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A PROBABILISTIC APPROACH ON STAR COLORING OF DEGREE SPLITTING GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring which states that every path on four vertices in G is in excess of two dissimilar colors. The star chromatic number $\chi_s(G)$ of G is the fewest number of colors that require to star color G. For a graph $G = (V, E)$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. Thus to construct the degree splitting graph of G, add new vertices $w_1, w_2, \dots w_t$ and join w_i to each vertex of S_i for $1 \leq i \leq t$. The degree splitting graph of G is denoted by $DS(G)$. In this short note, we show that if $DS(G)$ is a degree splitting graph with maximum degree $\Delta \geq 3$, then $\chi_s(DS(G)) \leq \left\lceil 12\Delta^{\frac{3}{2}} \right\rceil$. The proof of our theorem mainly rely on probabilistic logic.

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1. Introduction

In this note, we consider $G = (V(G), E(G))$ be a finite, simple, connected and undirected graphs. Grünbaum in 1973 introduced the star chromatic number [4]. A star coloring of a graph G is a proper vertex coloring which states that every path on four vertices in G is in excess of two dissimilar colors. The star chromatic number $\chi_s(G)$ of G is the fewest number of colors that require to star color G.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. Coleman et al. [3] proved that star coloring remains NP-hard problem even on bipartite graphs.

For a graph $G = (V, E)$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. Thus to construct the degree splitting graph [6] of G, add new vertices $w_1, w_2, \dots w_t$ and join w_i to each vertex of S_i for $1 \leq i \leq t$. The degree splitting graph of G is denoted by $DS(G)$. For some graph terminologies not defined in this note see [2, 5].

Albertson et al. [1] proved that $\chi_s(G) \leq \Delta(\Delta-1)+2$. In this note, we show that if $DS(G)$ is a degree splitting graph with maximum degree $\Delta \geq 3$, then $\chi_s(DS(G)) \leq \left\lceil 12\Delta^{\frac{3}{2}} \right\rceil$.

2. Observation

1. For any graph G, $\chi_s(G) \leq \chi_s(DS(G))$.

3. Main Result

We shall make use of the following Lovász local lemma to prove the main theorem:

Lemma 1. *[7][Lovász local lemma] Let $A_1, A_2 \cdots A_n$ be the events in an arbitrary probability space. Let the graph $H=(V,E)$ on the nodes $1,2,\cdots n$ be a dependency graph for the events A_i (that is, two events A_i and A_j will share an edge in H if and only if they are dependent). If there exists a real numbers $0 \leq y_i < 1$ such that for all i we have*

$$Pr(A_i) \leq y_i. \prod_{(i,j)\in E} (1-y_i)$$

then

$$Pr(\cap \overline{A_i}) \geq \prod_{(i=1)} (1-y_i) > 0$$

Theorem 2. *Let G be any graph with maximum degree $\Delta \geq 3$, then*

$$\chi_s(DS(G)) \leq \left\lceil 12\Delta^{\frac{3}{2}} \right\rceil.$$

Proof. Suppose that $s = 12\Delta^{\frac{3}{2}}$. For each vertex $v \in V(G)$ randomly and independently choose $c(v)$ from $\{1,2,\ldots,s\}$. For each edge $vw \in E(G)$, let $A_{v,w}$ be the type-I event then $c(v)=c(w)$. For each path of length 3 $vwxy$ in G, let $B_{v,w,x,y}$ be the type-II event then $c(v)=c(x)$ and $c(w)=c(y)$.

We will apply lemma to obtain colorings such that no type-I event and no type-II event occurs. No type-I event implies that we have a proper vertex coloring. No type-II event implies that no two disjoint edges share the same pair of colors. That is we have a star coloring.

For each type-I event A, $P(A) = \frac{1}{s}$. For each type-II event B, $P(B) = \frac{1}{s^2}$.

An event involving a particular set of vertices is dependent only on the events involving at least one of the vertices in that set. Each vertex is involved in atmost Δ

type-I events and atmost $2\Delta^3$ type-II events. A type-I events involves two vertices, and is thus mutually independent of all but at most 2Δ type-I events and at most $4\Delta^3$ type-II events. A type-I events involves four vertices, and is thus mutually independent of all but at most 4Δ type-I events and at most $8\Delta^3$ type-II events.

In order to apply Lovász local lemma, we need to choose y_1 as $\frac{2}{s}$ for the type-I events and y_2 as $\frac{2}{s^2}$ for the type-II events. Then we have to prove the following inequalities that arise the hypothesis of the generalised version of Lovász local lemma.

$$P(A) = \frac{1}{s} \leq \frac{2}{s}\left(1 - \frac{2}{s}\right)^{2\Delta}\left(1 - \frac{2}{s^2}\right)^{4\Delta} \tag{1}$$

$$P(B) = \frac{1}{s^2} \leq \frac{2}{s^2}\left(1 - \frac{2}{s}\right)^{4\Delta}\left(1 - \frac{2}{s^2}\right)^{8\Delta^3} \tag{2}$$

It is easy to check that if (2) is satisfied, then (1) holds as well. Let us check (2),

$$\begin{aligned}
\left(1 - \frac{2}{s}\right)^{4\Delta}\left(1 - \frac{2}{s^2}\right)^{8\Delta^3} &\geq \left(1 - \frac{8\Delta}{s}\right)\left(1 - \frac{16\Delta^3}{s^2}\right) \\
&\geq \left(1 - \frac{2}{3\sqrt{\Delta}}\right)\left(1 - \frac{1}{9}\right) \\
&> \frac{1}{2}
\end{aligned}$$

for any $\Delta \geq 3$. Hence by Lovász local lemma c is a star coloring of G with non-zero probability.

References

[1] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, R. Ramamurthi, *Coloring with no 2-colored P_4's*, The Electronic Journal of Combinatorics 11 (2004), Paper # R26.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, London, MacMillan 1976.

[3] T.F. Coleman, J. Moré, *Estimation of sparse Hessian matrices and graph coloring problems*, Mathematical Programming, 28(3)(1984), 243-270.

[4] B. Grünbaum, *Acyclic colorings of planar graphs*, Israel Journal of Mathematics, 14(1973), 390-408.

[5] F. Harary, *Graph theory*, Narosa Publishing home, New Delhi 1969.

[6] R. Ponraj, S. Somasundaram, *On the degree splitting graph of a graph*, National Academy Science Letters, 27(7-8)(2004), 275-278.

[7] J. Spencer, *Ten lectures on the Probabilistic method*, SIAM, Philadelphia, PA, 1987.

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