2^f, \pi(F)) \le h^*_{\rm pre}(2^f)$ implying $h^*_{\rm pre}(f, F) \le h^*_{\rm pre}(2^f)$. Therefore, $h^*_{\rm pre}(f) = \sup_{F\in K(X,f)} \{h^*_{\rm pre}(f|_F, F)\} \le h^*_{\rm pre}(2^f)$. □

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