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STABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN AIZERMANN DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we employ the direct method of Lyapunov which involves the use of a suitable scalar function known as Lyapunov function, to establish sufficient conditions for the stability, uniform stability, asymptotic stability, uniform asymptotic stability, boundedness, uniform boundedness and uniform-ultimate boundedness of solutions to certain Aizermann vector differential equations. The results of this paper are additions to the body of literature by improving and complementing some of the existing results. Finally, we provide two numerical examples and the graphical representations of the behaviour of the solutions of the examples using Maple software.

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1. Introduction

The problems of stability and boundedness of solutions of linear and nonlinear differential equations have been shown to be of great importance in the theory and application of differential equations. Thus, we shall examine the conditions for boundedness and stability of solutions of the following systems of first order vector differential equations known as Aizermann differential equation:

$$\dot{X} = F(X) + BY + P_1(t, X, Y), \ \dot{Y} = G(X) + DY + P_2(t, X, Y), \tag{1}$$

where $X,\ Y\ \in \mathbb{R}^n,\ B,$ and D are real $n \times n$ constant symmetric matrices, $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are one time continuously differentiable functions(C^1) satisfying $F(0) = G(0) = 0$. Conditions for existence and uniqueness of solutions of (1) are assumed.

In [50], Zhou obtained the conditions for boundedness and asymptotic behaviour of solutions of system

$$\dot{x} = \frac{1}{a(x)}[h(y) - F(x)],$$

$$\dot{y} = -a(x)[g(x) - e(t)].$$

Also, boundedness, convergence and asymptotic behaviour of solutions of the above systems have been considered by Qian [37] and Jiang [21]. The case where $a(x) \equiv 1$ and $h(y) = y$ in the above system has been examined for boundedness, asymptotic behaviour of solutions and global stability of zero solution by Burton [9], Graef [19], Sugie [39], LaSalle and Lefschetz [24], Huiqing [20] and Pan and Jiang [33].

Since the introduction of Aizermann problem in 1949, several authors and researchers have considered various forms of the problem. For instance, Erugin [12] examined the stability of the solution of the one dimensional system

$$\dot{x} = \psi(x) + by, \quad \dot{y} = cx + dy$$

arising in connection with the so-called Aizermann problem, in which b, c and d are constants and $\psi(x)$ is a continuous scalar function. Krasovskii ([22], [23]) considered the systems of first order equations of the type

$$\dot{x} = f(x) + by, \quad \dot{y} = cx + g(y),$$

and

$$\dot{x} = f(x) + g(y), \quad \dot{y} = cx + dy$$

where $f(x), g(y)$ are continuous scalar functions of x and b, c, d are constants. The author gave necessary and sufficient conditions for the asymptotic stability of the trivial solution.

Erugin [12] and Malkin [26] considered the system

$$\dot{x} = ax + f(y); \quad \dot{y} = bx + cy, \tag{2}$$

and showed that the trivial solution of the system (2) is asymptotically stable in the large under the conditions $a + c < 0$, $(acy - bf(y))y > 0$ for $y \neq 0$ and

$$\int_0^y (acy - bf(y))dy \to +\infty \text{ as } |y| \to +\infty. \tag{3}$$

Later, Mufti [27] solved the problem of Aizermann for the following systems of two equations

$$\dot{x} = ax + f(y); \quad \dot{y} = bx + cy \tag{4}$$

and

$$\dot{x} = f(x) + ay; \quad \dot{y} = bx + cy, \tag{5}$$

where a, b, c are constants and $f(y), f(x)$ are continuous scalar functions. He proved a similar theorem to that of Malkin for the system (2) but without the requirement of condition (3). In the case of system (5), the author gave a new result of a theorem which asserts that if $c^2 + ab \neq 0$, then the trivial solution is asymptotically stable in the large under the generalized Hurwitz conditions.

Much later, Ezeilo [14] extended some of the results on Aizermann scalar differential equation to the vector form by considering

$$\dot{X} = F(X) + BY,\ \dot{Y} = G(X) + DY, \tag{6}$$

in which $X,\ Y \in \mathbb{R}^n$, B and D are real $n \times n$ matrices and $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are C^1 functions. This equation is an n-dimensional version of Aizermann differential equations. Ezeilo was able to generalize the stability results established by Krasovskii to the corresponding n-dimensional case (6). To the best of our knowledge, Ezeilo is the only one so far who had worked on the vector version of Aizermann differential equation.

The motivation for this work is from the works of Ezeilo [14] and Krasovskii ([22], [23]). Our aim is to further study some qualitative properties (i.e. stability, boundedness and ultimate boundedness) of solutions of system (1) by using the direct method of Lyapunov. The boundedness and ultimate boundedness properties of (1) have not been considered to the best of our knowledge in the literature. Thus, this ascertains the originality of this research. For further and better understanding on the qualitative analysis of solutions of differential equations, interested readers can check the works of Ademola and Arawomo [1], Adeyanju ([2], [3]), Adeyanju and Tunc [4], Adeyanju et. al [5], Adeyanju and Adams [6], Cartwright [10], Erugin [11], Ezeilo [13], Ezeilo and Tejumola [15], Loud [25], Mufti ([27], [28]), Omeike *et al.* [29], Omeike ([30], [31]), Omeike and Afuwape [32], Pliss [34], Qian ([35],[36]), Tejumola [40], Tunç ([42], [43], [44]), and Yoshizawa ([45], [47], [48], [49]).

Consider a system of differential equations (Yoshizawa [46])

$$\dot{X} = f(t, X), \tag{7}$$

where X is an n-vector and $f(t, X)$ is an n- vector function which is defined in a region $\Omega \subset I \times \mathbb{R}^n$ and continuous in (t_0, X_0) so that for each (t_0, X_0) there is a solution $X(t; t_0, X_0)$ satisfying

$$X(t; t_0, X_0) = X \tag{8}$$

and

$$X(t_0; t_0, X_0) = X_0. \tag{9}$$

Let f be Lipschitz and continuous so as to ensure the existence of a unique solution of equation (7). Then, we can give the following definitions and theorems about the solutions of equation (7).

Definition 1. Stability and Asymptotic Stability *(Yoshizawa [46]).*
A solution $\phi(t)$ of (7) defined for $t \geq 0$, is said to be Lyapunov stable if given an $\epsilon > 0$, there exists a $\delta > 0$ such that any solution $\varphi(t)$ of (7) with:

$$\|\varphi(0) - \phi(0)\| < \delta \tag{10}$$

satisfies

$$\|\varphi(t) - \phi(t)\| < \epsilon \tag{11}$$

for all $t \geq 0$, where $\|.\|$ stands for norm.

If in addition to the definition of stability above, we have:

$$\|\varphi(t) - \phi(t)\| \to 0 \text{ as } t \to \infty, \tag{12}$$

then we say the solution $\phi(t)$ is asymptotically stable.

Definition 2. Boundedness *(Yoshizawa [46])*
A solution $\phi(t)$ of (7) is said to be bounded if there exist a $\beta > 0$ and a constant $M > 0$ such that $\|\phi(t, t_0, x_0)\| < M$ whenever $\|x_0\| < \beta,\ t \geq t_0$.

We now provide some theorems about the differential system (7) under the assumption that $f(t, X)$ is continuous on $0 \leq t < \infty$, $\|X\| < H$, and $f(t, 0) \equiv 0$.

Theorem 1. *(Yoshizawa [46])*
Suppose that there exists a Lyapunov function $V(t, X)$ defined on $0 \leq t < \infty,\ \|X\| < H$ which satisfies the following conditions:

(i) $V(t, 0) = 0$,

(ii) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, *where $a(r)$ and $b(r)$ are continuous-increasing positive definite function(CIP),*

(iii) $\dot{V}_{(7)}(t, X) \leq -c(\|X\|)$, *where $c(r)$ is continuous on $[0, H]$ and is positive definite.*

Then the zero solution of (7) is stable, asymptotically stable, uniformly stable, asymptotically stable, and uniformly-asymptotically stable.

Theorem 2. *(Yoshizawa [46])*
Suppose that there exists a Lyapunov function $V(t, X)$ defined on $I \times \mathbb{R}^n$ which satisfies the following conditions:

(i) $a(\|X\|) \leq V(t, X)$, *where $a(r)$ is continuous, monotone increasing function and $a(0) = 0$,*

(ii) $\dot{V}_{(7)}(t, X) \leq 0$,

Then, the solutions of equation (7) are bounded.

Theorem 3. *(Yoshizawa [46])*
Suppose that there exist a Lyapunov function $V(t, X)$ defined on $0 \leq t < \infty$, $\|X\| \geq D$, (where D may be large) which satisfies:

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, *where $a(r)$ and $b(r)$ are continuous, monotone increasing functions and*

(ii) $\dot{V}(t, X) \leq -c(\|X\|)$, *where $c(r)$ is positive and continuous.*

Then, the solutions of equation (7) are uniformly bounded and uniformly-ultimately bounded.

Theorem 4. *LaSalle's Invariance Principle(Tunc and Mohammed [41])*
If V is a Lyapunov function on a set G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geq 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in G^* : V(\dot{\psi}) = 0\}$, where G^* is the closure of set G and ω denotes the omega limit set of a solution.

2. Preliminary Results

Here, we state some known results that will be helpful in the proofs of our main results later.

Lemma 5. *([15], [18], [29], [42])*
Let A be a real $n \times n$ symmetric matrix and

$$\delta_a \leq \lambda_i(A) \leq \Delta_a, \ (i = 1, 2, ..., n),$$

where δ_a and Δ_a are constants representing the least and greatest eigenvalues of matrix A respectively. Then, for any $X \in \mathbb{R}^n$

$$\delta_a \langle X, X \rangle \leq \langle AX, X \rangle \leq \Delta_a \langle X, X \rangle.$$

Lemma 6. *([14])*
Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and suppose that $H(0) = 0$.

(i) Then, for any $X \in \mathbb{R}^n$,

$$H(X) = \int_0^1 J_h(sX)X ds,$$

where $J_h(X)$ is the Jacobian matrix of $H(X)$;

(ii) Let $J_h(X)$ be symmetric and commutes with a certain real symmetric $n \times n$ matrix E. Then

$$\frac{d}{dt}\int_0^1 \langle EH(sX), X\rangle ds = \langle EH(X), \dot{X}\rangle,$$

for any real differentiable vector $X = X(t) \in \mathbb{R}^n$.

Lemma 7. *([16], [17], [7]) Let A, B be any two real symmetric positive definite $n \times n$ matrices. Then,*

(i) the eigenvalues $\lambda_i(AB)$, $(i = 1, 2, \ldots, n)$, of the product matrix AB are real and satisfy

$$\min_{1\le j,k\le n} \lambda_j(A)\lambda_k(B) \le \lambda_i(AB) \le \max_{1\le j,k\le n} \lambda_j(A)\lambda_k(B);$$

(ii) the eigenvalues $\lambda_i(A + B)$, $(i = 1, 2, \ldots, n)$, of the sum of matrices A and B are real and satisfy

$$\{\min_{1\le j\le n} \lambda_j(A) + \min_{1\le k\le n} \lambda_k(B)\} \le \lambda_i(A + B) \le \{\max_{1\le j\le n} \lambda_j(A) + \max_{1\le k\le n} \lambda_k(B)\}.$$

3. Formulation of Main Results

In this section, we state and prove some results regarding the system (1). The following estimates, which are defined for the matrices in the brackets will be used in the the proofs of the theorems.

Let $\delta_2, \delta_3, \delta_4, \delta_5, \delta_7, \gamma_1, \gamma_2, \Delta_2, \Delta_3, \Delta_4$ and Δ_5 be some positive constants such that:

(i) $\delta_2 \le \lambda_i(DJ_f(X) - BJ_g(X) + J_g(X)) \le \Delta_2, \quad \delta_5 \le |\lambda_i(B(I - B))| \le \Delta_5,$

(ii) $\delta_3 \le \lambda_i(-B) \le \Delta_3, \quad \gamma_1 \le |\lambda_i(D^2 + DJ_f(X) - BJ_g(Y) + J_g(X))| \le \gamma_2,$

(iii) $-\gamma_3 \leq \lambda_i(J_g(X)J_f(X)) \leq -\gamma_4, \quad -\Delta_7 \leq \lambda_i(-BD) \leq -\delta_7,$
where $i = (1, 2, ..., n)$.

Theorem 8. *Let $J_f(X),\ J_g(X)$ denote the Jacobian matrices $\frac{\partial f_i}{\partial x_i},\ \frac{\partial g_i}{\partial x_i}$ of $F(X)$ and $G(X)$ respectively, such that $F(0) = 0, G(0) = 0,\ P_1(t, X, Y) = 0$ and $P_2(t, X, Y) = 0$. Furthermore, suppose that:*

(i) the matrices $B, D,\ J_f(X)$ are all symmetric and negative definite while matrix $J_g(X)$ is symmetric and positive definite;

(ii) the matrix B commutes with matrix D, also matrices J_g and J_f commute with each other;

(iii) the matrix $\{DJ_f(X) - BJ_g(X)\}$ is strictly positive definite;

(iv) the product matrix

$$\{BJ_g(X_2) - DJ_f(X_2)\}\{D + J_f(X_1)\},$$

is positive definite for arbitrary $X_1, X_2 \in \mathbb{R}^n$.

Then the trivial solution of system (1) is uniformly-asymptotically stable and satisfies

$$\| X(t) \| \to 0,\ \| \dot{X}(t) \| \to 0,\ \text{as } t \to \infty. \tag{13}$$

Theorem 9. *If in addition to the assumptions of Theorem 8, we have*

(v) $\|P_1(t, X, Y)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\|)$, and $\|P_2(t, X, Y)\| \leq \alpha_0 + \alpha_1(\|X\| + \|Y\|)$,

where $\delta_0, \delta_1, \alpha_0$, and α_1 are some positive constants. Then, all solutions of Eq.(1) are bounded, uniformly bounded and uniform-ultimately bounded.

Theorem 10. *Further to the assumptions (i)-(iv) of Theorem 8, we have*

(v) $\|P_1(t, X, Y)\| \leq \theta_1(t) + \theta_2(t)\{\|X\| + \|Y\|\}$, $\|P_2(t, X, Y)\| \leq \phi_1(t) + \phi_2(t)\{\|X\| + \|Y\|\}$ for all $t \geq 0$, $\max \theta_i(t) < \infty, \max \phi_i(t) < \infty$ and $\theta_1(t), \theta_2(t), \phi_1(t), \phi_2(t) \in L^1(0, \infty)$, where $L^1(0, \infty)$ is the space of integrable Lebesgue functions.

Then, any solution $(X(t), Y(t))$ of system (1) with initial condition

$$X(0) = X_0, Y(0) = Y_0 \tag{14}$$

satisfies

$$\|X(t)\| \leq K,\ \|Y(t)\| \leq K \tag{15}$$

for all $t \geq 0$, where $K > 0$ is a constant depending on $B,\ D,\ \theta_1(t), \theta_2(t), \phi_1(t), \phi_2(t)$, $t_0,\ X_0,\ Y_0$, and on the functions $P_1(t, X, Y)$ and $P_2(t, X, Y)$.

The main tool to be used in proving these theorems is the Lyapunov function defined as

$$2V(X,Y) = \| DX - BY \|^2 + 2\int_0^1 \langle DF(sX) - BG(sX), X\rangle ds + 2\int_0^1 \langle G(sX), X\rangle ds - \langle BY, Y\rangle. \tag{16}$$

The following lemmas are required in providing proofs of the three theorems above.

Lemma 11. *Suppose that under the assumptions of Theorem 8, there exist constants, say K_1 and K_2 both positive, such that the function V defined in equation* (16) *satisfies*

$$K_1\{\|X\|^2 + \|Y\|^2\} \leq 2V(X,Y) \leq K_2\{\|X\|^2 + \|Y\|^2\}, \tag{17}$$

and

$$V(X,Y) \to +\infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{18}$$

Furthermore, there exists a positive constant K_3 such that for any solution (X,Y) of (1),

$$\frac{dV}{dt}_{(1)} \leq -K_3\{\|X\|^2 + \|Y\|^2\}, \tag{19}$$

for all $X, Y \in \mathbb{R}^n$.

Proof. It is obvious from (16) that when $X = Y = 0$, $V(X,Y) = 0$. On applying Lemma 6 to the function defined in (16), we have

$$\begin{aligned} 2V =& \|DX - BY\|^2 + 2\int_0^1\int_0^1 \langle \{DJ_f(s_1s_2X) - BJ_g(s_1s_2X)\}X, X\rangle s_1 ds_1 ds_2 \\ &+ 2\int_0^1\int_0^1 \langle J_g(s_1s_2X)X, X\rangle s_1 ds_1 ds_2 - \langle BY, Y\rangle \\ \geq& 2\int_0^1\int_0^1 \langle \{DJ_f(s_1s_2X) - BJ_g(s_1s_2X) + J_g(s_1s_2X)\}X, X\rangle s_1 ds_1 ds_2 - \langle BY, Y\rangle. \end{aligned}$$

By the hypotheses (i) and (iii) of Theorem 8 and Lemma 5, we have

$$\langle \{DJ_f(s_1s_2X) - BJ_g(s_1s_2X) + J_g(s_1s_2X)\}X, X\rangle \geq \delta_2\|X\|^2$$

and

$$-\langle BY, Y\rangle \geq \delta_3\|Y\|^2.$$

Thus,

$$\begin{aligned}&\int_0^1\int_0^1 \langle\{DJ_f(s_1s_2X) - BJ_g(s_1s_2X) + J_g(s_1s_2X)\}X, X\rangle s_1ds_1ds_2\\ \geq&\delta_2\|X\|^2\int_0^1\int_0^1 s_1ds_1ds_2\\ =&\frac{1}{2}\delta_2\|X\|^2.\end{aligned}$$

Hence,

$$2V(X,Y) \geq \delta_4\|X\|^2 + \delta_3\|Y\|^2,$$

where $\delta_4 = \frac{1}{2}\delta_2$.
Therefore, there exists a positive constant $K_1 = \min\{\delta_3, \delta_4\}$, such that

$$2V(X,Y) \geq K_1(\|X\|^2 + \|Y\|^2), \tag{20}$$

for all $X, Y \in \mathbb{R}^n$. It then follows from (20) that $V(X,Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(X,Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \neq 0$, which implies that

$$V(X,Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{21}$$

Similarly, by the hypothesis (ii) of Theorem 8 and Lemma 5, we have

$$\langle\{DJ_f(s_1s_2X) - BJ_g(s_1s_2X) + J_g(s_1s_2X)\}X, X\rangle \leq \Delta_2\|X\|^2$$

and

$$-\langle BY, Y\rangle \leq \Delta_3\|Y\|^2.$$

Thus,

$$\int_0^1\int_0^1 \langle\{DJ_f(s_1s_2X) - BJ_g(s_1s_2X) + J_g(s_1s_2X)\}X, X\rangle s_1ds_1ds_2$$

$$\begin{aligned}&\leq \Delta_2\|X\|^2\int_0^1\int_0^1 s_1ds_1ds_2\\ &= \frac{1}{2}\Delta_2\|X\|^2.\end{aligned}$$

Hence,

$$2V(X,Y) \leq \Delta_4\|X\|^2 + \Delta_3\|Y\|^2,$$

where $\Delta_4 = \frac{1}{2}\Delta_2$. So, we can find a positive constant $K_2 = \max\{\Delta_3, \Delta_4\}$, such that

$$2V(X,Y) \leq K_2(\|X\|^2 + \|Y\|^2), \tag{22}$$

for all $X,\ Y \in \mathbb{R}^n$. Therefore, from the inequalities (20) and (22), we have

$$K_1\{\|X\|^2 + \|Y\|^2\} \le 2V \le K_2\{\|X\|^2 + \|Y\|^2\}.$$

This establishes inequality (17) of Lemma 11.

Next, we obtain the derivative of V with respect to t along the solution path of (1) such that it satisfies

$$\frac{dV}{dt} \equiv \frac{d}{dt}V(t, X, Y)|_{(1)} \le -K_4, \tag{23}$$

provided that $\|X\|^2+\|Y\|^2 \le K_5$, where both K_4 and K_5 are some positive constants.

The derivative of V with the aid of Lemma 6 is

$$\begin{aligned}\frac{dV}{dt} =& \langle DX - BY, DF(X) + BDY - BG(X) - BDY\rangle + \langle DF(X) - BG(X), F(X) + BY\rangle \\ &+ \langle G(X), F(X) + BY\rangle - \langle BY, G(X) + DY\rangle.\end{aligned}$$

On simplifying the above derivative and arranging terms, we obtain

$$\begin{aligned}\frac{dV}{dt} =& \langle DX + F(X), DF(X) - BG(X)\rangle + \langle G(X), F(X)\rangle - \langle BY, DY\rangle \\ =& \int_0^1\int_0^1 \langle DX + J_f(s_1X)X, DJ_f(s_2X)X - BJ_g(s_2X)X\rangle ds_1 ds_2 \\ &+ \int_0^1\int_0^1 \langle J_g(s_1X)X, J_f(s_2X)X\rangle ds_1 ds_2 - \langle BY, DY\rangle \\ =& \int_0^1\int_0^1 \langle \{D + J_f(s_1X)\}\{DJ_f(s_2X) - BJ_g(s_2X)\}X, X\rangle ds_1 ds_2 \\ &+ \int_0^1\int_0^1 \langle J_g(s_1X)X, J_f(s_2X)X\rangle ds_1 ds_2 - \langle BY, DY\rangle.\end{aligned}$$

By the hypotheses (i) and (iv) of Theorem 8, and Lemma 7, we have

$$\frac{dV}{dt} \le -\gamma_4\|X\|^2 - \delta_7\|Y\|^2.$$

Thus, there exists a constant $K_3 = \min\{\gamma_4, \delta_7\} > 0$ such that

$$\frac{dV}{dt} \le -K_3\{\|X\|^2 + \|Y\|^2\}, \tag{24}$$

for all $X, Y \in \mathbb{R}^n$. This completes the proof of Lemma 11.

Lemma 12. *Suppose that, under the assumptions of Theorem 9, there exist some positive constants K_6 and K_7 such that for any solution (X, Y) of the system* (1)*, the function V defined by equation* (16)*, satisfies*

$$\frac{dV}{dt} \leq -K_6\{\|X\|^2+\|Y\|^2\}+K_8(\|X\|^2+\|Y\|^2)^{\frac{1}{2}}\{\|P_1(t,X,Y)\|+\|P_2(t,X,Y)\|\} \quad (25)$$

for all $X, Y \in \mathbb{R}^n$.

Proof.
By following the same reasoning as used in the proof of Lemma 11, but this time $P_1(t, X, Y) \neq 0$, and $P_2(t, X, Y) \neq 0$, we have

$$\begin{aligned}\frac{dV}{dt} =&\langle DX - BY, DF(X) + BDY + DP_1(t, X.Y) - BG(X) - BDY - BP_2(t, X, Y)\rangle \\ &+ \langle DF(X) - BG(X), F(X) + BY + P_1(t, X, Y)\rangle \\ &+ \langle G(X), F(X) + BY + P_1(t, X, Y)\rangle - \langle BY, G(X) + DY + P_2(t, X, Y)\rangle.\end{aligned}$$

Simplifying the above derivative and arranging terms, we obtain

$$\begin{aligned}\frac{dV}{dt} =&\langle DX + F(X), DF(X) - BG(X)\rangle + \langle G(X), F(X)\rangle - \langle BY, DY\rangle \quad (26)\\ &+ \langle D^2X + DF - BG + G - BDY, P_1(t, X, Y)\rangle - \langle BDX + BY - B^2Y, P_2(t, X, Y)\rangle \\ =& \int_0^1 \int_0^1 \langle DX + J_f(s_1X)X, DJ_f(s_2X)X - BJ_g(s_2X)X\rangle ds_1 ds_2 \\ &- \langle BY, DY\rangle + \int_0^1 \langle\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X - BDY, P_1(t; X, Y)\rangle \\ &+ \int_0^1 \int_0^1 \langle J_g(s_1X)X, J_f(s_2X)X\rangle ds_1 ds_2 + \langle B^2Y - BDX - BY, P_2(t; X, Y)\rangle \\ =& \int_0^1 \int_0^1 \langle\{D + J_f(s_1X)\}\{DJ_f(s_2X)X - BJ_g(s_2X)\}X, X\rangle ds_1 ds_2 \\ &- \langle BY, DY\rangle + \int_0^1 \langle\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X - BDY, P_1(t, X, Y)\rangle ds_1 \\ &+ \int_0^1 \int_0^1 \langle J_g(s_1X)X, J_f(s_2X)X\rangle ds_1 ds_2 + \langle B^2Y - BY - BDX, P_2(t, X, Y)\rangle.\end{aligned}$$

By applying the hypotheses of Theorem 8, Lemma 7 and Lemma 11, we have

$$\begin{aligned}\frac{dV}{dt} \leq& -\gamma_4\|X\|^2 - \delta_7\|Y\|^2 + \langle B^2Y - BY - BDX, P_2(t, X, Y)\rangle \\ &+ \int_0^1 \langle\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X - BDY, P_1(t, X, Y)\rangle ds_1.\end{aligned}$$

But,

$$\begin{aligned}\langle B^2Y - BY - BDX, P_2(t,X,Y)\rangle \leq& |\langle B^2Y - BY - BDX, P_2(t,X,Y)\rangle| \\ \leq& \{\|BDX\| + \|B(I-B)Y\|\}\|P_2(t,X,Y)\| \\ \leq& \{\Delta_7\|X\| + \Delta_5\|Y\|\}\|P_2(t,X,Y)\|.\end{aligned}$$

Also,

$$\begin{aligned}&\langle\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X - BDY, P_1(t,X,Y)\rangle \\ \leq& |\langle\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X - BDY, P_1(t,X,Y)\rangle| \\ \leq& \Big(\|\{D^2 + DJ_f(s_1X) - BJ_g(s_1X) + J_g(s_1X)\}X\| + \|BDY\|\Big)\|P_1(t,X,Y)\| \\ \leq& \{\gamma_1\|X\| + \Delta_7\|Y\|\}\|P_1(t,X,Y)\|.\end{aligned}$$

Thus, by letting $K_7 = \max\{\Delta_7, \Delta_5, \gamma_1\}$ and $K_6 = \min\{\gamma_4, \delta_7\}$, we have

$$\frac{dV}{dt} \leq -K_6\{\|X\|^2 + \|Y\|^2\} + K_7\{\|X\| + \|Y\|\}\{\|P_1(t,X,Y)\| + \|P_2(t,X,Y)\|\}.$$

However, since

$$\{\|X\| + \|Y\|\} \leq 2^{\frac{1}{2}}(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}},$$

our estimate for $\frac{dV}{dt}$ becomes

$$\frac{dV}{dt} \leq -K_6\{\|X\|^2 + \|Y\|^2\} + K_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}\{\|P_1(t,X,Y)\| + \|P_2(t,X,Y)\|\}, \tag{27}$$

where $K_8 = 2^{\frac{1}{2}}K_7$, for all $t \geq 0$. This completes the proof of Lemma 12.

Proof of Theorem 8.
From inequalities (20), (22) and (24) of the proof of Lemma 11, the trivial solution of system (1) is uniformly stable.

The conclusion of the proof of Theorem 8 is based on LaSalle's invariance principle [See, Theorem 4.] and is as follows.

Let us consider the set W defined by

$$W = \{(X,Y) : \frac{dV}{dt}(X,Y) = 0\},$$

where $X, Y \in \mathbb{R}^n$. By using LaSalle's invariance principle, we observe that $(X,Y) \in W$ implies that $X = Y = 0$. Hence, this shows that the largest invariant set contained in W is $(0,0)$. Therefore, by Theorem 1 we conclude that the zero solution of system (1) is asymptotically stable and uniformly-asymptotically stable. This completes the proof of Theorem 8.

Proof of Theorem 9.
To prove the theorem, it is enough to prove that there exists a constant $K_9 > 0$ such that

$$\|X\|^2 + \|Y\|^2 \leq K_9, \text{ for } t \geq T(X_0, Y_0), \tag{28}$$

for any solution (X, Y) of the system (1) where $X_0 = X(0), Y_0 = Y(0)$.

From the proof of Lemma 11, it is clear that the function V defined in equation (16) satisfied

$$V(X, Y) \to \infty \text{ as } \|X\|^2 + \|Y\|^2 \to \infty. \tag{29}$$

Now, let (X, Y) be any solution of the system (1). Then, from Lemma 11, the derivative of V with respect to t along the solution path of (1) is

$$\begin{aligned}
\frac{dV}{dt} \leq& - K_6\{\|X\|^2 + \|Y\|^2\} + K_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}\{\|P_1(t, X, Y)\| + \|P_2(t, X, Y)\|\} \\
\leq& - K_6\{\|X\|^2 + \|Y\|^2\} \\
&+ K_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}\{\delta_0 + \delta_1(\|X\| + \|Y\|) + \alpha_0 + \alpha_1(\|X\| + \|Y\|)\} \\
\leq& - K_6\{\|X\|^2 + \|Y\|^2\} + K_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}\{M_0 + M_1(\|X\| + \|Y\|)\},
\end{aligned}$$

where $M_0 = \delta_0 + \alpha_0$ and $M_1 = \delta_1 + \alpha_1$.

If we apply the following inequality

$$(\|X\| + \|Y\|) \leq 2^{\frac{1}{2}}(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}},$$

to the above, we have

$$\begin{aligned}
\frac{dV}{dt} \leq& - K_6\{\|X\|^2 + \|Y\|^2\} + K_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}\{M_0 + M_1 2^{\frac{1}{2}}(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}}\} \\
\leq& - K_6\{\|X\|^2 + \|Y\|^2\} + K_8 M_0\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}} + K_8 M_1 2^{\frac{1}{2}}(\|X\|^2 + \|Y\|^2).
\end{aligned}$$

By letting $\delta_9 = \frac{1}{2}(K_6 - K_8 M_1 2^{\frac{1}{2}})$, $M_1 < K_6 K_8^{-1} 2^{-\frac{1}{2}}$ and $\delta_8 = K_8 M_0$, we obtain

$$\frac{dV}{dt} \leq -2\delta_9\{\|X\|^2 + \|Y\|^2\} + \delta_8\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}. \tag{30}$$

On choosing $(\|X\|^2 + \|Y\|^2)^{\frac{1}{2}} \geq \delta_{10} = \delta_8 \delta_9^{-1}$, then the inequality (30) above implies that

$$\frac{dV}{dt} \leq -\delta_9\{\|X\|^2 + \|Y\|^2\}. \tag{31}$$

Hence, there exists γ_7 such that,

$$\frac{dV}{dt} \leq -1, \text{ if } \|X\|^2 + \|Y\|^2 \geq \gamma_7^2.$$

Following the Yoshizawa's approach in Yoshizawa [48], we can establish that for any solution $(X(t), Y(t))$ of the system (1), we ultimately have

$$\|X(t)\|^2 + \|Y(t)\|^2 \leq K_{10}, \tag{32}$$

for some positive constant K_{10}. This means that, for any solution $(X(t), Y(t))$ of system (1), we cannot have

$$\|X(t)\|^2 + \|Y(t)\|^2 \geq \delta_{10}^2, \tag{33}$$

for all $t \geq 0$. But suppose on the contrary that (33) were true for all $t \geq 0$. Then, by (31), we should have

$$\frac{dV}{dt} \leq -\delta_9\delta_{10}^2 < 0 \text{ for all } t \geq 0, \tag{34}$$

which clearly means that $V(X(t), Y(t)) \to -\infty$ as $t \to \infty$. This contradicts the conclusion of Lemma 12 that V is non-negative. Thus, there exists a $t_1 \geq 0$ such that

$$\|X(t_1)\|^2 + \|Y(t_1)\|^2 < \delta_{10}^2. \tag{35}$$

In view of the conclusion of Lemma 11, there exists a constant $\delta_{11} > \delta_{10}$ such that

$$\max_{\|X\|^2+\|Y\|^2=\delta_{10}^2} V(X, Y) < \min_{\|X\|^2+\|Y\|^2=\delta_{11}^2} V(X, Y). \tag{36}$$

Then, it will be proven that any solution $(X(t),\ Y(t))$ of (1) satisfying (35) must necessarily satisfy

$$\|X\|^2 + \|Y\|^2 < \delta_{11}^2, \text{ for } t \geq t_1, \tag{37}$$

thereby validating our claim.

Let's assume that (37) is not true. Then in view of (35) there exist t_2 and t_3, $t_1 < t_2 < t_3$, such that

$$\|X(t_2)\|^2 + \|Y(t_2)\|^2 = \delta_{10}^2 \tag{38}$$

$$\|X(t_3)\|^2 + \|Y(t_3)\|^2 = \delta_{11}^2 \tag{39}$$

and such that

$$\delta_{10}^2 \leq \|X(t)\|^2 + \|Y(t)\|^2 \leq \delta_{11}^2 \tag{40}$$

for $t_2 \leq t \leq t_3$. By (31), inequality (40) implies that $V(t_2) > V(t_3)$ and this contradicts the claim that $V(t_2) < V(t_3)$ $(t_2 < t_3)$ which is obtained from (36), (38) and (39). Hence, any solution $(X(t), Y(t))$ of (1) must satisfy (37). This completes the proof of Theorem 9

Proof of Theorem 10.
Let $(X(t), Y(t))$ be any given solution of (1). We will use the strategy introduced in [8] to establish the proof of the theorem. Already, from the conditions of Theorem 10 and the conclusion of Lemma 11, we have

$$2V(t, X, Y) \geq K_1(\|X\|^2 + \|Y\|^2). \tag{41}$$

And also, we have from Lemma 12 that

$$\begin{aligned}
\frac{dV}{dt} \leq & -K_6\{\|X\|^2 + \|Y\|^2\} + K_7\{\|X\| + \|Y\|\}\{\|P_1(t, X, Y)\| + \|P_2(t, X, Y)\|\} \\
\leq & K_7\{\|X\| + \|Y\|\}\Big(\phi_1(t) + \theta_1(t) + (\phi_2(t) + \theta_2(t))\{\|X\| + \|Y\|\}\Big) \\
\leq & K_7(\phi_2(t) + \theta_2(t))\{\|X\| + \|Y\|\}\{\|X\| + \|Y\|\} + K_7(\phi_1(t) + \theta_1(t))\{\|X\| + \|Y\|\} \\
\leq & 2K_7(\phi_2(t) + \theta_2(t))\{\|X\|^2 + \|Y\|^2\} + K_7(\phi_1(t) + \theta_1(t))\{2 + \|X\|^2 + \|Y\|^2\},
\end{aligned}$$

after using the obvious inequalities

$$2|\langle X, Y\rangle| \leq \|X\|^2 + \|Y\|^2, \ \|X\| \leq 1 + \|X\|^2 \text{ and } \|Y\| \leq 1 + \|Y\|^2.$$

By applying inequality (30) and letting $\theta_3(t) = \theta_2(t) + \phi_2(t)$ and
$\theta_4(t) = \theta_1(t) + \phi_1(t)$, we obtain

$$\begin{aligned}
\dot{V} \leq & 2K_7K_2^{-1}\theta_3(t)V + 2\theta_4(t)K_7K_2^{-1}V + 2K_7\theta_4(t) \\
= & K_7K_2^{-1}\big(2\theta_3(t) + \theta_4(t)\big)V + 2K_7\theta_4(t).
\end{aligned} \tag{42}$$

On integrating both sides of inequality (42) from 0 to t $(t \geq 0)$ and taking $K_{11} = K_7K_2^{-1}$, we get

$$\begin{aligned}
V(t) - V(0) \leq & K_{11}\int_0^t V(s)\big(2\theta_3(s) + \theta_4(s)\big)ds + 2K_7\int_0^t \theta_4(s)ds \\
V(t) \leq & V(0) + K_{11}\int_0^t V(s)\big(2\theta_3(s) + \theta_4(s)\big)ds + 2K_7\int_0^t \theta_4(s)ds.
\end{aligned} \tag{43}$$

Now, by using Gronwall-Bellman inequality[38], we have

$$V(t) \leq K_{12}\exp\Big(K_{11}\int_0^t \big(2\theta_3(s) + \theta_4(s)\big)ds\Big) \leq K_{13}$$

where $K_{12} = V(0) + 2K_7 \int_0^t \theta_4(s)ds$ and K_{13} is a positive constant.
This implies that

$$\|X\| \leq K_{13}\ , \|Y\| \leq K_{13}.$$

This completes the proof of Theorem 10.

4. Examples

In this section, two examples are presented to illustrate the applications and correctness of the results proved in the previous sections.

Example 1. *As a special case of equation* (1) *but* $P_1(t, X, Y) \equiv 0$, *and* $P_2(t, X, Y) \equiv 0$. *Let us consider the case* $n = 2$ *such that*

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \dot{Y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix},$$

$$F(X) = \begin{pmatrix} \tan^{-1} x_1 - 1.01x_1 \\ -0.1x_2 \end{pmatrix}, G(X) = \begin{pmatrix} \sin x_1 + 2x_1 \\ \sin x_2 + 2x_2 \end{pmatrix}, B = \begin{pmatrix} -0.2 & 0 \\ 0 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.001 \end{pmatrix}.$$

From the above, the following systems of first order differential equations are obtained.

$$\begin{aligned}
\dot{x_1} &= \tan^{-1} x_1 - 1.00x_1 - 0.2y_1, \\
\dot{x_2} &= -0.1x_2 - y_1, \\
\dot{y_1} &= \sin x_1 + 2x_1 - 0.01y_1, \\
\dot{y_2} &= \sin x_2 + 2x_2 - 0.001y_2.
\end{aligned}$$

and the Jacobian matrices of vectors $F(X)$ *and* $G(X)$ *are respectively*

$$J_f(X) = \begin{pmatrix} \frac{1}{1+x_1^2} - 1.01 & 0 \\ 0 & -0.1 \end{pmatrix} \text{ and } J_g(X) = \begin{pmatrix} \cos x_1 + 2 & 0 \\ 0 & \cos x_2 + 2 \end{pmatrix}.$$

Also, the product matrices $\{DJ_f(X) - BJ_g(X)\}$ *and* $\{BJ_g(X) - DJ_f(X)\}\{D + J_f(X)\}$ *are given by*

$$\{DJ_f(X) - BJ_g(X)\} = B^* = \begin{pmatrix} \frac{-0.01}{1+x_1^2} + 0.2\cos x_1 + 0.4101 & 0 \\ 0 & 2.0001 + \cos x_2 \end{pmatrix}$$

and

$$\{BJ_g(X) - DJ_f(X)\}\{D + J_f(X)\} = C^* =$$

$$\begin{pmatrix} (-0.2\cos x_1 + \frac{0.01}{1+x^2} - 0.4101)(\frac{1}{1+x_1^2} - 1.02) & 0 \\ 0 & 0.101\cos x_2 + 0.2020101 \end{pmatrix}.$$

Clearly, with simple calculations it can be shown that the eigen values of matrices $B, D, J_f(X), J_g(X), B^*$ *and* C^* *are respectively:* $\delta_b = -1, \Delta_b = -0.2; \delta_d = -0.001, \Delta_d = -0.01; \delta_f = -1.01, \Delta_f = -0.01; \delta_g = 1, \Delta_g = 3; \delta_b^* = 0.2001, \Delta_b^* = 3.0001$ *and* $\delta_c^* = 0.004002, \Delta_c^* = 0.622302$.

Therefore, matrices B, D *and* $J_f(X)$ *are symmetric and negative definite while matrices* J_g, B^*, C^* *are symmetric and positive definite. Thus, all the conditions of Theorem 10 are fulfilled.*

Example 2. *Suppose, in addition to the Example 1 above, we have,*

$$P_1(t, X, Y) = \frac{1}{[t^2 + (x_1 + x_2)^2 + (x_2 + x_2)^2 + 1]^2} \begin{pmatrix} x_1 + y_1 + 1 \\ x_2 + x_2 + 1 \end{pmatrix}$$

and

$$P_2(t, X, Y) = \frac{1}{(e^t + 1)^2} \begin{pmatrix} x_1 + y_1 \sin x_1 + 1 \\ x_2 + y_2 \sin x_2 + 1 \end{pmatrix}.$$

Then,

$$\begin{aligned} \|P_1(t, X, Y)\| &= \frac{1}{t^2 + (x_1 + x_1)^2 + (x_2 + x_2)^2 + 1]} \sqrt{(x_1 + y_1 + 1)^2 + (x_2 + x_2 + 1)^2}, \\ &\leq \frac{\sqrt{3}}{(t^2 + 1)} \sqrt{(x_1^2 + x_2^2) + (y_1^2 + y_2^2) + 2}, \\ &\leq \frac{\sqrt{3}}{(t^2 + 1)} \sqrt{\|X\|^2 + \|Y\|^2 + 2}, \\ &\leq \frac{\sqrt{3}}{(t^2 + 1)} \{\|X\| + \|Y\| + \sqrt{2}\}, \\ &\leq \frac{\sqrt{6}}{(t^2 + 1)} + \frac{\sqrt{3}}{(t^2 + 1)} \{\|X\| + \|Y\|\}, \\ &\leq \theta_1(t) + \theta_2\{\|X\| + \|Y\|\} \leq \sqrt{6} + \sqrt{3}\{\|X\| + \|Y\|\}, \end{aligned}$$

where, $\theta_1(t) = \frac{\sqrt{6}}{(t^2+1)} \leq \sqrt{6} \leq \delta_0$ *and* $\theta_2 = \frac{\sqrt{3}}{(t^2+1)} \leq \sqrt{3} \leq \delta_1$.

Similarly,

$$
\begin{aligned}
\|P_2(t,X,Y)\| &= \frac{1}{(e^t+1)}\sqrt{(x_1+y_1\sin x_1+1)^2+(x_2+y_2\sin x_2+1)^2},\\
&\le \frac{\sqrt{3}}{(e^t+1)}\sqrt{(x_1^2+x_2^2)+(y_1^2+y_2^2)+2},\\
&\le \frac{\sqrt{3}}{(e^t+1)}\sqrt{\|X\|^2+\|Y\|^2+2},\\
&\le \frac{\sqrt{3}}{(e^t+1)}\{\|X\|+\|Y\|+\sqrt{2}\},\\
&\le \frac{\sqrt{6}}{(e^t+1)}+\frac{\sqrt{3}}{(e^t+1)}\{\|X\|+\|Y\|\},\\
&\le \phi_1(t)+\phi_2\{\|X\|+\|Y\|\}\le \sqrt{6}+\sqrt{3}\{\|X\|+\|Y\|\},
\end{aligned}
$$

where $\phi_1 = \frac{\sqrt{6}}{(e^t+1)} \le \sqrt{6} \le \alpha_0$ *and* $\phi_2 = \frac{\sqrt{3}}{(e^t+1)} = \sqrt{3} \le \alpha_1$.
Also, all the conditions of Theorems 8 and Theorem 9 are satisfied by this example.

5. Simulation of Solutions

Figure 1 and Figure 2 below show the graphs of the asymptotic stability of the zero solution and boundedness of all solutions for the system of equations considered in Example 1 when $P_1(t,X,Y) \equiv 0$, $P_2(t,X,Y) \equiv 0$, and $t \to \infty$.

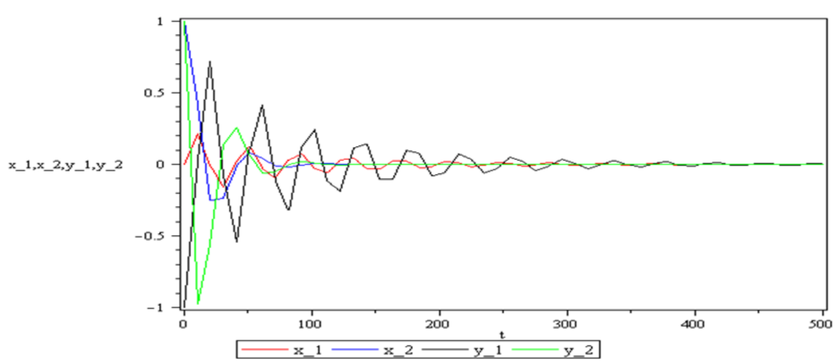

Figure 1:

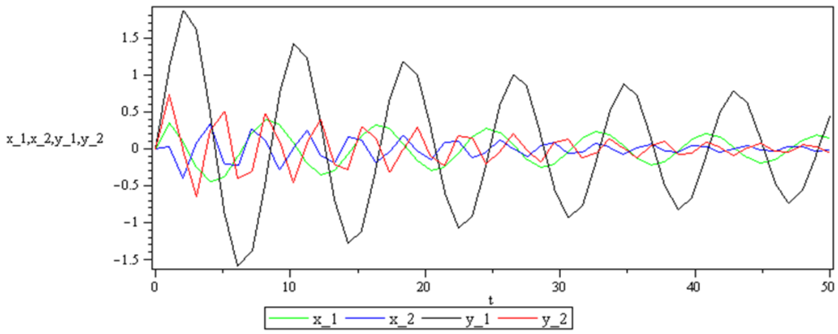

Figure 2:

On the other hand, Figure 3 shows the graph of the asymptotic stability of the zero solution and boundedness of all solutions for the system of equations considered in Example 2 when $P_1(t, X, Y) \neq 0$, $P_2(t, X, Y) \neq 0$ and $t \to \infty$.

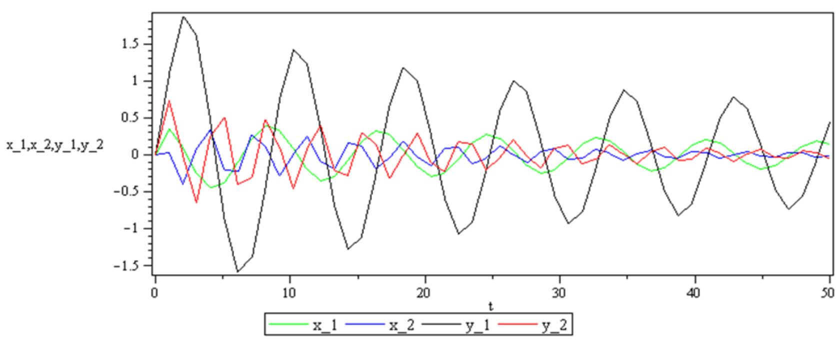

Figure 3:

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