

LYAPUNOV OPERATOR INEQUALITIES FOR EXPONENTIAL STABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS IN BANACH SPACES

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ABSTRACT. In the present paper we prove a sufficient condition and a characterization for the stability of linear skew-product semiflows by using Lyapunov function in Banach spaces. These are generalizations of the results obtained in [1] and [12] for the case of C_0 -semigroups. Moreover, there are presented the discrete variants of the results mentioned above.

1. INTRODUCTION

The theorem of A. M. Lyapunov establishes that if A is a $n \times n$ complex matrix then A has all its characteristics roots with real parts negative if and only if for any positive definite Hermitian matrix H , there exists a positive definite Hermitian matrix W satisfying the equation

$$(L_H) \quad A^*W + WA = -H$$

(where $*$ denotes the conjugate transpose of a matrix) (see [2]).

The use of the above Lyapunov operator equation is extended on the infinite-dimensional framework by Daleckij and Krein [4] for the case of semigroups $T(t) = e^{tA}$, where A is a bounded linear operator. The authors prove in [4] that $\{e^{tA}\}_{t \geq 0}$, with $A \in \mathcal{B}(X)$ is exponentially stable if and

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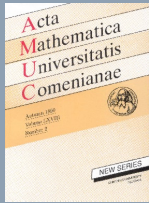


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only if there exists $W \in \mathcal{B}(X)$, $W \gg 0$ (i.e., there exists $m > 0$ such that $\langle Wx, x \rangle \geq m\|x\|^2$ for any $x \in X$), solution of the Lyapunov equation $A^*W + WA = -I$.

This result is extended by R. Datko [5], for the general case of C_0 -semigroups as it follows.

Theorem 1.1 ([5]). *A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W = W^*$, $W \geq 0$ such that*

$$(L) \quad \langle Ax, Wx \rangle + \langle Wx, Ax \rangle = -\|x\|^2$$

for all $x \in D(A)$, where A denotes the infinitesimal generator of $\{T(t)\}_{t \geq 0}$.

C. Chicone [3], Y. Latushkin [3], A. Pazy [9], J. Goldstein [6] and L. Pandolfi [8] studied the Lyapunov operator equations with unbounded A . All the above results are given in the setting of one-parameter semigroups acting on Hilbert spaces.

Moreover, in [10], an attempt to establish an equivalence between the solvability of the Lyapunov operator equation and the exponential stability of a C_0 -semigroup in the general context of Banach spaces is presented.

Also in [12], C. Preda and P. Preda studied the case of the Lyapunov operator equation for the exponential stability of one-parameter semigroups acting on Banach spaces by using the idea of N.U. Ahmed (see [1]).

For the case of linear skew-product semiflows on real Hilbert spaces, a result which presents an equality of Lyapunov type can be found in [15]. In that paper, Pham Viet Hai and Le Ngoc Thanh present some characterizations for the uniform exponential stability of linear skew-product semiflows using a variant of Lyapunov equality.

Some necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in Banach spaces are given in the paper [7]. The authors use Banach function spaces to obtain generalizations of some well-known results of Datko, Neerven, Rolewicz and Zabczyk.

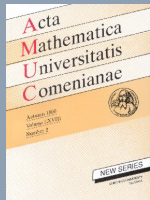


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On the other hand, in the paper [14], Pham Viet Hai extends the results of P. Preda, A. Pogan and C. Preda from [11] for the case of the uniform exponential stability of linear skew-product semiflows.

In the present paper, we try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows acting on Banach spaces.

This paper extends for the case of linear skew-product semiflows the results obtained in [12] for the case of strongly continuous, one-parameter semigroups acting on Banach spaces by using analogous techniques.

In order to do that, we need to recall some notions about the adjoint of a linear operator on a Banach space.

Let X be a real or complex Banach space and X' its (dual) conjugate space consisting of all bounded and antilinear functionals on X . Also X^* will denote the classic dual space of all bounded and linear functionals on X .

If Y is also a Banach space, we will denote by $\mathcal{B}(X, Y)$ the Banach space of all linear and bounded operators from X to Y . If $X = Y$, we will write $\mathcal{B}(X)$.

The norms on X , X' , Y and $\mathcal{B}(X, Y)$ will be denoted by the symbol $\|\cdot\|$.

We will use the symbols \mathbb{R} , \mathbb{R}_+ , \mathbb{N} to denote the set of real, nonnegative real and natural numbers respectively and $\mathbb{N}^* = \mathbb{N} - \{0\}$.

We will present some definitions in what follows.

Let Θ be a metric space.

Definition 1.1. A map $\sigma: \Theta \times \mathbb{R}_+ \rightarrow \Theta$ is said to be a *continuous semiflow* on Θ if the following conditions hold

- i) $\sigma(\theta, 0) = \theta$ for all $\theta \in \Theta$;
- ii) $\sigma(\theta, t + s) = \sigma(\sigma(\theta, s), t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$;

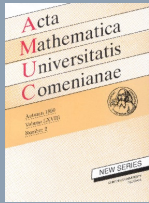


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iii) $(\theta, t) \mapsto \sigma(\theta, t)$ is continuous on $\Theta \times \mathbb{R}_+$.

If iii) holds for any $t, s \in \mathbb{R}$ then σ is said to be a *flow on Θ* .

Definition 1.2. Let σ be a continuous semiflow on Θ . A *strongly continuous cocycle over the continuous semiflow σ* is an operator-valued function

$$\Phi: \Theta \times \mathbb{R}_+ \rightarrow \mathcal{B}(X), \quad (\theta, t) \mapsto \Phi(\theta, t)$$

that satisfies the following properties

- i) $\Phi(\theta, 0) = I$ (I – the identity operator on X) for all $\theta \in \Theta$;
- ii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous for each $\theta \in \Theta$ and $x \in X$;
- iii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$ (the cocycle identity);

If, in addition,

- iv) there exist constants $M, \omega > 0$ such that

$$\|\Phi(\theta, t)\| \leq M e^{\omega t} \quad \text{for } t \geq 0 \text{ and } \theta \in \Theta,$$

then the strongly continuous cocycle is *exponentially bounded*.

Definition 1.3. The linear skew-product semiflow (LSPS) associated with the above cocycle is the dynamical system $\pi = (\Phi, \sigma)$ on $\varepsilon = X \times \Theta$ defined by

$$\pi: X \times \Theta \times \mathbb{R}_+ \rightarrow X \times \Theta, \quad (x, \theta, t) \mapsto \pi(x, \theta, t) = (\Phi(\theta, t)x, \sigma(\theta, t)).$$

We will give some examples of LSPS. First of all, we will define some notions used in the following examples.

Definition 1.4. A family $\{T(t)\}_{t \geq 0}$ of linear and bounded operators acting on X is said to be a C_0 -semigroup or a *strongly continuous semigroup* on X if the following conditions hold:

- i) $T(0) = I$;

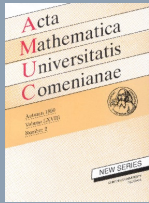


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- ii) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;
- iii) there exists $\lim_{t \rightarrow 0_+} T(t)x = x$ for all $x \in X$.

If the second property holds for any $t, s \in \mathbb{R}$, then $\{T(t)\}_{t \in \mathbb{R}}$ is called a C_0 -group. For a general presentation of the theory of C_0 -semigroups, we refer the reader to [9].

Definition 1.5. A family of linear and bounded operators $\{U(t, s)\}_{t \geq s \geq 0}$ is said to be a *two-parameter evolution family* if the following conditions hold:

- i) $U(t, t) = I$ for all $t \geq 0$;
- ii) $U(t, t_0)U(t_0, s) = U(t, s)$ for all $t \geq t_0 \geq s \geq 0$;
- iii) $U(\cdot, s)x$ is continuous on $[s, \infty)$ for all $s \geq 0, x \in X$;
 $U(t, \cdot)x$ is continuous on $[0, t)$ for all $t \geq 0, x \in X$;
- iv) there exist $M, \omega > 0$ such that

$$\|U(t, s)\| \leq M e^{\omega(t-s)} \quad \text{for all } t \geq s \geq 0.$$

For a general presentation of the theory of two-parameter evolution families, we refer the reader to [3] or [4].

Example 1.1. Let Θ be a metric space, σ a semiflow on Θ and $\{T(t)\}_{t \geq 0}$ a C_0 -semigroup on X . The pair $\pi_T = (\Phi_T, \sigma)$ where $\Phi_T(\theta, t) = T(t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ is a linear skew-product semiflow over σ on $\Theta \times X$.

Example 1.2. Let $\Theta = \mathbb{R}_+$, $\sigma(\theta, t) = \theta + t$ and let $\{U(t, s)\}_{t \geq s}$ be an evolution family on the Banach space X . We define

$$\Phi_U(\theta, t) = U(t + \theta, \theta) \quad \text{for all } (\theta, t) \in \Theta_+ \times \mathbb{R}_+.$$

Then $\{\Phi_U(\theta, t)\}_{\theta \in \Theta, t \geq 0}$ is an exponentially bounded, strongly continuous cocycle (over the above semiflow σ) and the linear skew-product semiflow associated with it is the pair $\pi = (\Phi_U, \sigma)$.

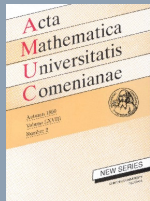


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Therefore, we can say that the notion of a cocycle generalizes the classic notion of a two-parameter evolution family.

Example 1.3. Let Θ be a metric space, σ a semiflow on Θ , X a Banach space and $A: \Theta \rightarrow \mathcal{B}(X)$ a continuous mapping. The problem

$$\begin{aligned}\dot{x}(t) &= A(\sigma(\theta, t))x(t) \\ x(t_0) &= x_0\end{aligned}$$

has an unique solution for all $t_0 \in \mathbb{R}_+$ and $x_0 \in X$. For details we refer the reader to [13].

Definition 1.6. A linear skew-product semiflow (LSPS) $\pi = (\Phi, \sigma)$ on a Banach bundle $\varepsilon = X \times \Theta$ is said to be *exponentially stable* if there exist constants $N, \nu > 0$ such that

$$\|\Phi(\theta, t)x\| \leq N e^{-\nu t} \|x\| \quad \text{for all } t \geq 0, \theta \in \Theta, x \in X.$$

All the results concerning the Lyapunov inequality for the exponential stability of linear skew-product semiflows (LSPS), were acting on Hilbert spaces. We will try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows (LSPS) acting on Banach spaces. This requires to recall some facts about the adjoint of a linear operator on a Banach space (see [12]).

Definition 1.7. Let X, Y be two Banach spaces and $A \in \mathcal{B}(X, Y)$. Then there exists an unique operator $A^* \in \mathcal{B}(Y', X')$ that satisfies $y(Ax) = A^*y(x)$ for all $x \in X$ and $y \in Y'$. A^* will be called *the adjoint of A*.

It can be easily checked that

- $\|A\| = \|A^*\|$;
- $(A + B)^* = A^* + B^*$;
- $(\lambda A)^* = \bar{\lambda}A^*$;

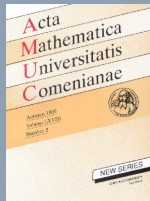


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- If X, Y are reflexive, then $A^{**} = A$.

It is worth to note that the above notion of the adjoint of a linear and bounded operator between two Banach spaces allows us to create a definition of the adjoint that directly generalizes the definition of the adjoint of an operator on Hilbert spaces. In other words, if X and Y are Hilbert spaces and $A \in \mathcal{B}(X, Y)$, then there is no difference of the adjoint between the adjoint A^* defined by considering X, Y to be Hilbert spaces, and the adjoint A^* defined by considering X, Y to be Banach spaces. If we chose that $A^*: Y^* \rightarrow X^*$, then we would obtain a different definition compared to the Hilbert space definition.

For defining the concept of a self-adjoint operator on a Banach space, we recall that X is isomorphic and isometric with a subspace of X'' .

Definition 1.8.

- (i) An operator $A \in \mathcal{B}(X, X')$ is said to be *self-adjoint* if the restriction of A^* to X is A , and therefore,

$$Ay(x) = \overline{Ax(y)} \quad \text{for all } x, y \in X.$$

- (ii) $A \in \mathcal{B}(X, X')$ is said to be *positive* if A is self-adjoint and $Ax(x) \geq 0$ for all $x \in X$.

Remark 1.1. It is easy to see that $A \in \mathcal{B}(X, X')$ is positive if and only if $Ax(x)$ is a positive real number for all $x \in X$.

In the following we will denote by

$$\mathcal{B}^+(X, X') = \{A \in \mathcal{B}(X, X') : A \text{ is positive}\}.$$

Following Lyapunov's idea, we obtain a Lyapunov-type operatorial equation for the case of linear skew-product semiflows acting on Banach spaces. Indeed, from the equation (L_H) and (L) , taking into account the fact that any C_0 -semigroup is a particular case of linear skew-product semiflows,

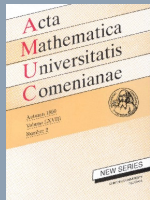


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we obtain for the case of Hilbert spaces that (see [15])

$$(L^*) \quad \langle A(\sigma(\theta, t))x, W(\sigma(\theta, t))x \rangle + \langle W(\sigma(\theta, t))x, A(\sigma(\theta, t))x \rangle = -\|x\|^2.$$

If we assume that (L^*) holds for some conditions, let f be the function defined by

$$f(t) = \langle W(\sigma(\theta, t))\Phi(\theta, t)x, \Phi(\theta, t)x \rangle.$$

It can be easily seen that $f'(t) = -\|\Phi(\theta, t)x\|^2$. Integrating with respect to τ on the interval $[0, t]$, we have

$$\langle W(\sigma(\theta, t))\Phi(\theta, t)x, \Phi(\theta, t)x \rangle - \langle W(\theta)x, x \rangle = - \int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau,$$

which implies

$$\Phi^*(\theta, t)W(\sigma(\theta, t))\Phi(\theta, t)x + \int_0^t \Phi^*(\theta, \tau)\Phi(\theta, \tau)x d\tau = W(\theta)x.$$

If we rewrite the equation above to the case of Banach spaces, using the considerations about the adjoint of an operator in Banach spaces, we have

$$(L') \quad W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau = W(\theta)x(x).$$

Remark 1.2. The bounded function $W: \Theta \rightarrow \mathcal{B}^+(X, X')$ from the equation (L') is said *Lyapunov function* corresponding to linear skew-product semiflow $\pi = (\Phi, \sigma)$.

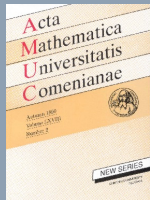


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2. RESULTS

In what follows it will be presented a sufficient condition for the exponential stability of linear skew-product semiflows acting on Banach spaces in terms of Lyapunov inequation.

Theorem 2.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS). If there exists $W : \Theta \rightarrow \mathcal{B}^+(X, X')$ bounded such that*

$$(1) \quad W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau \leq W(\theta)x(x)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$, then $\pi = (\Phi, \sigma)$ is exponentially stable.

Proof. Let $x \in X$, $\theta \in \Theta$ and $t \geq 0$. From (1) we have that

$$\begin{aligned} \int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau &\leq W(\theta)x(x) - W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) \\ &\leq W(\theta)x(x) = |W(\theta)x(x)| \leq K\|x\|^2 \end{aligned}$$

for all $\theta \in \Theta$, $x \in X$ and $t \geq 0$, where $K = \sup_{\theta \in \Theta} \|W(\theta)\| > 0$.

Thus we get that

$$\int_0^t \|\Phi(\theta, \tau)x\|^2 d\tau \leq K\|x\|^2$$

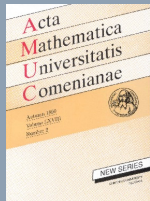


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for all $\theta \in \Theta$, $x \in X$ and $t \geq 0$, which implies the following relation for $t \rightarrow \infty$

$$\int_0^{\infty} \|\Phi(\theta, \tau)x\|^2 d\tau \leq K \|x\|^2 \quad \text{for all } \theta \in \Theta \text{ and } x \in X.$$

From [15, Lemma 2.4], it results that the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable. \square

In what follows, it will be presented the necessary condition which needs a stronger hypothesis.

Theorem 2.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W: \Theta \rightarrow \mathcal{B}^+(X, X')$ bounded such that*

$$(2) \quad W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau = W(\theta)x(x)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable and therefore we have from Definition 1.6 that there exist the constants $N, \nu > 0$ such that

$$\|\Phi(\theta, t)x\| \leq N e^{-\nu t} \|x\| \quad \text{for all } t \geq 0, \theta \in \Theta, x \in X.$$

Now we consider $x, y \in X$, $\theta \in \Theta$ and

$$W(\theta)x(y) = \int_0^{\infty} \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)d\tau.$$

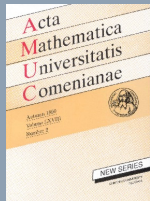


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Next we will show that $W \in \mathcal{B}^+(X, X')$.

Thus we have that

$$\begin{aligned}
 |W(\theta)x(y)| &= \left| \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y) d\tau \right| \leq \int_0^\infty |\Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)| d\tau \\
 &\leq \|\Gamma\| \int_0^\infty \|\Phi(\theta, \tau)x\| \|\Phi(\theta, \tau)y\| d\tau \leq \|\Gamma\| N^2 \int_0^\infty e^{-2\nu\tau} d\tau \|x\| \|y\| \\
 &= \frac{N^2}{2\nu} \|\Gamma\| \|x\| \|y\|,
 \end{aligned}$$

which shows that W is linear and bounded.

On the other hand,

$$\begin{aligned}
 \overline{W(\theta)y(x)} &= \int_0^\infty \overline{\Gamma(\Phi(\theta, \tau)y)(\Phi(\theta, \tau)x)} d\tau \\
 &= \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y) d\tau = W(\theta)x(y)
 \end{aligned}$$

for all $x, y \in X$ and $\theta \in \Theta$. Thus, W is self-adjoint.

Moreover,

$$W(\theta)x(x) = \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x) d\tau \geq \gamma \int_0^\infty \|\Phi(\theta, \tau)x\|^2 d\tau \geq 0,$$

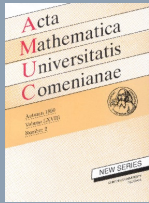


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which implies the fact that W is positive.

It results that $W \in \mathcal{B}^+(X, X')$. Now we have that

$$\begin{aligned} & W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) \\ &= \int_0^\infty \Gamma(\Phi(\sigma(\theta, t), \tau)\Phi(\theta, t)x)(\Phi(\sigma(\theta, t), \tau)\Phi(\theta, t)x)d\tau \\ &= \int_0^\infty \Gamma(\Phi(\theta, t + \tau)x)(\Phi(\theta, t + \tau)x)d\tau \\ &= \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau - \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau \\ &= W(\theta)x(x) - \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau \end{aligned}$$

and therefore, we get the relation (2) and the proof is complete. \square

As a result of the last two theorems, we now obtain the necessary and sufficient conditions for the exponential stability of a linear skew-product semiflow (LSPS) as follows.

Corollary 2.1. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$ for all*

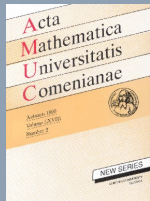


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$x \in X$, there exists $W : \mathbb{R}_+ \rightarrow \mathcal{B}^+(X, X')$ bounded such that

$$(3) \quad W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau = W(\theta)x(x)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. Necessity results from Theorem 2.2.

Sufficiency results analogously with Theorem 2.1, by considering in addition $\Gamma \in \mathcal{B}^+(X, X')$ with the same property as in Theorem 2.2. \square

In what follows we will also present the discrete versions of the above results.

A sufficient condition is given as follows

Theorem 2.3. Let $\pi = (\Phi, \sigma)$ be linear skew-product semiflow. If there exists $W : \mathbb{N} \rightarrow \mathcal{B}^+(X, X')$ bounded such that

$$(4) \quad W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \|\Phi(\theta, k)x\|^2 \leq W(\theta)x(x)$$

for all $\theta \in \Theta$, $n \in \mathbb{N}^*$ and $x \in X$, then the linear skew-product semiflow is exponentially stable.

Proof. We take $n \in \mathbb{N}^*$ and $x \in X$. From relation (4), we have that

$$\begin{aligned} \sum_{k=0}^{n-1} \|\Phi(\theta, k)x\|^2 &\leq W(\theta)x(x) - W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) \\ &\leq W(\theta)x(x) = |W(\theta)x(x)| \leq L\|x\|^2 \end{aligned}$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$, where $L = \sup_{\theta \in \Theta} \|W(\theta)\| > 0$.

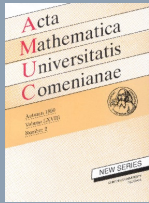


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For $n \rightarrow \infty$ in the previous relation we obtain that

$$\sum_{k=0}^{\infty} \|\Phi(\theta, k)x\|^2 \leq L\|x\|^2 < \infty \quad \text{for all } \theta \in \Theta \text{ and } x \in X.$$

Applying [15, Lemma 2.1 and Lemma 2.2], we get that the linear skew product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable. \square

The sufficient condition is given in the following theorem

Theorem 2.4. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma\|x\|^2$ for all $x \in X$, there exists $W: \Theta \rightarrow \mathcal{B}^+(X, X')$ bounded such that*

$$(5) \quad W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) = W(\theta)x(x)$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

Proof. As the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable, we have from Definition 1.6 that there exist the constants $N, \nu > 0$ such that

$$\|\Phi(\theta, n)x\| \leq N e^{-\nu n} \|x\|, \quad \text{for all } n \in \mathbb{N} \theta \in \Theta \text{ and } x \in X.$$

We take now $x, y \in X, n \in \mathbb{N}^*$ and

$$W(\theta)x(y) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)y).$$

Next it will be shown that $W \in \mathcal{B}^+(X, X')$.

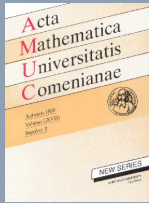


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Therefore, we have that

$$\begin{aligned}
 |W(\theta)x(y)| &= \left| \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)y) \right| \leq \sum_{k=0}^{\infty} |\Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)y)| \\
 &\leq \|\Gamma\| \sum_{k=0}^{\infty} \|\Phi(\theta, k)x\| \|\Phi(\theta, k)y\| \leq \|\Gamma\| N^2 \sum_{k=0}^{\infty} e^{-2\nu k} \|x\| \|y\| \\
 &\leq \frac{N^2}{1 - e^{-2\nu}} \|\Gamma\| \|x\| \|y\|,
 \end{aligned}$$

which shows that W is linear and bounded.

Moreover,

$$\begin{aligned}
 \overline{W(\theta)y(x)} &= \sum_{k=0}^{\infty} \overline{\Gamma(\Phi(\theta, k)y)(\Phi(\theta, k)x)} \\
 &= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)y) = W(\theta)x(y)
 \end{aligned}$$

for all $x, y \in X$ and $\theta \in \Theta$. Thus, W is self-adjoint.

On the other hand,

$$W(\theta)x(x) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) \geq \gamma \sum_{k=0}^{\infty} \|\Phi(\theta, k)x\|^2 \geq 0,$$

which implies the fact that W is positive.

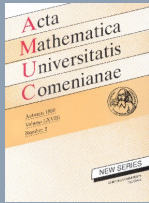


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It results that $W \in \mathcal{B}^+(X, X')$. Thus we have that

$$\begin{aligned}
 & W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) \\
 &= \sum_{k=0}^{\infty} \Gamma(\Phi(\sigma(\theta, n), k)\Phi(\theta, n)x)(\Phi(\sigma(\theta, n), k)\Phi(\theta, n)x) \\
 &= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, n+k)x)(\Phi(\theta, n+k)x) \\
 &= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) \\
 &= W(\theta)x(x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x)
 \end{aligned}$$

and therefore, we get the relation (5). □

As a result of Theorems 2.3 and 2.4, it can be obtained the following corollary

Corollary 2.2. *The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$ for all $x \in X$, there exists $W : \mathbb{R}_+ \rightarrow \mathcal{B}^+(X, X')$ bounded such that*

$$(6) \quad W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) = W(\theta)x(x)$$

for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

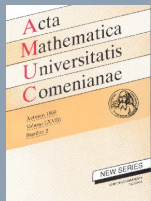


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Proof. *Necessity* results from Theorem 2.4.

Sufficiency results analogously with Theorem 2.3. □

Remark 2.1. As a conclusion, we can mention here that it is interesting to note that the sufficient condition can be easily obtained, but for the necessary condition, we need a stronger hypothesis. Thus, in terms of the the existence of $\Gamma \in \mathcal{B}^+(X, X')$ with the properties presented above, the exponential stability of a linear skew-product semiflow implies the existence of a Lyapunov function that verifies the Lyapunov-type equation.

Also, the sufficient condition holds in terms of the existence of $\Gamma \in \mathcal{B}^+(X, X')$.

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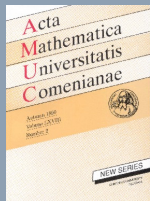


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