Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/
No. 68/2021
pp. 51-62
doi: 10.17114/j.aua.2021.68.05

APPLICATIONS OF QUASI-SUBORDINATION OF COEFFICIENTS ESTIMATES FOR NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS

W. G. Atshan, E. İ. Badawi, H. Ö. Güney

Abstract. The purpose of the present paper is to introduce and investigate a new subclasses of analytic and bi-univalent functions defined in the open unit disk,which are associated with the quasi-subordination. We obtain estimates on the initial coefficients $|a_2|$ and $|a_3|$ of functions in these subclasses. Also several known and new consequences of these results are pointed out.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: analytic function; univalent function; bi-univalent function; subordination; quasi-subordination.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of analytic functions defined on the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and normalized with

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad , \ z \in \mathbb{U}. \tag{1}$$

Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ consisting of form (1) which are univalent in $\mathbb{U}$. We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function w such that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$. If $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z),$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike function and convex function, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

From Koebe one quarter theorem [10], it is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4), \tag{2}$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ when both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ (see details in [12]). However, the familiar Koebe function is not bi-univalent. Lewin [7] investigated the class of bi-univalent functions Σ and obtained a bound $|a_2| \leqq 1.51$. Motivated by the work of Lewin [7], Brannan and Clunie [3] conjectured that $|a_2| \leqq \sqrt{2}$. Later Netanyahu [9] proved that $max|a_2| = \frac{4}{3}$ for $f \in \Sigma$. Brannan and Taha [3] also worked on certain subclasses of the bi-univalent function class Σ and obtained estimates for their initial coefficients. Various classes of bi-univalent functions were introduced and studied. In recent times, the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al.[12]. Motivated by this, many researchers (see [3, 4, 8, 12, 13, 14] also the references cited there in) recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients.

In 1970, the concept of quasi subordination was first defined by Robertson in [11]. Certain subclasses of bi-univalent functions associated with quasi-subordination were introduced and studied. [2, 5, 6].

For the functions f and φ, if there exists analytic functions h and w, with $|h(z)| \leq 1, w(0) = 0$ and $|w(z)| < 1$ such that the equality

$$f(z) = h(z)\varphi(w(z))$$

holds, then the function f is said to be quasi-subordinate to φ demonstrated by

$$f(z) \prec_q \varphi(z), \quad z \in \mathbb{U}. \tag{4}$$

Especially, prefering $h(z) \equiv 1$, the quasi-subordination given in (3) turns into the subordination $f(z) \prec \varphi(z)$. Thus, the quasi-subordination is a universality of the well known subordination and majorization (see [11]).

Ma and Minda have given a unified treatment of various subclass consisting of starlike and convex functions for either one of the quantities $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. The class $\mathcal{S}^*(\varphi)$ introduced by Ma and Minda [8] consists of starlike functions $f \in \mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathbb{U}$ and corresponding class $\mathcal{K}(\varphi)$ of convex functions $f \in \mathcal{A}$ satisfying $1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\ z \in \mathbb{U}$. For this purpose, they considered φ an analytic function with positive real part in the unit disc $\mathbb{U}$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with the respect to the real axis. The functions in the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are called starlike function of Ma-Minda type or convex function of Ma-Minda type respectively. By $\mathcal{S}^*_\Sigma(\varphi)$ and $K_\Sigma(\varphi)$, we denote to bi-starlike function of Ma-Minda type and bi-convex function of Ma-Minda type respectively [1].
In this investigation, we assume that

$$h(z) = A_0 + A_1 z + A_2 z^2 + \cdots, (|h(z)| \leq 1,\ z \in \mathbb{U}) \tag{5}$$

and

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, (B_1 > 0). \tag{6}$$

In order to derive our main results, we shall need the following lemma.

Lemma 1. *([10]) If $p \in \mathcal{P}$, then $|p_i| \leq 2$ for each i, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which*

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

In this paper, we will define three subclasses of the function class Σ by method of quasi-subordination and obtain the bounds for the modulus of initial coefficients of the functions in these classes. Some interesting results are also pointed out.

2. The subclass $\mathcal{M}^{\alpha}_{q,\Sigma}(\beta, \varphi)$

Definition 1. *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{M}^{\alpha}_{q,\Sigma}(\beta, \varphi)$ if the following quasi-subordination conditions are satisfied:*

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] - 1 \prec_q (\varphi(z) - 1) \quad , z \in \mathbb{U} \tag{7}$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]^{\beta}\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]-1\prec_q(\varphi(w)-1)\quad,w\in\mathbb{U}\quad(8)$$

where $0\le\alpha\le1$, $0\le\beta\le1$ *and* $g=f^{-1}$ *is given by (3).*

For $\beta=0$, we have the following subclass which was introduced and studied by Goyal and Kummar in [5]. Especially, the case $h(z)\equiv1$ was studied by Ali et.al in [1].

Remark 1. *A function* $f\in\Sigma$ *given by (1) is said to be in the class* $\mathcal{M}^{\alpha}_{q,\Sigma}(0,\varphi)=\mathcal{M}^{\alpha}_{q,\Sigma}(\varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-1\prec_q(\varphi(z)-1)\quad,z\in\mathbb{U}\qquad(9)$$

and

$$\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]-1\prec_q(\varphi(w)-1)\quad,w\in\mathbb{U}\qquad(10)$$

where $0\le\alpha\le1$ *and* $g=f^{-1}$ *is given by (3).*

For $\alpha=0$ and $\beta=0$, we have the following subclass which was introduced and studied by Brannan and Clunie et.al in [3].

Remark 2. *A function* $f\in\Sigma$ *given by (1) is said to be in the class* $\mathcal{M}^{0}_{q,\Sigma}(0,\varphi)=\mathcal{S}^{*}_{q,\Sigma}(\varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$\left[\frac{zf'(z)}{f(z)}\right]-1\prec_q(\varphi(z)-1)\quad,z\in\mathbb{U}\qquad(11)$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]-1\prec_q(\varphi(w)-1)\quad,w\in\mathbb{U}\qquad(12)$$

where $g=f^{-1}$ *is given by (3).*

Theorem 2. *If the function* f *belongs to the class* $\mathcal{M}^{\alpha}_{q,\Sigma}(\beta,\varphi)$, *then we have*

$$|a_2|\le min\left\{\frac{\sqrt{2}\sqrt{|A_0|(B_1+|B_2-B_1|)}}{\sqrt{\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)}},\right.$$
$$\left.\frac{\sqrt{2}|A_0|B_1\sqrt{B_1}}{\sqrt{\left|\left[\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)\right]A_0B_1^2-2(1+\alpha+\beta)^2(B_2-B_1)\right|}}\right\}\qquad(13)$$

and

$$|a_3| \leq min\left\{\frac{2|A_0|(B_1+|B_2-B_1|)}{\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)}+\frac{B_1(|A_0|+|A_1|)}{2(1+2\alpha+\beta)},\right.$$

$$\left.\frac{2A_0^2B_1^3}{\left|\left[\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)\right]A_0B_1^2-2(1+\alpha+\beta)^2(B_2-B_1)\right|}+\frac{B_1(|A_0|+|A_1|)}{2(1+2\alpha+\beta)}\right\} \tag{14}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ *and* $\varphi(z)$ *is given by (6).*

Proof. Let $f \in \mathcal{M}^{\alpha}_{q,\Sigma}(\beta,\varphi)$ and $g = f^{-1}$ given by (3). Then, there exists two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1$ and a function h defined by (5) satisfies

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-1 = h(z)(\varphi(u(z)-1)) \quad ,z\in\mathbb{U} \tag{15}$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]^{\beta}\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]-1 = h(z)(\varphi(v(w)-1)) \quad ,w\in\mathbb{U}. \tag{16}$$

Determine the functions $p(z)$ and $q(w)$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \ldots \tag{17}$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + \ldots. \tag{18}$$

Or equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right] \tag{19}$$

and

$$v(w) := \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[d_1 z + \left(d_2 - \frac{d_1^2}{2}\right)z^2 + \cdots\right]. \tag{20}$$

Using (19) and (20) in (15) and (16), respectively, we have

$$\left[\frac{zf'(z)}{f(z)}\right]^{\beta}\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]-1 = h(z)(\varphi\left(\frac{p(z)-1}{p(z)+1}\right)-1)) \quad ,z\in\mathbb{U} \tag{21}$$

and

$$\left[\frac{wg'(w)}{g(w)}\right]^{\beta}\left[(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)\right]-1=h(z)(\varphi\left(\frac{q(w)-1}{q(w)+1}\right)-1)) \quad ,w\in\mathbb{U}. \tag{22}$$

Using (5) and (6) in the right hands of the relations (21) and (22), we obtain

$$h(z)\left(\varphi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right)=\frac{1}{2}A_0B_1c_1z+\left\{\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\right\}z^2+\cdots \tag{23}$$

$$h(z)\left(\varphi\left(\frac{q(w)-1}{q(w)+1}\right)-1)\right)=\frac{1}{2}A_0B_1d_1w+\left\{\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2\right\}w^2+\cdots. \tag{24}$$

By equalizing (15), (16) and (24),respectively, we get

$$(1+\alpha+\beta)a_2=\frac{1}{2}A_0B_1c_1, \tag{25}$$

$$2(2\alpha+\beta+1)a_3+\left[\frac{1}{2}(\beta(\beta-1)+2\beta(1+\alpha))-(3\alpha+\beta+1)\right]a_2^2$$
$$=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2 \tag{26}$$

and

$$-(1+\alpha+\beta)a_2=\frac{1}{2}A_0B_1d_1, \tag{27}$$

$$\left[(5\alpha+3\beta+3)+\frac{1}{2}\left(\beta(\beta-1)+2\beta(1+\alpha)\right)\right]a_2^2-2(2\alpha+\beta+1)a_3$$
$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2. \tag{28}$$

From (25) and (27), we have

$$c_1=-d_1, \tag{29}$$

and

$$8(1+\alpha+\beta)^2a_2^2=A_0^2B_1^2((c_1^2+d_1^2)). \tag{30}$$

By summing (26) and (28) and using $|c_i|\leq 2, |d_i|\leq 2$, we obtain

$$|a_2|\leq\frac{\sqrt{2}\sqrt{|A_0|(B_1+|B_2-B_1|)}}{\sqrt{\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)}}. \tag{31}$$

Now, by summing (26) and (28) and using $|c_i|\leq 2, |d_i|\leq 2$ and (30)we obtain

$$|a_2|\leq\frac{\sqrt{2}|A_0|B_1\sqrt{B_1}}{\sqrt{\left|\left[\beta(\beta-1)+2\beta(1+\alpha)+2(1+\alpha+\beta)\right]A_0B_1^2-2(1+\alpha+\beta)^2(B_2-B_1)\right|}}. \tag{32}$$

From (31) and (32), we get the desired inequality (13). Next, for the bound on $|a_3|$, by subtracting (28) from (26), we obtain

$$a_3 = a_2^2 + \frac{2A_1B_1c_1 + A_0B_1(c_2 - d_2)}{8(1+2\alpha+\beta)}. \tag{33}$$

Using (31) with $|c_i| \leq 2$ and $|d_i| \leq 2$, we get

$$|a_3| \leq \frac{2|A_0|(B_1 + |B_2 - B_1|)}{\beta(\beta-1) + 2\beta(1+\alpha) + 2(1+\alpha+\beta)} + \frac{B_1(|A_0| + |A_1|)}{2(1+2\alpha+\beta)}. \tag{34}$$

Now, using (32) with $|c_i| \leq 2$ and $|d_i| \leq 2$, we get

$$|a_3| \leq \frac{2A_0^2B_1^3}{\left|[\beta(\beta-1) + 2\beta(1+\alpha) + 2(1+\alpha+\beta)]\, A_0B_1^2 - 2(1+\alpha+\beta)^2(B_2 - B_1)\right|} + \frac{B_1(|A_0| + |A_1|)}{2(1+2\alpha+\beta)}. \tag{35}$$

From (34) and (35), we get the desired inequality (14).

By putting $\beta = 0$ in the above theorem, we have the following corollary.

Corollary 3. *If the function f given by (1) belongs to the class $\mathcal{M}_{q,\Sigma}^{\alpha}(\varphi)$, then*

$$|a_2| \leq min\left\{\frac{\sqrt{2}\sqrt{|A_0|(B_1 + |B_2 - B_1|)}}{\sqrt{2(1+\alpha)}}, \frac{\sqrt{2}|A_0|B_1\sqrt{B_1}}{\sqrt{\left|[2(1+\alpha)]\, A_0B_1^2 - 2(1+\alpha)^2(B_2 - B_1)\right|}}\right\} \tag{36}$$

and

$$|a_3| \leq min\left\{\frac{2|A_0|(B_1 + |B_2 - B_1|)}{2(1+\alpha)} + \frac{B_1(|A_0| + |A_1|)}{2(2\alpha+\beta+1)}, \frac{2A_0^2B_1^3}{\left|[2(1+\alpha)]\, A_0B_1^2 - 2(1+\alpha)^2(B_2 - B_1)\right|} + \frac{B_1(|A_0| + |A_1|)}{2(2\alpha+\beta+1)}\right\} \tag{37}$$

where $0 \leq \alpha \leq 1$ and $\varphi(z)$ is given by (6).

By putting $\alpha = 0$ and $\beta = 0$ in the above theorem, we have the following corollary.

Corollary 4. *If the function f given by (1) belongs to the class $\mathcal{S}_{q,\Sigma}^{*}(\varphi)$, then*

$$|a_2| \leq min\left\{\sqrt{|A_0|(B_1 + |B_2 - B_1|)}, \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{\left|A_0B_1^2 - (B_2 - B_1)\right|}}\right\} \tag{38}$$

and

$$|a_3| \leq min\left\{|A_0|(B_1 + |B_2 - B_1|) + \frac{B_1(|A_0| + |A_1|)}{2}, \frac{A_0^2B_1^3}{|A_0B_1^2 - (B_2 - B_1)|} + \frac{B_1(|A_0| + |A_1|)}{2}\right\} \tag{39}$$

where $\varphi(z)$ is given by (6).

3. The subclass $\mathcal{S}^{\delta}_{q,\Sigma}(\gamma,\lambda,\varphi)$

Definition 2. *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}^{\delta}_{q,\Sigma}(\gamma,\lambda,\varphi)$ if the following quasi-subordination conditions are satisfied:*

$$\frac{1}{\gamma}\left[\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}+\delta z f''(z)-1\right] \prec_q (\varphi(z)-1) \quad , z\in\mathbb{U} \tag{40}$$

and

$$\frac{1}{\gamma}\left[\frac{wg'(w)}{(1-\lambda)w+\lambda g(w)}+\delta w g''(w)-1\right] \prec_q (\varphi(w)-1) \quad , w\in\mathbb{U} \tag{41}$$

where $0\le\lambda\le1$, $0\le\delta\le1$, $\gamma\in\mathbb{C}-\{0\}$ and $g=f^{-1}$ is given by (3).

For $\delta=0$, $\lambda=1$ and $\gamma=1$ we have the subclass $\mathcal{S}^{0}_{q,\Sigma}(1,1,\varphi)=\mathcal{S}^{*}_{q,\Sigma}(\varphi)$ given by Remark 2.

Theorem 5. *If the function f belongs to the class $\mathcal{S}^{\delta}_{q,\Sigma}(\gamma,\lambda,\varphi)$, then we have*

$$|a_2| \le min\left\{\frac{\sqrt{|\gamma||A_0|(B_1+|B_2-B_1|)}}{\sqrt{3(1+2\delta)+\lambda(\lambda-3)}},\right.$$
$$\left.\frac{|\gamma||A_0|B_1\sqrt{B_1}}{\sqrt{\left|[3(1+2\delta)+\lambda(\lambda-3)]\,\gamma A_0B_1^2-(2(1+\delta)-\lambda)^2(B_2-B_1)\right|}}\right\} \tag{42}$$

and

$$|a_3| \le min\left\{\frac{|\gamma||A_0|(B_1+|B_2-B_1|)}{3(1+2\delta)+\lambda(\lambda-3)}+\frac{|\gamma|B_1(|A_0|+|A_1|)}{3(1+2\delta)-\lambda},\right.$$
$$\left.\frac{|\gamma|A_0^2B_1^3}{\left|[3(1+2\delta)+\lambda(\lambda-3)]\,\gamma A_0B_1^2-(2(1+\delta)-\lambda)^2(B_2-B_1)\right|}+\frac{|\gamma|B_1(|A_0|+|A_1|)}{3(1+2\delta)-\lambda}\right\} \tag{43}$$

where $0\le\sigma\le1$, $\gamma\in\mathbb{C}-\{0\}$ and $\varphi(z)$ is given by (6).

Proof. Proceedings as in the proof of Theorem 2, we can get the relations as follows:

$$\frac{(2(1+\delta)-\lambda)}{\gamma}a_2=\frac{1}{2}A_0B_1c_1, \tag{44}$$

$$\frac{(3(1+2\delta)-\lambda)}{\gamma}a_3-\frac{\lambda(2-\lambda)}{\gamma}a_2^2=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2 \tag{45}$$

and

$$-\frac{(2(1+\delta)-\lambda)}{\gamma}a_2 = \frac{1}{2}A_0B_1d_1, \tag{46}$$

$$\frac{(6(1+2\delta)+\lambda(\lambda-4))}{\gamma}a_2^2 - \frac{(3(1+2\delta)-\lambda)}{\gamma}a_3 = \frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2}{4}d_1^2. \tag{47}$$

From (44) and (46), we have

$$c_1 = -d_1, \tag{48}$$

and

$$8(2(1+\delta)-\lambda)^2a_2^2 = A_0^2B_1^2\gamma^2((c_1^2+d_1^2)). \tag{49}$$

By summing (45) and (47) and using $|c_i| \leq 2, |d_i| \leq 2$, we obtain

$$|a_2| \leq \frac{\sqrt{|\gamma||A_0|(B_1+|B_2-B_1|)}}{\sqrt{3(1+2\delta)+\lambda(\lambda-3)}}. \tag{50}$$

Now, by summing (45) and (47) and using $|c_i| \leq 2, |d_i| \leq 2$ and (49), we obtain

$$|a_2| \leq \frac{|\gamma|^2|A_0|B_1\sqrt{B_1}}{\sqrt{\left|\left[3(1+2\delta)+\lambda(\lambda-3)\right]\gamma A_0B_1^2 - (2(1+\delta)-\lambda)^2(B_2-B_1)\right|}}. \tag{51}$$

From (50) and (51), we get the desired inequality (42). Next, for the bound on $|a_3|$, by subtracting (45) from (47), we obtain

$$a_3 = a_2^2 + \frac{\gamma\left(2A_1B_1c_1 + A_0B_1(c_2-d_2)\right)}{4(3(1+2\delta)-\lambda))}. \tag{52}$$

Using (50) with $|c_i| \leq 2$ and $|d_i| \leq 2$, we get

$$|a_3| \leq \frac{|\gamma||A_0|(B_1+|B_2-B_1|)}{3(1+2\delta)+\lambda(\lambda-3)} + \frac{|\gamma|B_1(|A_0|+|A_1|)}{3(1+2\delta)-\lambda}. \tag{53}$$

Now, using (51) with $|c_i| \leq 2$ and $|d_i| \leq 2$, we get

$$|a_3| \leq \frac{|\gamma|A_0^2B_1^3}{\left|\left[3(1+2\delta)+\lambda(\lambda-3)\right]\gamma A_0B_1^2 - (2(1+\delta)-\lambda)^2(B_2-B_1)\right|} + \frac{|\gamma|B_1(|A_0|+|A_1|)}{3(1+2\delta)-\lambda}. \tag{54}$$

From (53) and (54), we get the desired inequality (43).

For $\delta = 0$, $\lambda = 1$ and $\gamma = 1$ we have the subclass $\mathcal{S}^0_{q,\Sigma}(1,1,\varphi) = \mathcal{S}^*_{q,\Sigma}(\varphi)$ given by Corollary 4.

4. The subclass $\mathcal{H}^{\sigma}_{q,\Sigma}(\gamma,\varphi)$

Definition 3. *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{H}^{\sigma}_{q,\Sigma}(\gamma,\varphi)$ if the following quasi-subordination conditions are satisfied:*

$$\frac{1}{\gamma}\left[\frac{(1-\sigma)z^2f''(z)+zf'(z)}{(1-\sigma)zf'(z)+\sigma f(z)}-1\right] \prec_q (\varphi(z)-1) \quad , z \in \mathbb{U} \tag{55}$$

and

$$\frac{1}{\gamma}\left[\frac{(1-\sigma)w^2g''(w)+wg'(w)}{(1-\sigma)wg'(w)+\sigma g(w)}-1\right] \prec_q (\varphi(w)-1) \quad , w \in \mathbb{U} \tag{56}$$

where $0 \leq \sigma \leq 1$, $\gamma \in \mathbb{C}-\{0\}$ and $g = f^{-1}$ is given by (3).

For $\sigma = 0$ and $\gamma = 1$, we have the subclass $\mathcal{H}^{0}_{q,\Sigma}(1,\varphi) = \mathcal{S}^{*}_{q,\Sigma}(\varphi)$ given by Remark 2.

Theorem 6. *If the function f belongs to the class $\mathcal{H}^{\sigma}_{q,\Sigma}(\gamma,\varphi)$, then we have*

$$|a_2| \leq min\left\{\frac{\sqrt{|\gamma||A_0|(B_1+|B_2-B_1|)}}{\sqrt{|2(3-2\sigma)-(2-\sigma)^2|}},\right.$$
$$\left.\frac{|\gamma||A_0|B_1\sqrt{B_1}}{\sqrt{\left|\left[2(3-2\sigma)-(2-\sigma)^2\right]\gamma A_0 B_1^2-(2-\sigma)^2(B_2-B_1)\right|}}\right\} \tag{57}$$

and

$$|a_3| \leq min\left\{\frac{|\gamma||A_0|(B_1+|B_2-B_1|)}{2(3-2\sigma)-(2-\sigma)^2}+\frac{|\gamma|B_1(|A_0|+|A_1|)}{2(3-2\sigma)},\right.$$
$$\left.\frac{|\gamma|^2A_0^2B_1^3}{\left|\left[2(3-2\sigma)-(2-\sigma)^2\right]\gamma A_0B_1^2-(2-\sigma)^2(B_2-B_1)\right|}+\frac{|\gamma|B_1(|A_0|+|A_1|)}{2(3-2\sigma)}\right\} \tag{58}$$

where $0 \leq \sigma \leq 1$, $\gamma \in \mathbb{C}-\{0\}$ and $\varphi(z)$ is given by (6).

Proof. The proof of theorem is similar to above proofs.

For $\sigma = 0$ and $\gamma = 1$, we obtain the subclass $\mathcal{H}^{0}_{q,\Sigma}(1,\varphi) = \mathcal{S}^{*}_{q,\Sigma}(\varphi)$ given by Corollary 4.

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Waggas Galib Atshan
Department of Mathematics,,
College of Science,
University of Al-Qadisiyah,,
Diwaniyah, Iraq
email: *waggas.galib@qu.edu.iq*

Elaf İbrahim Badawi
Department of Mathematics,,
College of Science,
University of Al-Qadisiyah,,
Diwaniyah, Iraq
email: *eilafibraheem911994@gmail.com*

Hatun Özlem Güney
Department of Mathematics,
Faculty of Science,
Dicle University,
21280 Diyarbakır, Turkey
email: *ozlemg@dicle.edu.tr*