Acta Universitatis Apulensis No. 66/2021
ISSN: 1582-5329 pp. 71-88
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.65.06

ON MEROMORPHIC MULTIVALENT FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR

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Abstract. The purpose of this article is to define and investigate a new subclass of meromorphic starlike functions by using Liu-Srivastava operator. A number of sufficient conditions for function belonging to this class are derived.

2010 *Mathematics Subject Classification*: Primary 30C45, Secondary 30C50

Keywords: Meromorphic functions, convolution, integral operator.

1. Introduction

Let Σ_p denotes the class of p-valent meromorphic function of the form:

$$\lambda(\omega) = \frac{1}{\omega^p} + \sum_{t=p}^{\infty} a_t \omega^t, \tag{1}$$

which are analytic in the punctured open unit disc $U^* = \{\omega : \omega \in \text{ and } 0 < |\omega| < 1\} = U - \{0\}$, where $U = U^* \cup \{0\}$. In particular, $\Sigma_1 = \Sigma$ is the class of meromorphic functions defined in U^* and has simple pole at $\omega = 0$. Here we are listing some important subclasses of meromorphic functions which will be used in our subsequal useful work. In 1936, Roberston [22] define these classes of order α. By $MS_p^*(\alpha)$ we mean the subclass of Σ_p consisting of all meromorphically p-valent starlike functions of order α defined by

$$\lambda(\omega) \in MS_p^*(\alpha) \Leftrightarrow \Re\left(\frac{\omega\lambda'(\omega)}{p\lambda(\omega)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{2}$$

A function $\lambda(\omega) \in NS_p^*(\alpha)$ of meromorphically p-valent starlike functions of reciprocal order α if and only if

$$\lambda(\omega) \in NS_p^*(\alpha) \Leftrightarrow \Re\left(\frac{p\lambda(\omega)}{\omega\lambda'(\omega)}\right) < -\alpha \quad (0 \leq \alpha < 1;\ \omega \in U^*). \tag{3}$$

A closely related class of meromorphic p-valent convex functions of order α is denoted by $MK_p(\alpha)$ and defined as:

$$\lambda(\omega) \in MK_p(\alpha) \Leftrightarrow \Re\left(\frac{(\omega\lambda'(\omega))'}{p\lambda'(\omega)}\right) < -\alpha, \quad (\ \omega \in U^*). \tag{4}$$

It is readily verified from (2) and (3) that

$$\lambda\left(\omega\right) \in MK_p(\alpha) \Leftrightarrow -\frac{\omega\lambda^{'}\left(\omega\right)}{p} \in MS_p^*(\alpha). \tag{5}$$

For simplicity, we write

$$MS_p^*(0) = MS_p^*, \quad MK_p(0) = MK_p.$$

Many differential and integral operators can be written in terms of convolution of certain analytic functions. Let $\delta\left(\omega\right) \in \sum_p$ and having series representation of the form $\delta\left(\omega\right) = \frac{1}{\omega^p} + \sum\limits_{t=0}^{\infty} b_t\omega^t$, then convolution (Hadamard product) is denoted by $\lambda * \delta$ and defined as

$$\left(\lambda * \delta\right)\left(\omega\right) = \frac{1}{\omega^p} + \sum_{t=0}^{\infty} a_t b_t \omega^t = \left(\delta * \lambda\right)\left(\omega\right), \tag{6}$$

where $\lambda\left(\omega\right)$ is given in (1). A function $\lambda\left(\omega\right)$ is subordinate to $\delta\left(\omega\right)$ in U and written as $\lambda\left(\omega\right) \prec \delta\left(\omega\right)$, if there exists a Schwarz function $k(\omega)$, which is holomorphic in U^* with $|k(\omega)| < 1$, such that $\lambda\left(\omega\right) = \delta(k\left(\omega\right))$. Furthermore, if the function $\delta\left(\omega\right)$ is univalent in U^*, then we have the following equivalence (see [8, 15, 17, 24]):

$$\lambda\left(\omega\right) \prec \delta\left(\omega\right) \text{ and } \lambda\left(U\right) \subset \delta\left(U\right).$$

Further, $\lambda\left(\omega\right)$ is quasi-subordinate to $\delta\left(\omega\right)$ in U^* and written is

$$\lambda\left(\omega\right) \prec_q \delta\left(\omega\right) \quad \left(\ \omega \in U^*\right),$$

if there exist two analytic functions $\varphi\left(\omega\right)$ and $k\left(\omega\right)$ in U^* such that $\frac{\lambda(\omega)}{\varphi(\omega)}$ is analytic in U^* and

$$\left|\varphi\left(\omega\right)\right| \leq 1 \text{ and } k\left(\omega\right) \leq |\omega| < 1 \quad \omega \in U^*,$$

satisfying

$$\lambda\left(\omega\right) = \varphi\left(\omega\right)\delta\left(k\left(\omega\right)\right) \quad \omega \in U^*. \tag{7}$$

Remark 1. *In view of the fact that*

$$\Re\left(p\left(\omega\right)\right)<0\Rightarrow\Re\left(\frac{1}{p\left(\omega\right)}\right)=\Re\left(\frac{p\left(\omega\right)}{\left|p\left(\omega\right)\right|^{2}}\right)<0.$$

It follows that a meromorphically p-valent starlike function of reciprocal order 0 *is same as a meromorphically p-valent starlike function. When* $0<\alpha<1$*, the function* $\lambda\left(\omega\right)\in\sum_{p}$ *is meromorphically p-valent starlike of reciprocal order if and only if*

$$\left|\frac{p\lambda^{'}\left(\omega\right)}{p\lambda\left(\omega\right)}+\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}.$$

For $p=1$*, this class was studied by Sun et al. [26]. For arbitrary fixed real numbers A and B* ($-1\leq B<A\leq1$)*, we denote by* $P(A,B)$ *the class of the functions of the form*

$$q\left(\omega\right)=1+c_{1}\omega+c_{2}\omega^{2}+...\quad,$$

which are analytic in the unit disk U and satisfy the condition

$$q\left(\omega\right)\prec\frac{1+A\omega}{1+B\omega}. \tag{8}$$

The class $P(A,B)$ *was introduced and studied by Janowski [13]. We also observe from (8) (see also [23]) that a function* $q(z)\in P(A,B)$ *if and only if*

$$\left|\,q\left(\omega\right)-\frac{1-AB}{1-B^{2}}\right|<\frac{A-B}{1-B^{2}},\quad\left(B\neq-1\right), \tag{9}$$

and

$$Req\left(\omega\right)>\frac{1-A}{2},\quad\left(B=-1\right), \tag{10}$$

In [16] Liu and Srivastava defined the following operator $M_{p}^{m}(a,b)$ *such that* $\sum_{p}$ *to (11) (see also [1]-[7] and [29], [30]).*

$$M_{p}^{m}(a,b)\lambda\left(\omega\right)=\frac{1}{\omega^{p}}+\sum_{t=n}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^{m}a_{t}\omega^{t}\quad\left(ab>0,\ p\in\mathbb{N}\right). \tag{11}$$

The above integral operator was studied by $M_{1}^{m}(a,b)$ *for* $p=1$.

$$M_{1}^{m}(a,b)\lambda\left(\omega\right)=\frac{1}{\omega}+\sum_{t=1}^{\infty}\left[\frac{a}{a+b(1+t)}\right]^{m}a_{t}\omega^{t}\quad\left(a>0,b\geq0,m\in\mathbb{N}\right). \tag{12}$$

It is easily verified from (12) that

$$\lambda\left(\omega\right)\left(M_{1}^{m}(a,b)\lambda\left(\omega\right)\right)^{'}=aM_{1}^{m}(a,b)\lambda\left(\omega\right)-(a+b)M_{1}^{m+1}(a,b)\lambda\left(\omega\right)\qquad(b>0).$$

Motivation from the above cited work we refer [3, 11, 19, 21]. Using the operator $M_p^m(a,b)$, *we introduce the following new class.*

Definition 1. *A function* $\lambda(\omega) \in \sum_p$ *is said to be in the class* $Q_p^m(\alpha, \beta, \eta; A_1, B)$, *if it satisfies the subordination*

$$\frac{p}{1-p\beta}\left\{\frac{(1-2\eta)\,\omega\left(M_p^m(a,b)\lambda(\omega)\right)' - \eta\omega^2\left(M_p^m(a,b)\lambda(\omega)\right)''}{(1-\eta)\,M_p^m(a,b)\lambda(\omega) - \eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'} + \beta\right\} \prec -\frac{1+A_1\omega}{1+B\omega}, \tag{13}$$

where $A_1 = (1-\alpha)A + \alpha B$, $0 \le \alpha < 1$, $0 \le \eta \le 1$, $-1 \le B < A \le 1$, $0 \le pB < 1$ *and* $\left(M_p^m(a,b)\lambda(\omega)\right)$ *is defined in (11).*

Remark 2. *Using (9), (10) and for* $B \neq -1$, *the Definition 1.2 is equivalent to*

$$\left|\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right\} + \frac{1-A_1B}{1-B^2}\right| < \frac{A_1-B}{1-B^2}, \tag{14}$$

and for $B = -1$,

$$\Re\left[\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right\}\right] < \frac{1-A_1}{2}, \tag{15}$$

also, for $B = -1$, $A_1 \neq 1$, *(15) reduces to*

$$\left|\frac{1-p\beta}{p}\left(\frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)' + \beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right) + \frac{1}{1-A_1}\right| < \frac{1}{1-A_1}, \tag{16}$$

and for $B = -1$, $A_1 = 1$, *we obtain*

$$\left|\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right\} + 1\right| < 1, \tag{17}$$

where

$$\chi_\eta(\omega) = (1-\eta)\,\lambda(\omega) - \eta\omega\lambda(\omega)' \tag{18}$$

In recent years, more and more researchers are interested in the reciprocal case of the starlike functions (see [9, 10, 14, 20, 25, 28]). In the present investigation, we give some sufficient conditions for the function belonging to the class $Q_p^m(\alpha, \beta, \eta; A_1, B)$. *In order to establish our main results, we need the following lemmas.*

2. A Set of Lemmas

To derive our main results, we need the following lemmas.

Lemma 1. *(Jack's lemma [12]) Let the (nonconstant) function $k(\omega)$ be analytic in U with $k(0)=0$. If $|k(\omega)|$ attains its maximum value on the circle $|\omega|=r<1$ at a point $\omega_0\in U$, then $\omega_0 k(\omega_0)' = \gamma k(\omega_0)$, where γ is a real number and $\gamma\geq 1$.*

Lemma 2. *[18] Let Ω be a set in the complex plane C and suppose that ϕ is a mapping from $C^2\times U$ to C which satisfies $\phi(ix;y;z)\notin\Omega$ for $\omega\in U$, and for all real x,y such that $y\leq -\frac{1+x^2}{2}$. If the function $p(\omega)=1+c_1\omega+c_2\omega^2+...$ is analytic in U and $\phi(p(\omega),\,\omega p'(\omega),\,\omega)\in\Omega$ for all $\omega\in U$, then $Re(p(\omega))>0$.*

Lemma 3. *[27] Let $p(\omega)=1+b_1\omega+b_2\omega^2+...$ be analytic in U and ϑ be analytic and starlike (with respect to the origin) univalent in U with $\vartheta(0)=0$. If $\omega p'(\omega)\prec\vartheta(\omega)$ then $p(\omega)\prec 1+\int_0^\omega \frac{\vartheta(t)}{t}dt$.*

Unless otherwise mentioned, we shall assume that $A_1=(1-\alpha)A+\alpha B$, $0\leq\alpha<1$, $0\leq\eta\leq 1$, $-1\leq B<A\leq 1$, $0\leq pB<1$ and $p\in N$.

3. Main Results

We begin by stating the following result.

Theorem 4. *Let $\lambda(\omega)\in\sum_p$. Then $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$ if and only if*

$$\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}\prec -\frac{1+A_1\omega}{1+B\omega}. \tag{19}$$

Proof. Let $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$, then it follows from definition that

$$\frac{p}{1-p\beta}\left\{\frac{(1-2\eta)\,\omega\left(M_p^m(a,b)\lambda(\omega)\right)'-\eta\omega^2\left(M_p^m(a,b)\lambda(\omega)\right)''}{(1-\eta)\,M_p^m(a,b)\lambda(\omega)-\eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'}+\beta\right\}\prec -\frac{1+A_1\omega}{1+B\omega}. \tag{20}$$

Let

$$\chi_\eta(\omega)=(1-\eta)\,\lambda(\omega)-\eta\omega\lambda(\omega)'.$$

Mulitiplying $M_p^m(a,b)$ both side

$$\left(M_p^m(a,b)\chi_\eta(\omega)\right)=(1-\eta)\left(M_p^m(a,b)\lambda(\omega)\right)-\eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'. \tag{21}$$

Differentiate equation (21) by ω,

$$\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)' = (1-2\eta)\,\omega\left(M_p^m(a,b)\lambda(\omega)\right)' - \eta\omega^2\left(M_p^m(a,b)\lambda(\omega)\right)''. \quad (22)$$

Using (21), (22), (20) and after some simplifications we get (19). The converse is straight forward.

Theorem 5. *Let $\lambda(\omega)\in\sum_p$. Then $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$, where $M_p^m(a,b)\lambda(\omega)$ is defined in (11), if the the following conditions are satisfied (i) for $B\neq -1$*

$$\sum_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|1-\eta+\eta t|\,\left|p\left(1-B^2\right)(A_1-B)(\beta+t)+(1-p\beta)(1-A_1)(1+B)\right|$$
$$< \ |1-\eta+\eta p|\,\left|(1-p\beta)(1+B)(A_1-1)-p\left(1-B^2\right)(A_1-B)(\beta-p)\right|,$$

(ii) for $B=-1$, $A_1\neq 1$

$$\sum_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|(1-p\beta)(1-A_1)(1-\eta t)| < |[2p(1-\beta)-(1-A_1)(1-p\beta)](1-\eta+\eta p)|,$$

(iii) for $B=-1$, $A_1=1$

$$\sum_{t=p}^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right| |a_t|\,|(1-\eta-\eta t)\,p\,(t+\beta)| < |(1-\eta+\eta p)\,p\,(p-\beta)|.$$

Proof. (i) For $B\neq -1$, then by the condition (14) we only need to show that

$$\left|\frac{p\left(1-B^2\right)}{(1-p\beta)(A_1-B)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+\frac{1-A_1B}{A_1-B}\right|<1. \quad (23)$$

We first observe the

$$\left|\frac{p\left(1-B^2\right)}{(1-p\beta)(A_1-B)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+\frac{1-A_1B}{A_1-B}\right|$$
$$= \left|\frac{p\left(1-B^2\right)(A_1-B)\begin{pmatrix}\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)' \\ +\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)\end{pmatrix}+(1-p\beta)(1-A_1B)\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)(1-p\beta)(A_1-B)}\right| \quad (24)$$

Using (21), (22) in (24), we get

$$\left| \frac{p\left(1-B^2\right)\left(A_1-B\right)\left(\begin{array}{c}(1-2\eta)\,\omega\left(M_p^m(a,b)\lambda(\omega)\right)' - \eta\omega^2\left(M_p^m(a,b)\lambda(\omega)\right)'' \\ +\beta\left((1-\eta)\left(M_p^m(a,b)\lambda(\omega)\right)-\eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'\right)\end{array}\right) + (1-p\beta)(1-A_1B)\left((1-\eta)\left(M_p^m(a,b)\lambda(\omega)\right)-\eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'\right)}{(1-p\beta)(A_1-B)\left((1-\eta)\left(M_p^m(a,b)\lambda(\omega)\right)-\eta\omega\left(M_p^m(a,b)\lambda(\omega)\right)'\right)} \right|$$

$$\leq \frac{\begin{array}{c}|1-\eta+\eta p|\left|p\left(1-B^2\right)(A_1-B)(\beta-p)+(1-p\beta)(1-A_1B)\right| \\ +\left|\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}(1-\eta+\eta t)\left(p\left(1-B^2\right)(A_1-B)(\beta+t)+(1-p\beta)(1-A_1B)\right)\right|\end{array}}{|1-\eta+\eta p|\,|(1-p\beta)(A_1-B)|+\left|\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}(1-\eta+\eta t)(1-p\beta)(A_1-B)\right|}$$

$$< \frac{\begin{array}{c}|1-\eta+\eta p|\left|p\left(1-B^2\right)(A_1-B)(\beta-p)+(1-p\beta)(1-A_1B)\right| \\ +\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|1-\eta+\eta t|\left|p\left(1-B^2\right)(A_1-B)(\beta+t)+(1-p\beta)(1-A_1B)\right|\end{array}}{|1-\eta+\eta p|\,|(1-p\beta)(A_1-B)|+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|1-\eta+\eta t|\,|(1-p\beta)(A_1-B)|}. \quad (25)$$

Now by using the inequality (23), we have

$$\frac{\begin{array}{c}|1-\eta+\eta p|\left|p\left(1-B^2\right)(A_1-B)(\beta-p)+(1-p\beta)(1-A_1B)\right| \\ +\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|1-\eta+\eta t|\left|p\left(1-B^2\right)(A_1-B)(\beta+t)+(1-p\beta)(1-A_1B)\right|\end{array}}{|1-\eta+\eta p|\,|(1-p\beta)(A_1-B)|+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\,|1-\eta+\eta t|\,|(1-p\beta)(A_1-B)|} < 1,$$

which, in conjunction with (25), completes the proof of (i) for Theorem 3.2.

(ii): If $B=-1$, $A_1 \neq 1$, by the virtue of the condition (16) we only need to show that

$$\left| \frac{(1-A_1)(1-p\beta)}{p}\left(\frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right)+1\right| < 1. \quad (26)$$

We first observe that

$$\left|\frac{(1-A_1)(1-p\beta)}{p}\left(\frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right)+1\right|$$

$$\leq \frac{\begin{array}{c}|1-\eta+\eta p|\,|(1-A_1)(1-p\beta)-p(1-\beta)| \\ +\left|\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}p\left[\left(1-\eta t+\eta t^2\right)+\beta(1-\eta+\eta t)\right]+(1-p\beta)(1-A_1)(1-\eta t)\right|\end{array}}{|p(1-\beta)(1-\eta+\eta p)|+\left|\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}p\left[(1-\eta t+\eta t^2)+\beta(1-\eta+\eta t)\right]\right|}$$

$$< \frac{\begin{array}{c}|1-\eta+\eta p|\,|(1-A_1)(1-p\beta)-p(1-\beta)| \\ +\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\left|p\left[\left(1-\eta t+\eta t^2\right)+\beta(1-\eta+\eta t)\right]\right|+|(1-p\beta)(1-A_1)(1-\eta t)|\end{array}}{|p(1-\beta)(1-\eta+\eta p)|+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m |a_t|\left|p\left[(1-\eta t+\eta t^2)+\beta(1-\eta+\eta t)\right]\right|} \tag{27}$$

By using the inequality (26), we have

$$\frac{\begin{array}{c}|1-\eta+\eta p|\,|(1-A_1)(1-p\beta)-p(1-\beta)| \\ +\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(n+t)}\right]^m |a_t|\left|p\left[\left(1-\eta t+\eta t^2\right)+\beta(1-\eta+\eta t)\right]\right|+|(1-p\beta)(1-A_1)(1-\eta t)|\end{array}}{|p(1-\beta)(1-\eta+\eta p)|+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(n+t)}\right]^m |a_t|\left|p\left[(1-\eta t+\eta t^2)+\beta(1-\eta+\eta t)\right]\right|}<1,$$

which, in conjunction with (27) completes the proof of (ii) for Theorem 3.2.

(iii): If $B=-1$, $A_1=1$, by virtue of the condition (17), we only need to show that

$$\left|\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+1\right|<1. \tag{28}$$

We first observe that

$$\left|\frac{p}{1-p\beta}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+1\right|$$

$$= \left|\frac{(1-\eta+\eta p)\left(1-p^2\right)+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}(1-\eta-\eta t)(1+pt)}{(1-p\beta)(1-\eta+\eta p)+\sum\limits_{t=p}^{\infty}\left[\frac{a}{a+b(p+t)}\right]^m a_t\omega^{t+p}(1-\eta-\eta t)(1-p\beta)}\right|$$

$$\leq \frac{|1-\eta+\eta p|\left|1-p^2\right|+\sum\limits_{t=p}^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t|\left|\omega^{t+p}\right||(1-\eta-\eta t)(1+pt)|}{|(1-p\beta)(1-\eta+\eta p)|+\sum\limits^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t|\left|\omega^{t+p}\right||(1-\eta-\eta t)(1-p\beta)|}$$

$$< \frac{|1-\eta+\eta p|\left|1-p^2\right|+\sum\limits_{t=p}^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t||(1-\eta-\eta t)(1+pt)|}{|(1-p\beta)(1-\eta+\eta p)|+\sum\limits^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t||(1-\eta-\eta t)(1-p\beta)|}. \tag{29}$$

Now by using the inequality (28) we have

$$\frac{|1-\eta+\eta p|\left|1-p^2\right|+\sum\limits_{t=p}^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t||(1-\eta-\eta t)(1+pt)|}{|(1-p\beta)(1-\eta+\eta p)|+\sum\limits^{\infty}\left|\left[\frac{a}{a+b(p+t)}\right]^m\right||a_t||(1-\eta-\eta t)(1-p\beta)|}<1.$$

which, in conjunction with (29) completes the proof of (iii) for Theorem 3.2.

Theorem 6. *If $\lambda(\omega)\in\sum_p$ satisfies any one of the following conditions*
(i) for $B\neq -1$

$$\left|\pounds_p^m(a,b)\chi_\eta(\omega)\right|<\frac{(1-p\beta)(A_1-B)}{(1-p\beta)(A_1-B)-1+|B|}, \tag{30}$$

(ii) for $B=-1$, $-1<A_1\leq 0$

$$\left|\pounds_p^m(a,b)\chi_\eta(\omega)\right|<\frac{(1-p\beta)(1-A_1)(1+A_1)}{2p\beta(1+A_1)+2(1-A_1)}, \tag{31}$$

(iii) for $B=-1$, $A_1=1$

$$\left|\pounds_p^m(a,b)\chi_\eta(\omega)\right|<\frac{(1-p\beta)}{2-p\beta}, \tag{32}$$

then $\lambda(\omega) \in Q_p^m(\alpha, \beta, \eta; A_1, B)$, *where*

$$\pounds_p^m(a,b)\chi_\eta(\omega) = 1 + \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)''}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'} - \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}.$$

Proof. (i) for $B \neq -1$. Let

$$k(\omega) = \frac{1 + \frac{1+|B|}{1+|B|+A_1-B}\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right)}{1 - \frac{1+|B|}{1+|B|+A_1-B}} - 1, \quad (\omega \in U), \tag{33}$$

then the function k is analytic in U with $k(0) = 0$. Using (33) and after some simplifications, we obtain

$$\frac{p\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} = \frac{(1-p\beta)(A_1-B)k(\omega) - 1 + |B|}{1+|B|}. \tag{34}$$

Differentiating both sides of (34) logarithmically we get

$$1 + \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)''}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'} - \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} = \frac{(1-p\beta)(A_1-B)\omega k'(\omega)}{(1-p\beta)(A_1-B)k(\omega) - 1 + |B|}. \tag{35}$$

By virtue of (30) and (35), we find that

$$\begin{aligned} &\left|1 + \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)''}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'} - \frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right| \\ = \;& (1-p\beta)(A_1-B)\left|\frac{\omega k'(\omega)}{(1-p\beta)(A_1-B)k(\omega) - 1 + |B|}\right|, \end{aligned}$$

and

$$\left|\pounds_p^m(a,b)\chi_\eta(\omega)\right| < \frac{(1-p\beta)(A_1-B)}{(1-p\beta)(A_1-B) - 1 + |B|}.$$

Next, we claim that $|k(\omega)| < 1$. Indeed if not there exists a point $\omega_0 \in U$ such that

$$\max_{|\omega|\leq|\omega_0|}|k(\omega)| = |k(\omega_0)| = 1, \quad \omega_0 \in U.$$

Applying Lemma 2.1 to $k(\omega)$ at the point ω_0, we have

$$\omega_0 k'(\omega_0) = \gamma k(\omega_0), \; (\gamma \geq 1).$$

By writing

$$k(\omega_0) = e^{i\theta}, \ (0 \leq \theta \leq 2\pi),$$

and setting $\omega = \omega_0$ in (35), we get

$$\left|\mathcal{L}_p^m(a,b)\chi_\eta(\omega_0)\right| = (1-p\beta)(A_1 - B)\left|\frac{\gamma}{(1-p\beta)(A_1-B) - (1+|B|)e^{-i\theta}}\right|,$$

which implies

$$\left|\mathcal{L}_p^m(a,b)\chi_\eta(\omega_0)\right| \geq (1-p\beta)(A_1 - B)\left|\frac{1}{(1-p\beta)(A_1-B) - (1+|B|)e^{-i\theta}}\right|.$$

This implies that

$$\left|\mathcal{L}_p^m(a,b)\chi_\eta(\omega_0)\right|^2 \geq \frac{[(1-p\beta)(A_1-B)]^2}{[(1-p\beta)(A_1-B)]^2 + (1+|B|)^2 - 2(1-p\beta)(A_1-B)(1+|B|)\cos\theta}. \tag{36}$$

Since the right hand side of (36) takes its minimum value for $\cos\theta = -1$, we have

$$\left|\mathcal{L}_p^m(a,b)\chi_\eta(\omega_0)\right|^2 \geq \frac{[(1-p\beta)(A_1-B)]^2}{[(1-p\beta)(A_1-B) + (1+|B|)]^2}.$$

This contradicts our condition (30) of Theorem 2.4. Therefore, we conclude that $|k(\omega)| < 1$, which shows that

$$\left|\frac{1 + \frac{1+|B|}{1+|B|+A_1-B}\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right)}{1 - \frac{1+|B|}{1+|B|+A_1-B}} - 1\right| < 1, \quad (B \neq -1, \ \omega \in U).$$

This implies that

$$\left|\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)} + \beta\right) + 1\right| < \frac{(A_1-B)}{(1+|B|)},$$

then, we have

$$\begin{aligned}
&\left|\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)+\frac{1-A_1B}{(1-B^2)}\right| \\
\leq\ & \left|\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)+1\right|+\left|\frac{1-A_1B}{1-B^2}-1\right| \\
<\ & \frac{A_1-B}{1+|B|}+\frac{|B|\,(A_1-B)}{1-B^2} \\
=\ & \frac{A_1-B}{1-B^2},\ \ (B\neq-1,\ \ \omega\in U).
\end{aligned}$$

Therefore, we conclude that $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$, for $B\neq-1$.

(ii) For $B=-1$, $-1<A_1\leq0$, analogously to Theorem 2.2 we let

$$k(\omega)=\frac{1+\frac{1-A_1}{2}\frac{1}{\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)}}{1-\frac{1-A_1}{2}}-1. \tag{37}$$

Working on the similar lines as in Theorem 3.3 in (i), we have

$$\left|\left(\frac{1-p\beta}{p}\right)\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+1\right|<\frac{2}{1-A_1}-1.$$

This implies that

$$\begin{aligned}
&\left|\left(\frac{1-p\beta}{p}\right)\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\frac{1}{1-A_1}\right| \\
\leq\ & \left|\left(\frac{1-p\beta}{p}\right)\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+1\right|+\left|\frac{1}{1-A_1}-1\right|, \\
<\ & \frac{2}{1-A_1}-1-\frac{1}{1-A_1}+1, \\
=\ & \frac{1}{1-A_1},\ \ (B=-1,-1<A_1\leq0,\ \ \omega\in U).
\end{aligned}$$

Therefore, we conclude that $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$ for $B=-1$, $-1<A_1\leq 0$.

(iii) For $B=-1$, $A_1=1$

$$k(\omega)=\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)+1. \tag{38}$$

Working on the similar lines as in Theorem 3.3 in (i), we have

$$\left|\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)+1\right|<1.$$

This implies that

$$\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)<-\frac{1+\omega}{1-\omega}.$$

Therefore, we conclude that $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$ for $B=-1$, $A_1=1$.

Theorem 7. *If* $\lambda(\omega)\in\sum_p$ *satisfies*

$$\Re\left(\pounds_p^m(a,b)\chi_\eta(\omega)\right)<\begin{cases}-\frac{(1-A_1)+p\beta(A_1-B)}{2(1-p\beta)(A_1-B)}, & \text{for } B+\frac{1-B}{2(1-p\beta)}\le A_1\le 1\\ -\frac{(1-p\beta)(A_1-B)}{2[(1-A_1)+p\beta(A_1-B)]}, & \text{for } B<A_1\le B+\frac{1-B}{2(1-p\beta)}\end{cases}, \tag{39}$$

then $\lambda(\omega)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$.

Proof. Suppose that

$$g(\omega)=\frac{-\frac{p}{1-p\beta}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right)-\frac{1-A_1}{1-B}}{1-\frac{1-A_1}{1-B}}-1,\quad(\omega\in U). \tag{40}$$

Then $g(\omega)$ is analytic in U. It follows from (40) that

$$\frac{-p\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}=\frac{(1-p\beta)(A_1-B)g(\omega)+(1-A_1)+p\beta(A_1-B)}{1-B}, \tag{41}$$

Differentiating (41) logarithmically, we obtain

$$-\pounds_p^m(a,b)\chi_\eta(\omega)=\frac{(1-p\beta)(A_1-B)g'(\omega)}{(1-p\beta)(A_1-B)g(\omega)+(1-A_1)+p\beta(A_1-B)}=\left(g(\omega),\ \omega g'(\omega),\omega\right),$$

where

$$\Phi(r,\ s,t)=\frac{(1-p\beta)(A_1-B)s}{(1-p\beta)(A_1-B)r+(1-A_1)+p\beta(A_1-B)}.$$

For all real x and y satisfying $y \leq -\frac{1+x^2}{2}$, we have

$$\begin{aligned}\Re\left(\Phi\left(ix,\ y,\omega\right)\right) &= \frac{\left(1-p\beta\right)\left(A_1-B\right)y}{i\left(1-p\beta\right)\left(A_1-B\right)x+\left(1-A_1\right)+p\beta\left(A_1-B\right)}\\ &\leq -\frac{1+x^2}{2}\ \frac{\left(1-p\beta\right)\left(A_1-B\right)\left[\left(1-A_1\right)+p\beta\left(A_1-B\right)\right]}{i\left[\left(1-p\beta\right)\left(A_1-B\right)\right]^2x+\left[\left(1-A_1\right)+p\beta\left(A_1-B\right)\right]^2}\\ &\leq \begin{cases} -\frac{(1-A_1)+p\beta(A_1-B)}{2(1-p\beta)(A_1-B)}, & \left(B+\frac{1-B}{2(1-p\beta)}\leq A_1\leq 1\right)\\ -\frac{(1-p\beta)(A_1-B)}{2[(1-A_1)+p\beta(A_1-B)]}, & \left(B<A_1\leq B+\frac{1-B}{2(1-p\beta)}\right)\end{cases}.\end{aligned}$$

We now put

$$\Omega=\left\{Re\left(\xi\right)\begin{cases} -\frac{(1-A_1)+p\beta(A_1-B)}{2(1-p\beta)(A_1-B)}, & \text{for } B+\frac{1-B}{2(1-p\beta)}\leq A_1\leq 1\\ -\frac{(1-p\beta)(A_1-B)}{2[(1-A_1)+p\beta(A_1-B)]}, & \text{for } B<A_1\leq B+\frac{1-B}{2(1-p\beta)}\end{cases}\right\},$$

then $\Phi\left(ix,\ y,\omega\right)\notin\Omega$ for all real x, y such that $y\leq -\frac{1+x^2}{2}$. Moreover, in view of (39), we know that $\Phi\left(g\left(\omega\right),\omega g'\left(\omega\right),\omega\right)\in\Omega$. Thus, by Lemma 2.2, we deduce that $Re\left(g\left(\omega\right)\right)>0$, which shows that the desired assertion of Theorem 3.4 holds.

Theorem 8. *If $\lambda\left(\omega\right)\in\sum_p$ satisfies any one of the following conditions*
(i) for $B\neq -1$

$$\left|\left\{\frac{p\left(1-B^2\right)}{\left(1-p\beta\right)\left(A_1-B\right)}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)'}{\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)}+\beta\right)+\frac{1-A_1B}{A_1-B}\right\}'\right|\leq\vartheta\left|\omega\right|^\tau, \quad (42)$$

(ii) for $B=-1,\ A_1\neq 1$

$$\left|\left\{1+\frac{\left(1-A_1\right)\left(1-p\beta\right)}{p}\left(\frac{\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)}{\omega\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)'+\beta\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)}\right)\right\}'\right|\leq\vartheta\left|\omega\right|^\tau, \quad (43)$$

(iii) for $B=-1,\ A_1=1$

$$\left|\left\{\frac{p}{\left(1-p\beta\right)}\left(\frac{\omega\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)'}{\left(M_p^m(a,b)\chi_\eta\left(\omega\right)\right)}+\beta\right)+1\right\}'\right|<1\leq\vartheta\left|\omega\right|^\tau, \quad (44)$$

then $\lambda\left(\omega\right)\in Q_p^m(\alpha,\beta,\eta;A_1,B)$, where $0<\vartheta\leq\tau+1$, $\tau\geq 0$.

Proof. (i) for $B \neq -1$,we define the function $\psi(\omega)$ by

$$\psi(\omega) = \omega\left[\frac{p\left(1-B^2\right)}{(1-p\beta)(A_1-B)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+\frac{1-A_1B}{A_1-B}\right],$$

then $\psi(\omega)$ is regular in U and $\psi(0)=0$. The condition of theorem gives us that

$$\left|\left(\frac{\psi(\omega)}{\omega}\right)'\right| \leq \vartheta\,|\omega|^{\tau}.$$

It follows that

$$\left|\left(\frac{\psi(\omega)}{\omega}\right)\right| = \left|\int_0^{\omega}\left(\frac{\psi(t)}{t}\right)' dt\right| \leq \int_0^{|\omega|}\vartheta\,|\omega|^{\tau}\,d\,|t| = \frac{\vartheta}{\tau+1}\,|\omega|^{\tau+1}.$$

This implies that

$$\left|\left(\frac{\psi(\omega)}{\omega}\right)\right| \leq \frac{\vartheta}{\tau+1}\,|\omega|^{\tau+1} < 1, \quad (0<\vartheta\leq\tau+1,\ \tau\geq 0).$$

Therefore, by the definition of $\psi(\omega)$, we conclude that

$$\left|\frac{p\left(1-B^2\right)}{(1-p\beta)(A_1-B)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+\frac{1-A_1B}{A_1-B}\right|<1,$$

which is equivalent to

$$\left|\frac{p}{(1-p\beta)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+\frac{1-A_1B}{A_1-B}\right|<\frac{(A_1-B)}{(1-B^2)}.$$

Therefore, we conclude that $\lambda(\omega) \in Q_p^m(\alpha,\beta,\eta;A_1,B)$.

(ii) for $B=-1$, $A_1 \neq 1$, we define the function

$$\psi(\omega) = \left[1+\frac{(1-A_1)(1-p\beta)}{p}\left\{\frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right\}\right].$$

Then $\psi(\omega)$ is regular in U and $\psi(0)=0$. Working on the similar lines as in Theorem 3.5 in (i) we can be easily verified.

$$\left|\frac{(1-p\beta)}{p}\left\{\frac{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'+\beta\left(M_p^m(a,b)\chi_\eta(\omega)\right)}\right\}+\frac{1}{1-A_1}\right|<\frac{1}{1-A_1}.$$

(iii) for $B=-1$, $A_1=1$

$$\psi(\omega)=\omega\left[\frac{p}{(1-p\beta)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+1\right],$$

Then $\psi(\omega)$ is regular in U and $\psi(0)=0$. Using similar arguments as in proof of (iii) can be easily get.

$$\left|\frac{p}{(1-p\beta)}\left\{\frac{\omega\left(M_p^m(a,b)\chi_\eta(\omega)\right)'}{\left(M_p^m(a,b)\chi_\eta(\omega)\right)}+\beta\right\}+1\right|<1.$$

This completes the proof.

Acknowledgements. he present investigation was supported by UKM under Grant no. GUP-2019-032 and the *Key Project of Education Department of Hunan Province* under Grant no. 19A097 of the P. R. China.

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