Acta Universitatis Apulensis No. 68/2021
ISSN: 1582-5329 pp. 99-107
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2021.68.09

ON AN INTEGRAL OPERATOR

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Abstract. In this paper we define an integral operator for analytic functions in the open unit disk and we obtain properties of this integral operator.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: analytic function, integral operator, univalence.

1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in U.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. For $c \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[c, n] = \{f \in \mathcal{H}(U) : f(z) = c + a_n z^n + \cdots\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots\right\},$$

with $\mathcal{A}_1 = A$.

Let use $S_\alpha(\rho)$ the class spiral functions of type α and order ρ, where $\alpha, \rho \in \mathbb{R}$,

$$S_\alpha(\rho) = \left\{f \in A : \mathrm{Re}\frac{e^{i\alpha} z f'(z)}{f(z)} > \rho \cos\alpha, |\alpha| < \frac{\pi}{2}, \rho < 1, z \in U\right\}$$

We have $S_\alpha(0) = S_\alpha$, where S_α is the class spiral functions of type α.

In this paper we define the integral operator $W : E \to \mathcal{H}(U), E \subseteq \mathcal{H}(U)$,

$$W(f)(z) = \left[\frac{\lambda+\mu}{z^{\gamma}\phi^{\eta}(z)}\int_0^z f^{\nu}(t)t^{\delta-1}\varphi^{\sigma}(t)dt\right]^{\frac{1}{\beta}}, \tag{1}$$

where $\lambda, \mu, \eta, \nu, \gamma, \beta, \delta, \sigma \in \mathbb{C}$, $\beta \neq 0$, $\phi, \varphi \in \mathcal{H}[1, n]$ with $\phi(z)\varphi(z) \neq 0$, $z \in U$, $f \in \mathcal{H}(U)$, $\lambda + \mu \neq 0$.

We have the next remarks.

i_1) For $\lambda + \mu = \beta + \gamma$, $\beta \neq 0$, $\eta = \sigma = 1$, $\varphi, \phi \in \mathcal{H}[1, n], f \in \mathcal{A}_n$, from (1) we have the integral operator Miller-Mocanu-Reade,

$$I_{\nu,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\nu}(t)t^{\delta-1}\varphi(t)dt\right]^{\frac{1}{\beta}}. \tag{2}$$

The integral operator $I_{\nu,\beta,\gamma,\delta}$ had defined by S.S. Miller, P.T. Mocanu, M.O. Reade in the year 1978 and studied in [8].

i_2) For $\lambda + \mu = 1$, $\beta = \nu = 1$, $\gamma = \delta$, $\eta = \sigma = 0$, the function $f \in \mathcal{A}_n$, from (1) we obtain the integral operator Hallenbeck-Ruscheweyh, which had studied in [3],

$$J_{\gamma}(f)(z) = \frac{1}{z^{\gamma}}\int_0^z t^{\gamma-1}f(t)dt. \tag{3}$$

i_3) If $\lambda + \mu = e^{i\alpha} + \gamma$, $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\beta = \nu = e^{i\alpha}$, $\delta = \gamma$, $\eta = \sigma = 0$, $f \in S_{\alpha}(\rho)$, from (1) we have the integral operator Bajpai,

$$B_{\alpha,\gamma}(f)(z) = \left[\frac{e^{i\alpha}+\gamma}{z^{\gamma}}\int_0^z [f(t)]^{e^{i\alpha}}t^{\gamma-1}dt\right]^{e^{-i\alpha}}. \tag{4}$$

S.K.Bajpai in [1] proved that, if $f \in S_{\alpha}(\rho)$, $0 \leq \rho < 1$, $Re\gamma > -\rho\cos\alpha$, $|\alpha| < \frac{\pi}{2}$, then $B_{\alpha,\gamma} \in S_{\alpha}(\rho)$.

i_4) We take $\lambda + \mu = 1$, $\nu = \beta$, $\beta \neq 0$, $\gamma = \delta = 0$, $\eta = \sigma = 0$, the function $f \in \mathcal{A}_n$, from (1) we obtain the integral operator Miller-Mocanu [7],

$$T_{\beta}(f)(z) = \left[\int_0^z t^{-1}f^{\beta}(t)dt\right]^{\frac{1}{\beta}}. \tag{5}$$

i_5) For $\lambda + \mu = \gamma + 1 > 0$, $\gamma \in \mathbb{N}^*$, $\beta = 1$, $\delta = \gamma$, $\eta = \sigma = 0$, $f \in \mathcal{A}_n$ from (1) we obtain the integral operator Bernardi-Libera [2],

$$D_{\gamma}(f)(z) = \frac{\gamma+1}{z^{\gamma}}\int_0^z t^{\gamma-1}f(t)dt. \tag{6}$$

i_6) If $\beta \neq 0$, $\lambda + \mu = \beta$, $\gamma = 0$, $\eta = \sigma = 0$, $\delta = \beta - \nu$, $\nu = \alpha$, $\alpha, \in \mathbb{C}$, $f \in A$, we have the integral operator Pascu-Pescar [12],

$$H_{\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\alpha} dt\right]^{\frac{1}{\beta}}. \qquad (7)$$

i_7) For $\beta = \lambda + \mu = 1$, $\gamma = 0$, $\eta = \sigma = 0$, $\alpha = \nu = 1 - \delta$, $f \in A$, we get the integral operator Kim-Merkes [4],

$$K_{\alpha}(f)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt. \qquad (8)$$

i_8) We take $\lambda + \mu = 1$, $\delta = \gamma$ $\nu = \beta$, $\beta \neq 0$, $\eta = \sigma = 0$ and $f \in \mathcal{H}(U)$, we have the integral operator

$$G_{\beta,\gamma}(f)(z) = \left[\frac{\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \qquad (9)$$

which had studied by S.S.Miller, P.T.Mocanu, M.O.Reade in [9].

If $\beta \geq 1$, $Re\gamma > 0$, $G_{\beta,\gamma}(f)$ is the averaging integral operator.

i_9) For $\lambda + \mu = 2$, $\nu = \beta = \gamma = \delta = 1$, $\eta = \delta = 0$ and $\phi(z) = \varphi(z) = 1$, $f \in \mathcal{H}(U)$, we obtain the integral operator Libera [5],

$$L(f)(z) = \frac{2}{z} \int_0^z f(t) dt. \qquad (10)$$

In this paper we obtain certain properties of general integral operator defined by (1) and applications.

2. Preliminaries

We need the following lemmas.

Lemma 1 (Pascu [11]). *Let α be a complex number, $Re\alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad (11)$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t)dt\right]^{\frac{1}{\alpha}} \tag{12}$$

is regular and univalent in U.

Lemma 2 (Mocanu and Şerb, [10]). *Let $M_0 = 1,5936\ldots$ be the positive solution of equation*

$$(2 - M)e^M = 2 .$$

If $f \in A$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \le M_0, \quad z \in U, \tag{13}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{14}$$

The edge M_0 is sharp.

Lemma 3 (General Schwarz Lemma, [6]). *Let f be the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If the function $f(z)$ has in $z = 0$ one zero with multiply $\ge m$, then*

$$|f(z)| \le \frac{M}{R^m}|z|^m, \quad z \in U_R, \tag{15}$$

the equality (in the inequality (15) for $z \ne 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3. Main results

Theorem 4. *Let $\lambda, \mu, \eta, \nu, \beta, \delta, \sigma$ be complex numbers, $\beta \ne 0$, $a = Re(\nu + \delta) > 0$, M_1, M_2 be positive real numbers and the functions $f \in \mathcal{A}_n$, $f(z) = z + a_{n+1}z^{n+1} + \ldots$, $\phi, \varphi \in \mathcal{H}[1, n]$, $\phi(z) = 1 + c_n z^n + \ldots$, $\varphi(z) = 1 + d_n z^n + \ldots$.*
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M_1, z \in U, \tag{16}$$

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| < M_2, z \in U \tag{17}$$

and

$$|\nu|M_1 + |\sigma|M_2 \leq \frac{(n+2a)^{\frac{n+2a}{2a}}}{2n^{\frac{n}{2a}}}, \tag{18}$$

then the integral operator W *defined by* (1) *is*

$$W(f)(z) = \left(\frac{\lambda+\mu}{\nu+\delta}\right)^{\frac{1}{\beta}} \frac{1}{z^{\gamma}\phi^{\eta}(z)}(z + b_2z^2 + \ldots)^{\frac{\nu+\delta}{\beta}}, z \in D \subseteq U \tag{19}$$

Proof. From (1) we have

$$W(f)(z) = \left(\frac{\lambda+\mu}{\nu+\delta}\right)^{\frac{1}{\beta}} \frac{1}{z^{\gamma}\phi^{\eta}(z)} \left\{\left[(\nu+\delta)\int_0^z t^{\nu+\delta-1}\left(\frac{f(t)}{t}\right)^{\nu}\varphi^{\sigma}(t)dt\right]^{\frac{1}{\nu+\delta}}\right\}^{\frac{\nu+\delta}{\beta}}, \tag{20}$$

for all $z \in U$.

We consider the function

$$G(z) = \left[(\nu+\delta)\int_0^z t^{\nu+\delta-1}\left(\frac{f(t)}{t}\right)^{\nu}\varphi^{\sigma}(t)dt\right]^{\frac{1}{\nu+\delta}}, z \in U. \tag{21}$$

Let's the function

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\nu}\varphi^{\sigma}(t)dt, z \in U, \tag{22}$$

which is regular in U and $g(0) = g'(0) - 1 = 0$.
We have

$$g'(z) = \left(\frac{f(z)}{z}\right)^{\nu}\varphi^{\sigma}(z)$$

and

$$g''(z) = \nu\left(\frac{f(z)}{z}\right)^{\nu-1}\frac{zf'(z)-f(z)}{z^2}\varphi^{\sigma}(z) + \left(\frac{f(z)}{z}\right)^{\nu}\sigma(\varphi(z))^{\sigma-1}\varphi'(z),$$

for all $z \in U$.
We obtain

$$\frac{zg''(z)}{g'(z)} = \nu\left(\frac{zf'(z)}{f(z)} - 1\right) + z\sigma\frac{\varphi'(z)}{\varphi(z)}, z \in U. \tag{23}$$

From (23) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\left[|\nu|\left|\frac{zf'(z)}{f(z)} - 1\right| + |z||\sigma|\left|\frac{\varphi'(z)}{\varphi(z)}\right|\right], \tag{24}$$

for all $z \in U$.

Applying Lemma 3, from (16) and (17) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M_1|z|^n, z \in U, \tag{25}$$

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq M_2|z|^{n-1}, z \in U \tag{26}$$

and hence, by (24) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}|z|^n \left[|\nu|M_1 + |\sigma|M_2\right] \tag{27}$$

for all $z \in U$.
We consider the function $Q : [0,1] \to \mathbb{R}$, $Q(x) = \frac{(1-x^{2a})x^n}{a}$, where $x = |z|, x \in [0,1]$.
We have

$$\max_{x\in[0,1]} Q(x) = \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{2a+n}{2a}}}, n \in \mathbb{N} - \{0\}. \tag{28}$$

By (18), (28) and (27) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \tag{29}$$

for all $z \in U$.
Now, from (29) and Lemma 1, it results that the function $G(z)$ belongs to the class $\mathcal{S}$ and we have

$$G(z) = z + b_2z^2 + \ldots, z \in U. \tag{30}$$

From (30) and (20) we obtain

$$W(f)(z) = \left(\frac{\lambda+\mu}{\nu+\delta}\right)^{\frac{1}{\beta}} \frac{1}{z^\gamma \phi^\eta(z)}(z + b_2z^2 + \ldots)^{\frac{\nu+\delta}{\beta}}, \tag{31}$$

for all $z \in D \subseteq U$ and Theorem 4 is proof.

If $\lambda + \mu + \eta = \nu + \delta = \beta = 1$ and $\gamma = \eta = 0$, then the integral operator $W(f)(z)$ is in the class $\mathcal{S}$.

Using Theorem 4 we have the next applications.

Corollary 1. Let $\beta, \gamma, \nu, \delta \in \mathbb{C}$, $\beta \neq 0$, $a = Re(\nu+\delta) > 0$, the functions $\phi, \varphi \in \mathcal{H}[1,n]$, $f \in \mathcal{A}_n$, M_1, M_2 positive real numbers.
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M_1, z \in U, \tag{32}$$

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| < M_2, z \in U \tag{33}$$

and

$$|\nu|M_1 + M_2 \leq \frac{(n+2a)^{\frac{n+2a}{2a}}}{n^{\frac{n}{2a}}}, \tag{34}$$

the integral operator Miller-Mocanu-Reade defined by (2) is

$$I_{\nu,\beta,\gamma,\delta}(z) = \frac{\beta+\gamma}{(\nu+\delta)z^{\gamma}\phi(z)}(z+b_2z^2+\ldots)^{\frac{\nu+\delta}{\beta}}, z \in D \subseteq U. \tag{35}$$

Proof. For $\lambda+\mu=\beta+\gamma$, $\beta \neq 0$, $\eta=\sigma=1$, from Theorem 4 we obtain Corollary 1.

Corollary 2. Let γ be a complex number, $a = Re(\gamma+1) > 0$, the function $f \in \mathcal{A}_n$. If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq n\left(1+\frac{2a}{n}\right)^{\frac{n+2a}{2a}}, z \in U, \tag{36}$$

then the integral operator Hallenbeck-Ruscheweyh, defined by (3) is

$$J_{\gamma}(f)(z) = \frac{1}{\gamma+1}z(1+b_2z+\ldots)^{\gamma+1}, z \in D \subseteq U. \tag{37}$$

Proof. We take in Theorem 4, $\lambda+\mu=1$, $\beta=\nu=1$, $\gamma=\delta$, $\eta=\sigma=0$, the function $f \in \mathcal{A}_n$ and we obtain Corollary 2.

Remark 1. For $\gamma = 0$, from Corollary 2 we obtain $J_0(f)(z) = z + b_2z^2 + \ldots$, for all $z \in U$ and here the integral operator

$$J_0(f)(z) = \int_0^z t^{-1}f(t)dt, f \in \mathcal{A}_n, \tag{38}$$

is in the class $\mathcal{S}$.

Corollary 3. Let γ be a complex number, α be a real number, $a = Re(e^{i\alpha}+\gamma) > 0$, the function $f \in S_{\alpha}(\rho)$. If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq (1+2a)^{\frac{1+2a}{2a}}, z \in U, \tag{39}$$

then

$$B_{\alpha,\gamma}(f)(z) = \frac{1}{z^{\gamma}}(z+b_2z^2+\ldots)^{\frac{e^{i\alpha}+\gamma}{e^{i\alpha}}}, z \in D \subseteq U. \tag{40}$$

Proof. Applying Theorem 4 for $n = 1$, $\lambda + \mu = e^{i\alpha} + \gamma$, $\alpha \in \mathbb{R}$, $\beta = \nu = e^{i\alpha}$, $\delta = \gamma$, $\eta = \sigma = 0$, we obtain Corollary 3.

Corollary 4. Let β be a complex number, $\beta \neq 0$, $a = Re\beta > 0$ and the function $f \in \mathcal{A}_n$, M_1 be positive real number.
If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M_1, z \in U \tag{41}$$

and

$$|\beta| M_1 \leq n\left(1 + \frac{2a}{n}\right)^{\frac{n+2a}{2a}}, \tag{42}$$

then the integral operator Miller-Mocanu defined by (5) is

$$T_\beta(f)(z) = \frac{1}{\beta^{,\frac{1}{\beta}}}(z + b_2 z^2 + \ldots), z \in D \subseteq U. \tag{43}$$

Proof. For $\lambda + \mu = 1$, $\nu = \beta$, $\beta \neq 0$, $\gamma = \delta = 0$, $\eta = \sigma = 0$ from Theorem 4, we obtain Corollary 4.

Remark 2. For $\beta = 1$, from Corollary 4, we have $T_1(f) \in \mathcal{S}$, $T_1(f)(z) = z + b_2 z^2 + \ldots$, for all $z \in U$.

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