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DOUBLE FUZZY BASICALLY DISCONNECTED SPACES

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Abstract. In this paper we introduce and study a new class of double fuzzy topological space called (r, s)-fuzzy b-extremely disconnected space. Several characterizations and some interesting properties of these spaces are also given.

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1. Introduction

The concept of fuzzy sets was introduce by Zadeh [15]. Later on, Chang [2] introduced the concept of fuzzy topology, then the generalizations of the concept of fuzzy topology have been done by many authors. In [1], Atanassove introduced the idea of intuitionistic fuzzy sets, then Coker [3, 4], introduced the concept of intuitionistic fuzzy topological spaces. On the other hand, as a generalization of fuzzy topological spaces Samanta and Mondal [14], introduced the concept of intuitionistic gradation of openness. In 2005, the term intuitionistic is ended by Garcia and Rodabaugh [11]. They proved that the term intuitionistic is unsuitable in mathematics and applications and they replaced it by double. Many other topologies (see [7, 8, 9, 10, 12, 16]) studied various notions in double fuzzy fuzzy topological space. The purpose of this paper is to introduce a new class of double fuzzy topological space called (r, s)-fuzzy b-extremely disconnected space. Several characterizations and some interesting properties of these spaces are also given.

2. Preliminaries

Throughout this paper, Let X be a non-empty set, I the unit interval [0, 1], $I_0 = (0, 1]$ and $I_1 = [0, 1)$. The family of all fuzzy sets on X is denoted by I^X. By $\overline{0}$ and $\overline{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set

$\lambda \in I^X, \bar{1}-\lambda$ denotes its complement. Given a function $f : I^X \longrightarrow I^Y$ and its inverse $f^{-1} : I^Y \longrightarrow I^X$ are defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\mu)(x) = \mu(f(x))$, for each $\lambda \in I^X, \mu \in I^Y$ and $x \in X$, respectively. All other notations are standard notations of fuzzy set theory.

Definition 1. *[4, 14] A double fuzzy topology on X is a pair of maps $\tau, \tau^\star : I^X \to I$, which satisfies the following properties:*

1. *$\tau(\lambda) \leq \underline{1} - \tau^\star(\lambda)$ for each $\lambda \in I^X$.*
2. *$\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^X$.*
3. *$\tau(\bigvee_{i\in\Gamma} \lambda_i) \geq \bigwedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^\star(\bigvee_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Gamma$.*

The triplet $(X, \tau, \tau^\star)$ is called a double fuzzy topological space.

Definition 2. *[4, 14] A fuzzy set λ is called an (r, s)-fuzzy open if $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$, λ is called an (r, s)-fuzzy closed if, and only if $\underline{1} - \lambda$ is an (r, s)-fuzzy open set.*

Definition 3. *[4, 14] A function $f : (X, \tau_1, \tau_1^\star) \to (Y, \tau_2, \tau_2^\star)$ is said to be a double fuzzy continuous if, and only if $\tau_1(f^{-1}(\nu)) \geq \tau_2(\nu)$ and $\tau_1^\star(f^{-1}(\nu)) \leq \tau_2^\star(\nu)$ for each $\nu \in I^Y$.*

Theorem 1. *[13, 5] Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. Then the double fuzzy closure operator and the double fuzzy interior operator of $\lambda \in I^X$ are defined by $C_{\tau,\tau^\star}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\underline{1} - \mu) \geq r, \tau^\star(\underline{1} - \mu) \leq s\}$, $I_{\tau,\tau^\star}(\lambda, r, s) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq s\}$, where $r \in I_0$ and $s \in I_1$ such that $r + s \leq 1$.*

Definition 4. *[6] Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. For each $\lambda, \mu \in I^X, r \in I_0$ and $s \in I_1$,*

1. *λ is called (r, s)-fuzzy b-open if $\lambda \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda, r, s), r, s) \vee C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda, r, s), r, s)$.*
2. *λ is called an (r, s)-fuzzy b-closed set if $\underline{1} - \lambda$ is an (r, s)-fuzzy b-open set.*
3. *An (r, s)-fuzzy b-closure of λ is defined by $BC_{\tau,\tau^\star}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu$ and μ is (r, s)-fuzzy b-closed$\}$.*
4. *An (r, s)-fuzzy b-interior of λ is defined by $BI_{\tau,\tau^\star}(\lambda, r, s) = \vee\{\mu \in I^X \mid \lambda \leq \mu$ and μ is (r, s)-fuzzy b-closed$\}$.*

3. Properties of (r,s)-fuzzy b-extremely disconnected spaces

Definition 5. *A double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be (r,s)-fuzzy b-extremely disconnected if $BC_{\tau,\tau^\star}(\lambda,r,s)$ is (r,s)-fuzzy b-open for every (r,s)-fuzzy b-open set λ of $(X,\tau,\tau^\star)$.*

Example 1. *Let $I=[0,1], X=\{a,b\}$. The fuzzy subset λ is defined as $\lambda(a)=0.5, \lambda(b)=0.5$. Let $\tau,\tau^\star : I^X \to I$ be defined as follows:*

$$\tau(\alpha)=\begin{cases}\overline{1} & \text{if } \alpha=\underline{0} \text{ or } \underline{1}\\ \frac{1}{2} & \text{if } \alpha=\lambda\\ \overline{0} & \text{otherwise}\end{cases} \qquad \tau^\star(\alpha)=\begin{cases}\overline{0}, & \text{if } \alpha=\underline{0} \text{ or } \underline{1}\\ \frac{1}{2}, & \text{if } \alpha=\lambda\\ \overline{1}, & \text{otherwise}\end{cases}$$

Clearly, $(X,\tau,\tau^\star)$ is an (r,s)-fuzzy b-extremely disconnected space.

Proposition 1. *For a double fuzzy topological space $(X,\tau,\tau^\star)$, the following statements are equivalent:*

1. *$(X,\tau,\tau^\star)$ is an (r,s)-fuzzy b-extremely disconnected space.*
2. *For each (r,s)-fuzzy b-closed set λ, $BI_{\tau,\tau^\star}(\lambda,r,s)$ is (r,s)-fuzzy b-closed.*
3. *For each (r,s)-fuzzy b-open set λ, $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\overline{1}-BC_{\tau_1,\tau_1^\star}(\lambda,r,s))=\overline{1}$.*
4. *For every pair of (r,s)-fuzzy b-open sets λ and μ such that $BC_{\tau,\tau^\star}(\lambda,r,s)+\mu=\overline{1}$, $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\mu,r,s)=\overline{1}$.*

Proof. (1)⇒(2): Let λ be any (r,s)-fuzzy b-closed set. Then $\overline{1}-\lambda$ is (r,s)-fuzzy b-open. Now $BC_{\tau,\tau^\star}(\overline{1}-\lambda,r,s)=\overline{1}-BI_{\tau,\tau^\star}(\lambda,r,s)$, By (1), $BC_{\tau,\tau^\star}(\overline{1}-\lambda,r,s)$ is (r,s)-fuzzy b-open, which implies that $BI_{\tau,\tau^\star}(\lambda,r,s)$ is (r,s)-fuzzy b-closed.
(2)⇒(3): Let λ be any (r,s)-fuzzy b-open set. Then $\overline{1}-\lambda$ is (r,s)-fuzzy b-closed. By (2), we have $BI_{\tau,\tau^\star}(\overline{1}-\lambda,r,s)$ is (r,s)-fuzzy b-closed. Now $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s),r,s)=BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(BI_{\tau,\tau^\star}(\overline{1}-\lambda,r,s),r,s)(*)$. Therefore by $(*)$, $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s),r,s)=BC_{\tau,\tau^\star}(\lambda,r,s)+BI_{\tau,\tau^\star}(\overline{1}-\lambda,r,s)=BC_{\tau,\tau^\star}(\lambda,r,s)+\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s)$. Hence $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s),r,s)=\overline{1}$.
(3)⇒(4): Let λ and μ be (r,s)-fuzzy b-open sets with $BC_{\tau,\tau^\star}(\lambda,r,s)+\mu=\overline{1}(**)$. By (3), $\overline{1}=BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s),r,s)$. By $(**)$, $\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s)=\mu$. Then $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\mu,r,s)=\overline{1}$.
(4)⇒(1): Let λ be any (r,s)-fuzzy b-open set. Put $\mu=\overline{1}-BC_{\tau,\tau^\star}(\lambda,r,s)$. Then clearly μ is (r,s)-fuzzy b-open and $BC_{\tau,\tau^\star}(\lambda,r,s)+\mu=\overline{1}$. Therefore by (4), $BC_{\tau,\tau^\star}(\lambda,r,s)+BC_{\tau,\tau^\star}(\mu,r,s)=\overline{1}$. Then $BC_{\tau,\tau^\star}(\lambda,r,s)$ is (r,s)-fuzzy b-open and so $(X,\tau,\tau^\star)$ is (r,s)-fuzzy b-extremely disconnected.

Proposition 2. *A double fuzzy topological space $(X, \tau, \tau^\star)$ is (r, s)-fuzzy b-extremely disconnected if and only if for all (r, s)-fuzzy b-open set λ and an (r, s)-fuzzy b-closed set μ such that $\lambda \leq \mu$, $BC_{\tau,\tau^\star}(\lambda, r, s) \leq BI_{\tau,\tau^\star}(\mu, r, s)$.*

Proof. Let $(X, \tau, \tau^\star)$ be an (r, s)-fuzzy b-extremely disconnected space. Let λ be a fuzzy (r, s)-fuzzy b-open and μ be (r, s)-fuzzy b-closed with $\lambda \leq \mu$. Then by (2) of Proposition 1, $BI_{\tau,\tau^\star}(\mu, r, s)$ is an (r, s)-fuzzy b-closed set. Also since λ is (r, s)-fuzzy b-open and $\lambda \leq \mu$, $\lambda \leq BI_{\tau,\tau^\star}(\mu, r, s)$. Since $BI_{\tau,\tau^\star}(\mu, r, s)$ is (r, s)-fuzzy b-closed, we have $BC_{\tau,\tau^\star}(\lambda, r, s) \leq BI_{\tau,\tau^\star}(\mu, r, s)$. Conversely let μ be any (r, s)-fuzzy b-closed set. Then $BI_{\tau,\tau^*}(\mu, r, s)$ is (r, s)-fuzzy b-open in (X, τ, τ^*) and $BI_{\tau,\tau^*}(\mu, r, s) \leq \mu$. Then $BC_{\tau,\tau^*}(BI_{\tau,\tau^*}(\mu, r, s), r, s) \leq BI_{\tau,\tau^*}(\mu, r, s)$. This implies that $BI_{\tau,\tau^*}(\mu, r, s)$ is (r, s)-fuzzy b-closed. Hence by (2) of Proposition 1, (X, τ, τ^*) is (r, s)-fuzzy b-extremely disconnected.

Remark 1. *Let $(X, \tau, \tau^\star)$ be an (r, s)-fuzzy b-extremely disconnected space. Let $\{\lambda_i, \overline{1} - \mu_i : i \in N\}$ be a collection such that λ_i's are (r, s)-fuzzy b-open and μ_i's are (r, s)-fuzzy b-closed and let λ, μ be (r, s)-fuzzy b-clopen sets, respectively. If $\lambda_i \leq \lambda \leq \mu_j$ and $\lambda_i \leq \mu \leq \mu_j$ for all $i, j \in \mathbb{N}$, then there exists an (r, s)-fuzzy b-clopen set γ such that $BC_{\tau,\tau^\star}(\lambda_i, r, s) \leq \gamma \leq BI_{\tau,\tau^\star}(\mu_j, r, s)$ for all $i, j \in \mathbb{N}$.*

Proof. By Proposition 1, we have $BC_{\tau,\tau^\star}(\lambda_i, r, s) \leq BC_{\tau,\tau^\star}(\lambda, r, s) \wedge BI_{\tau,\tau^\star}(\lambda, r, s) \leq BI_{\tau,\tau^\star}(\mu_j, r, s)$ for all $i, j \in \mathbb{N}$. So $\gamma = BC_{\tau,\tau^\star}(\lambda, r, s) \wedge BI_{\tau,\tau^\star}(\lambda, r, s)$ is an (r, s)-fuzzy b-clopen set satisfying the required conditions.

Proposition 3. *Let $(X, \tau, \tau^\star)$ be an (r, s)-fuzzy b-extremally disconnected space. Let $\{\lambda_q\}_{q\in Q}$ and $\{\mu_q\}_{q\in Q}$ be monotone increasing collections of fuzzy (r, s)-fuzzy b-open sets and (r, s)-fuzzy b-closed sets of $(X, \tau, \tau^\star)$, respectively and suppose that $\lambda_{q_1} \leq \mu_{q_2}$ whenever $q_1 < q_2$, where Q denoted the set of rational numbers. Then there exists a monotone increasing collection $\{\eta_q\}_{q\in Q}$ of (r, s)-fuzzy b-clopen sets of $(X, \tau, \tau^\star)$ such that $BC_{\tau,\tau^\star}(\lambda_{q_1}, r, s) \leq \theta_{q_2}$ and $\theta_{q_1} \leq BI_{\tau,\tau^\star}(\mu_{q_2}, r, s)$ whenever $q_1 < q_2$.*

Proof. Let us arrange into a sequence $\{q_n\}$ of all rational numbers without repetition. For every $n \geq 2$ we shall define a collection $\{\eta_{q_i} : 1 \leq i < n\} \subset I^X$ such that (A_n) $BC_{\tau,\tau^\star}(\lambda_q, r, s) \leq \theta_{q_i}$ if $q < q_i$, $\theta_{q_i} \leq BI_{\tau,\tau^\star}(\mu_q, r, s)$ if $q_i < q$ for all $i < n$. It is clear that the countable collections $\{BC_{\tau,\tau^\star}(\lambda_q, r, s)\}$ and $\{BI_{\tau,\tau^\star}(\mu_q, r, s)\}$ satisfying $BC_{\tau,\tau^\star}(\lambda_{q_1}, r, s) \leq BI_{\tau,\tau^\star}(\mu_{q_2}, r, s)$ if $q_1 < q_2$. Then there exists an (r, s)-fuzzy b-clopen set δ_1 such that $BC_{\tau,\tau^\star}(\lambda_{q_1}, r, s) \leq \delta_1 \leq BI_{\tau,\tau^\star}(\mu_{q_2}, r, s)$. Setting $\eta_{q_1} = \delta_1$, we get (A_2). Assume that fuzzy subsets η_{q_i} are already defined for $i < n$ and satisfy (A_n). Define $\sum = \vee\{\eta_{q_i} : i < n, q_i < q_n\} \vee \lambda_{r_n}$ and $\Phi = \wedge\{\eta_{q_j} : j < n, q_j > q_n\} \wedge \mu_{q_n}$. Then we have $BC_{\tau,\tau^\star}(\eta_{q_i}, r, s) \leq BC_{\tau,\tau^\star}(\sum, r, s) \leq BI_{\tau,\tau^\star}(\eta_{q_j}, r, s)$ and $BC_{\tau,\tau^\star}(\eta_{q_i}, r, s) \leq BI_{\tau,\tau^\star}(\Phi, r, s) \leq BI_{\tau,\tau^\star}(\eta_{q_j}, r, s)$ whenever $q_i < q_n < q_j$ $(i, j < n)$ as well as $\lambda_q \leq BC_{\tau,\tau^\star}(\sum, r, s) \leq \mu_{q'}$ and $\lambda_q \leq BI_{\tau,\tau^\star}(\Phi, r, s) \leq \mu_{q'}$. This shows that the countable collections $\{\eta_{q_i} : i < n, q_i < q_n\} \cup \{\lambda_q : q < q_n\}$ and $\{\eta_{q_j} : j < n, q_j > q_n\} \cup \{\mu_q : q > q_n\}$ together with $\sum$ and Φ satisfy all hypotheses of Remark 1. Hence there exists an (r, s)fuzzy b-clopen set δ_n such that $BC_{\tau,\tau^\star}(\delta_n, r, s) \leq \mu_q$ if $q_n < q$, $\lambda_q \leq BI_{\tau,\tau^\star}(\delta_n, r, s)$ if $q < q_n$, $BC_{\tau,\tau^\star}(\eta_{q_i}, r, s) \leq BI_{\tau,\tau^\star}(\delta_n, r, s)$ if $q_i < q_n$, $BC_{\tau,\tau^\star}(\delta_n, r, s) \leq BI_{\tau,\tau^\star}(\eta_{q_j}, r, s)$ if $q_n < q_j$, where $1 \leq i, j \leq n-1$. Letting $\eta_{q_n} = \delta_n$ we obtain the fuzzy sets $\eta_{q_1}, \eta_{q_2} ... \eta_{q_n}$ that satisfy (A_{n+1}). Therefore, the collection $\{\eta_{q_i} : i = 1, 2, ...\}$ has required property.

Definition 6. *A function $f : (X, \tau_1, \tau_1^\star) \to (Y, \tau_2, \tau_2^\star)$ is called*

1. *(r, s)-fuzzy b-irresolute if $f^{-1}(\lambda)$ is (r, s)-fuzzy b-open set of $(X, \tau, \tau^\star)$ for every (r, s)-fuzzy b-open set λ of $(Y, \sigma, \sigma^\star)$.*

2. *(r, s)-fuzzy b-open if $f(\lambda)$ is (r, s)-fuzzy b-open set of $(Y, \sigma, \sigma^\star)$ for every (r, s)-fuzzy b-open set λ of $(X, \tau_1, \tau_1^\star)$.*

Proposition 4. *Let $(X, \tau_1, \tau_1^\star)$ and $(Y, \tau_2, \tau_2^\star)$ be any two double fuzzy topological spaces. A function $f : (X, \tau_1, \tau_1^\star) \to (Y, \tau_2, \tau_2^\star)$ is (r, s)-fuzzy b-irresolute if, and only if $f(BC_{\tau_1,\tau_1^\star}(\lambda, r, s)) \leq BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s)$ for every fuzzy set λ in $(Y, \tau_2, \tau_2^\star)$, $r \in I_0$, $s \in I_1$.*

Proof. Let λ be any fuzzy set in I^X and f be an (r, s)-fuzzy b-irresolute function such that $r \in I_0$, $s \in I_1$. Then $BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s)$ is an (r, s)-fuzzy b-closed set in I^Y. Since f is an (r, s)-fuzzy b-irresolute function, $f^{-1}(BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s))$ is an (r, s)-fuzzy b-closed set in I^X. We have $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s))$. Also, by the definition of (r, s)-fuzzy b-closure, $BC_{\tau_1,\tau_1^\star}(\lambda, r, s) \leq f^{-1}(BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s))$, that is, $f(BC_{\tau_1,\tau_1^\star}(\lambda, r, s)) \leq BC_{\tau_2,\tau_2^\star}(f(\lambda), r, s)$. Conversely, let λ be an (r, s)-fuzzy b-closed set in I^Y such that $f(BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda), r, s)) \leq BC_{\tau_1,\tau_1^\star}(f(f^{-1}(\lambda)), r, s)$. Then $BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda), r, s) \leq f^{-1}(\lambda)$. So that $f^{-1}(\lambda) = BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda), r, s)$. That is, $f^{-1}(\lambda, r, s)$ is an (r, s)-fuzzy b-closed and hence, f is (r, s)-fuzzy b-irresolute function.

Proposition 5. *Let $(X, \tau_1, \tau_1^\star)$ and $(Y, \tau_2, \tau_2^\star)$ be two double fuzzy topological spaces and $f : (X, \tau_1, \tau_1^\star) \to (Y, \tau_2, \tau_2^\star)$ be an (r, s)-fuzzy b-open surjective function. Then $f^{-1}(BC_{\tau_2,\tau_2^\star}(\lambda, r, s)) \leq BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda), r, s)$ for every fuzzy set λ in $(Y, \tau_2, \tau_2^\star)$, $r \in I_0$, $s \in I_1$.*

Proof. Let $\lambda \in I^Y$, $r \in I_0$, $s \in I_1$ such that $\mu = f^{-1}(1 - \lambda)$. Then $BI_{\tau_1,\tau_1^\star}(f^{-1}(\overline{1} - \lambda), r, s) = BI_{\tau_1,\tau_1^\star}(\mu, r, s)$ is (r, s)-fuzzy b-open in I^X. But $BI_{\tau_1,\tau_1^\star}(\mu, r, s) \leq \mu$, hence $f(BI_{\tau_1,\tau_1^\star}(\mu, r, s)) \leq f(\mu)$, that is, $BI_{\tau_2,\tau_2^\star}(f(BI_{\tau_1,\tau_1^\star}(\mu, r, s)), r, s) \leq BI_{\tau_2,\tau_2^\star}(f(\mu), r, s)$. Since f is (r, s)-fuzzy b-open, $f(BI_{\tau_1,\tau_1^\star}(\mu, r, s))$ is an (r, s)-fuzzy b-open in I^Y,$r \in I_0$, $s \in I_1$. Therefore, $f(BI_{\tau_1,\tau_1^\star}(\mu, r, s)) \leq BI_{\tau_2,\tau_2^\star}(f(\mu), r, s) = BI_{\tau_2,\tau_2^\star}(\overline{1} - \lambda, r, s)$.

Hence, $BI_{\tau_1,\tau_1^\star}(f^{-1}(\overline{1}-\lambda),r,s) = BI_{\tau_1,\tau_1^\star}(\mu,r,s) \leq f^{-1}(BI_{\tau_2,\tau_2^\star}(\overline{1}-\lambda),r,s)$. Therefore, $\overline{1}-BI_{\tau_1,\tau_1^\star}(f^{-1}(\overline{1}-\lambda),r,s) = \overline{1}-BI_{\tau_1,\tau_1^\star}(\mu,r,s) \geq \overline{1}-f^{-1}(BI_{\tau_2,\tau_2^\star}(\overline{1}-\lambda,r,s))$. Hence, $f^{-1}(\overline{1}-BI_{\tau_2,\tau_2^\star}(\overline{1}-\lambda,r,s) \leq BC_{\tau_1,\tau_1^\star}(\overline{1}-f^{-1}(\overline{1}-\lambda),r,s)$. Therefore, $f^{-1}(BC_{\tau_2,\tau_2^\star}(\lambda,r,s)) \leq BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s)$.

Proposition 6. *The image $(Y,\tau_2,\tau_2^\star)$ of an (r,s)-fuzzy b-extremely disconnected space $(X,\tau_1,\tau_1^\star)$ under (r,s)-fuzzy b-irresolute (r,s)-fuzzy b-open surjective mapping is also (r,s)-fuzzy b-extremely disconnected.*

Proof. Let $\lambda \in I^Y$ be an (r,s)-fuzzy b-open fuzzy set, $r \in I_0$, $s \in I_1$ such that f is an (r,s)-fuzzy b-irresolute function, so $f^{-1}(\lambda)$ is an (r,s)-fuzzy b-open set in I^X. But $(X,\tau_1,\tau_1^\star)$ is (r,s)-fuzzy b-extremely disconnected, $BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s)$ is an (r,s)-fuzzy b-open set in I^X. Also, f is (r,s)-fuzzy b-open surjective, $BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s))$ is (r,s)-fuzzy b-open in I^Y. Then $f^{-1}(BC_{\tau_2,\tau_2^\star}(\lambda,r,s)) \leq BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s)$ and $f^{-1}(BC_{\tau_2,\tau_2^\star}(\lambda,r,s)) = BC_{\tau_2,\tau_2^\star}(\lambda,r,s) \leq f^{-1}(BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s)) \leq$ $BC_{\tau_2,\tau_2^\star}(f(f^{-1}(\lambda),r,s)) = BC_{\tau_2,\tau_2^\star}(\lambda,r,s)$; $BC_{\tau_2,\tau_2^\star}(\lambda,r,s) = f(BC_{\tau_1,\tau_1^\star}(f^{-1}(\lambda),r,s))$. Then $BC_{\tau_2,\tau_2^\star}(\lambda,r,s)$ is an (r,s)-fuzzy b-open set in I^Y which implies $(Y,\tau_2,\tau_2^\star)$ is an (r,s)-fuzzy b-extremely disconnected.

Definition 7. *Let $(X,\tau,\tau^\star)$ be a double fuzzy topological space. A mapping $f: X \to R(L)$ is called lower (resp. upper) (r,s)-fuzzy b-continuous if $f^{-1}(R_t)$ (resp. $f^{-1}(L_t)$) is (r,s)-fuzzy b-open (resp. L-fuzzzy b-closed) for each $t \in R$.*

Proposition 7. *Let $(X,\tau,\tau^\star)$ be any double fuzzy topological space; let $\lambda \in L^X$ and let $f: X \to R(L)$ be such that*

$$f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda(x) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

for all $x \in X$. Then f is lower (resp. upper) (r,s)-fuzzy b-continuous if and only if λ is (r,s)-fuzzy b-open (resp. (r,s)-fuzzy b-closed).

Proof. It suffices to observe that

$$f^{-1}(R_t) = \begin{cases} 1 & t < 0 \\ \lambda & 0 \leq t < 1 \\ 0 & t \geq 1. \end{cases}$$

implies that f is lower (r,s)-fuzzy b-continuous if and only if λ is (r,s)-fuzzy b-open.

$$f^{-1}(L_t') = \begin{cases} 1 & t \leq 0 \\ \lambda & 0 < t \leq 1 \\ 0 & t > 1. \end{cases}$$

implies that f is upper (r,s)-fuzzy b-continuous if and only if λ is (r,s)-fuzzy b-closed.

Definition 8. *The characteristic function of* $\lambda \in I^X$ *is the map* $\chi_\lambda : X \to I(L)$ *defined by* $\chi_\lambda(x) = (\lambda(x)), x \in X$.

Proposition 8. *Let* $(X,\tau,\tau^\star)$ *be a double fuzzy topological space and let* $\lambda \in I^X$. *Then* χ_λ *is lower (resp. upper)* (r,s)*-fuzzy* b*-continuous if and only if* λ *is* (r,s)*-fuzzy* b*-open (resp.* (r,s)*-fuzzy* b*-closed).*

Proof. The proof follows from Propositon 7.

Definition 9. *Let* $(X,\tau,\tau^\star)$ *and* $(Y,\sigma,\sigma^\star)$ *be any two smooth fuzzy topological spaces. A mapping* $f : (X,\tau_1,\tau_1^\star) \to (Y,\tau_2,\tau_2^\star)$ *is called strong* (r,s)*-fuzzy* b*-continuous if* $f^{-1}(\lambda)$ *is* (r,s)*-fuzzy* b*-clopen set of* $(X,\tau,\tau^\star)$ *for every* (r,s)*-fuzzy* b*-open set* λ *of* $(Y,\sigma,\sigma^\star)$.

Proposition 9. *Let* $(X,\tau,\tau^\star)$ *be a double fuzzy topological space. Then the following statements are equivalent:*

1. $(X,\tau,\tau^\star)$ *is an* (r,s)*-fuzzy* b*-extremely disconnected space.*

2. *If* $g,h : X \to R(L)$ *where* g *is lower* (r,s)*-fuzzy* b*-continuous,* h *is upper* (r,s)*-fuzzy* b*-continuous, then there exits a strong* (r,s)*-fuzzy* b*-continuous function* f *on* X *with values in* $R(L)$ *such that* $g \leq f \leq h$.

3. *If* $\overline{1}-\lambda$, μ *are* (r,s)*-fuzzy* b*-open sets such that* $\mu \leq \lambda$, *then there exists a strong* (r,s)*-fuzzy* b*-continuous function* $f : X \to I^X$ *such that* $\mu \leq (\overline{1}-L_1)f \leq R_0 f \leq \lambda$.

Proof. (1)⇒(2): Define two functions $\lambda,\mu : Q \to I^X$ by $\lambda(r) = \lambda_r = h^{-1}(R'_r)$ and $\mu(r) = \mu_r = g^{-1}(L_r)$ for all $r \in Q$. Clearly, λ and μ are monotonic increasing families of (r,s)-fuzzy b-closed and (r,s)-fuzzy b-open sets of $(X,\tau,\tau^\star)$. Moreover, $\lambda_r < \mu_{r'}$ if $r < r'$. Now, by Proposition 3 there exists a function $\eta : Q \to I^X$ such that $\lambda_r \leq BI_{\tau,\tau^*}(\eta_{r'},r,s)$, $BC_{\tau,\tau^*}(\eta_r,r,s) \leq BI_{\tau,\tau^*}(\eta_{r'},r,s)$, $BC_{\tau,\tau^*}(\eta_r,r,s) \leq \mu_{r'}$ whenever $r < r'$ $(r,r' \in Q)$. Letting $\omega_t = \bigwedge_{r<T} \eta_{r'}$ for each $t \in \mathbb{R}$, we define a monotone decreasing family $\bigwedge_{r<t} \{\omega_t : t \in \mathbb{R}\} \subset I^X$. Moreover, $BC_{\tau,\tau^*}(\omega_t,r,s) \leq BC_{\tau,\tau^*}(\omega_s,r,s)$ whenever $s < t$. Indeed, for $s < r < r' < t$ $(s,t \in \mathbb{R}$ and $r,r' \in Q)$ we have $\omega'_s \leq BC_{\tau,\tau^*}(\eta_r,r,s) \leq BI_{\tau,\tau^*}(\eta_{r'},r,s) \leq \omega'_t$, hence $BC_{\tau,\tau^*}(\omega_t,r,s) \leq BI_{\tau,\tau^*}(\omega_s,r,s)$. We have also

$$\begin{aligned}\bigvee_{t\in\mathbb{R}} \omega_t &= \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} \eta'_r \\ &\geq \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} \mu'_r \\ &= \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} g^{-1}(L'_r) \\ &= g^{-1}(\bigvee_{t\in\mathbb{R}} L'_r) \\ &= 1.\end{aligned}$$

Similarly, $\bigwedge_{t\in\mathbb{R}} \omega_t = 0$. We now define a function $f : X \to \mathbb{R}(I)$ satisfying the required properties. Let $f(x)(t) = \omega_t(x)$ for all $x \in X$ and $t \in \mathbb{R}$. Then above discussion shows that f is well defined, that is $f(x) \in \mathbb{R}(I)$ for every $x \in X$. To prove f is (r, s)-fuzzy b-continuous, observe that

$$\bigvee_{s>t} \omega_s = \bigvee_{s>t} BI_{\tau,\tau^*}(\omega_s, r, s)$$

and

$$\bigwedge_{s<t} \omega_s = \bigwedge_{s<t} BC_{\tau,\tau^*}(\omega_s, r, s).$$

Then $f^{-1}(R_t) = \bigvee_{s>t} \omega_s = \bigvee_{s>t} BI_{\tau,\tau^*}(\omega_s, r, s)$ is (r, s)-fuzzy b-open. Now $f^{-1}(L'_t) = \bigwedge_{s<t} \omega_s = \bigwedge_{s<t} BC_{\tau,\tau^*}(\omega_s, r, s)$, so that f is (r, s)-fuzzy b-continuous. To conclude the proof it remains to show that $g \leq f \leq h$, that is,

$$g^{-1}(L'_t) \leq f^{-1}(L'_t) \leq h^{-1}(L'_t)$$

and

$$g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t)$$

for each $t \in \mathbb{R}$. We have

$$\begin{aligned}g^{-1}(L'_t) &= \bigwedge_{s<t} g^{-1}(L'_s) \\ &= \bigwedge_{s<t} \bigwedge_{r<s} g^{-1}(L'_s) \\ &= \bigwedge_{s<t} \bigwedge_{r<s} \mu'_r \\ &\leq \bigwedge_{s<t} \bigwedge_{r<s} \eta'_r \\ &= \bigwedge_{s<t} \omega_s \\ &= f^{-1}(L'_t)\end{aligned}$$

and

$$\begin{aligned} f^{-1}(L'_t) &= \bigwedge_{s<t} \omega_s \\ &= \bigwedge_{s<t} \bigwedge_{r<s} \eta'_r \\ &\leq \bigwedge_{s<t} \bigwedge_{r<s} \eta'_r h^{-1}(R_r) \\ &= \bigwedge_{s<t} h^{-1}(L'_s) \\ &= h^{-1}(L'_t). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} g^{-1}(R_t) &= \bigvee_{s>t} g^{-1}(R_s) \\ &= \bigvee_{s>t} \bigvee_{r>s} g^{-1}(L'_r) \\ &= \bigvee_{s>t} \bigvee_{r>s} \mu'_r \\ &\leq \bigvee_{s>t} \bigvee_{r>s} \eta'_r \\ &= \bigvee_{s>t} \omega_s \\ &= f^{-1}(R_t) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(R_t) &= \bigvee_{s>t} \omega_s \\ &= \bigvee_{s>t} \bigvee_{r>s} \eta'_r \\ &\leq \bigvee_{s>t} \bigvee_{r>s} \lambda'_r h^{-1}(R_r) \\ &= \bigvee_{s>t} h^{-1}(R_s) \\ &= h^{-1}(R_t). \end{aligned}$$

(2)⇒(3): Suppose $1-\lambda$ is an (r,s)-fuzzy b-open set and μ is an (r,s)-fuzzy b-open set, $\mu \leq \lambda$. Then $\chi_\mu \leq \chi_\lambda$ and χ_μ, χ_λ are lower and upper (r,s)-fuzzy b-continuous functions, respectively. Hence by (2), there exists an (r,s)-fuzzy b-continuous function $f : (X,\tau,\tau^\star) \to \mathbf{R}(I)$ such that$\chi_\mu \leq f \leq \chi_\lambda$. Clearly, $f(x) \in [0,1](I)$ for all $x \in X$ and $\mu = (1-L_1)\chi_\mu \leq (1-L_1)f \leq R_0 f \leq R_0\chi_\lambda = \lambda$.
(3)⇒(1): This follows from Proposition 2 and the fact that $(1-L_1)f$ and $R_0 f$ are (r,s)-fuzzy b-closed and (r,s)-b-open sets, respectively.

Proposition 10. *Let $(X,\tau,\tau^\star)$ be an (r,s)-fuzzy b-extremely disconnected space and let $A \subset X$ be such that χ_A is (r,s)-fuzzy b-open. Let $f : (A,\tau|A) \to I^X$ be strong (r,s)-fuzzy b-continuous. Then f has a strong (r,s)-fuzzy b-continuous extension over $(X,\tau,\tau^\star)$.*

Proof. Let $g, h : X \to I^X$ be such that $g = f = h$ on A and $g(x) = 0, h(x) = 1$ if $x \notin A$. We now have

$$R_t g = \begin{cases} \mu_t \wedge \chi_A & \text{if t} \geq 0, \\ 1 & \text{if } t < 0. \end{cases}$$

where μ_t is (r, s)-fuzzy b-open and is such that $\mu_t|A = R_t f$ and

$$L_t h = \begin{cases} \lambda_t \wedge \chi_A & \text{if t} \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

where λ_t is (r, s)-fuzzy b-clopen and is such that $\lambda_t|A = L_t f$. Thus g is lower (r, s)-fuzzy b-continuous h is upper (r, s)-fuzzy b-continuous and $g \leq h$. By Proposition 9, there is an (r, s)-fuzzy strong b-continuous function $F : X \to I^X$ such that $g \leq F \leq h$. Hence $F \equiv f$ on A.

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