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THE LIE GROUP OF UMBRELLA MATRICES AND PAIRWISE COMPARISONS MATRICES

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ABSTRACT. In this paper, we work with the umbrella matrix Lie group and its Lie algebra. We show that the Lie subalgebra of the PC matrix is the Lie algebra of the umbrella matrix. Furthermore, we obtain an isomorphism between the Lie subgroup of PC matrices and the umbrella matrix Lie group.

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1. Introduction

Pairwise comparisons occur when we compare two things, and we can express these comparisons with matrices. These matrices are called Pairwise Comparisons (PCs) matrices. Studies on PC matrices are old, and nowadays, they are the matrices we use to make more precise decisions about anything [2]. There are two types of PC matrices: multiplicative and additive. The group property of these matrices has been studied [6]. and some significant results have been obtained through studies on their Lie group structure [5]. The Lie group and Lie algebra of PC matrices have also been studied [3, 4, 7].
Geometrically, we can define umbrella matrices as orthogonal rotation matrices whose determinant is not -1 and which leave the $S = \begin{bmatrix} 1 & 1\dots & 1 \end{bmatrix}$ axis constant. These matrices were first discussed in the study [8] and brought to the literature. One of them was examined in the [9] study for the curvature matrix of the curve-hypersurface binary. A feature that makes umbrella matrices unique and essential is that they are Lie group structures, so we have information about their Lie algebra structure. In addition, the Lie group of this matrix group has been studied in detail in [8, 10] studies.
This paper begins with a section of preliminaries presented to make the paper as self-contained as possible. Some significant results of the Lie group structure and Lie algebra of Umbrella matrices are presented in section 3. In the study in [5], the complement space dimension of the additive part of the (3×3)- type skew-symmetric matrix set was

calculated. However, for $n > 3$, this situation has not been examined. In section 4, we generalize this situation for all $(n \times n)$ type matrices using a different method. Finally, we identify an isomorphism from the $A(n)$ Umbrella matrix Lie subgroup to the H PC matrix Lie subgroup.

2. Preliminaries

Firstly, let us give this section's necessary definitions and theorems.

Defination 2.1. A is an orthogonal matrix, for which -1 is not an eigenvalue, may be written as

$$A = (I_n - B)^{-1}(I_n + B)$$

in which B is skew-symmetric matrix; Cayley Formula. [11]

Defination 2.2. Let A orthogonal matrix. If

$$AS = S$$

then A is called an umbrella matrix, where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$. [8]

Defination 2.3. Let V be given as vector space and

$$[\ ,\ ]: \quad V \times V \to \quad V$$

transformation as

- 2-linear
- $[X,Y] = -[Y,X], \ \forall X,Y,Z \in V$
- $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \ \forall X,Y,Z \in V.$

The $[,]$ transform is called a Lie operator on V, and the vector space V is called a Lie algebra. [12]

Defination 2.4. Let V be a Lie algebra and W a subspace of V such that $[X,Y] \in W$ for all $X,Y \in W$; then W is called a subalgebra of V. [13]

Teorem 2.1. Let $A(n)$ is the set of umbrella matrices, $O(n)$ is the set of orthogonal matrices and $SO(n)$ is the set of orthogonal matrices whose determinant is not -1, then

- $A(n) \subseteq O(n)$ is subgroup.
- $A(n) \subseteq SO(n)$ is subgroup.
- $A(n)$ and $A(n,\mathbb{C})$ are the Lie group. [8]

3. Umbrella Matrices Lie Algebra of the Lie Group

In the paper [8], the Lie algebra of the Lie group of $A(n)$ matrices is calculated. We will obtain when use a different approach the Lie algebra of the Lie group of umbrella matrices in the following theorem, assuming that g is the Lie algebra of $(n \times n)$-type skew-symmetric matrices.

Teorem 3.1. Let $B(n)$ be the space of skew-symmetric matrices whose row sums are zero

$$B(n) = \{B \in g \mid BS = 0,\ S = \begin{bmatrix} 1 & 1 \ldots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n\}$$

then, $B(n)$ is the Lie algebra of the Lie group $A(n)$.

Proof. Let $A \in A(n)$. Therefore, we have

$$AS = S$$

where $AA^T = I_n$ and $S = \begin{bmatrix} 1 & 1 \ldots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n$. We consider the curve $A(t)$ as a curve in the Lie group $A(n)$. Hence, let $A(0) = I_n$ is curve for $t = 0$.

$$A(t)A(t)^T = I_n. \tag{1}$$

Differentiating Eq.(1) with respect to t, we have

$$A^{'}(t)A(t)^T + A(t)A'(t)^T = 0.$$

For $t = 0$, we obtain

$$A^{'}(0) + A'(0)^T = 0. \tag{2}$$

Therefore, due to Eq.(2), $A^{'}(0)$ is skew-symmetric matrix. In addition to differentiating $A(t)S = 0$ with respect to t, we have

$$A^{'}(t)S = 0. \tag{3}$$

Thus, we have write $t = 0$ in Eq.(3) and we get

$$A^{'}(0)S = 0 \tag{4}$$

Consequently, because of the equations Eq.(3) and Eq.(4), the elements of A(n) Lie group Lie algebra are skew-symmetric matrices whose row sums are zero.

Proposition 3.1. The Lie algebra of the space of matrices $B(n)$ is the Lie subalgebra of g.

Proof. Let $B_1, B_2 \in B(n)$. Then,

$$B_1 S = 0 \tag{5}$$

$$B_2 S = 0 \tag{6}$$

where $S = \begin{bmatrix} 1 & 1 \dots & 1 \end{bmatrix}^T \in \mathbb{R}^n_1$. Therefore, from Eq.(5) and (6), we can write

$$\begin{aligned}(B_1 + B_2)S &= B_1 S + B_2 S \\ &= 0.\end{aligned}$$

So, $(B_1 + B_2) \in B(n)$.

- For $c \in \mathbb{R}$ and $B \in B(n)$, we can see easily that $cB \in B(n)$.
- For the Lie brackets $[\ ,\ ]$, we have

$$\begin{aligned}[\ ,\ ]: \quad B(n) \times B(n) &\to \quad B(n) \\ (A,B) \quad &\to \quad [A,B] = AB - BA.\end{aligned}$$

  Hence, we obtain

$$(AB - BA)S = S.$$

  Consequently, $B(n)$ is the Lie subalgebra of g.

Teorem 3.2. The dimension of the A(n) Lie Group is $\frac{(n-1)(n-2)}{2}$. [8]

Proof. We know that the dimension of a manifold is equal to the dimension of the tangent space. Then, let $I_n \in A(n)$ and $\{x_{11}, x_{21}, ..., x_{nn}\}$ be a coordinate system for a coordinate neighborhood of I_n. Accordingly, the set

$$\psi = \{\psi_{ij}(I_n) \mid \ \psi_{ij} = E_{ij} - E_{ji} - E_{in} + E_{ni} + E_{jn} - E_{nj},\ 1 \le i \le j < n\}$$

is a base of set $T_{A_n}(I_n)$ and where the $E = \{E_{ij} \in GL(n, \mathbb{R}) \mid E_{ij} = \frac{\partial}{\partial x_{ij}},\ 1 \le i \le j < n\}$ set denotes the standard base of $\mathbb{R}^n_n$. Therefore, we can write

$$T_{A(n)}(I_n) = \frac{(n-1)(n-2)}{2}$$

and since dim$T_{A(n)}(I_n)$=dim$A(n)$. Thus we obtain

$$dimA(n) = \frac{(n-1)(n-2)}{2}.$$

4. PC Matrices and Umbrella Matrices

Let us assume that

$$G = \{A = [a_{ij}]_{n\times n} \mid A \odot A^T = I,\ a_{ij} > 0,\ 1 \le i, j \le n\}$$

is the abelian PC matrices of type $(n \times n)$. Operation of this abelian group is defined as

$$\begin{array}{rccl} \odot : & G \times G & \longrightarrow & G \\ & (A,B) & \longrightarrow & \odot(A,B) = A \odot B = [a_{ij}b_{ij}] \end{array}$$

where $''\odot''$ is the Hadamart product. The Lie algebra of the G Lie group is g, which is the space of skew-symmetric matrices. Hence, let us consider the space of

$$C_g = \{A \in g \mid a_{ik} + a_{kj} = a_{ij},\ 1 \le i, j, k \le n\}$$

additive consistent matrices. C_g is the Lie subgroup of g and we can write the equation

$$g = C_g \oplus C_g^{\perp}.$$

In this equation, the $C_g^{\perp}$ space is called the orthogonal complement space of g and is written as

$$C_g^{\perp} = \{B \in g \mid BS = 0,\ S = \begin{bmatrix} 1 & 1\ldots & 1 \end{bmatrix}^T \in \mathbb{R}_1^n\}.$$

Proposition 4.1. Let $C_g^{\perp}$ be the orthogonal complement of g and $B(n)$ be the Lie algebra of the Lie group A(n) . Then,

$$C_g^{\perp} = B(n).$$

Proof. From Teo.(3.1), we can easily see $C_g^{\perp} = B(n)$.

We will examine the orthogonal complement of a (3×3)-type skew-symmetric matrix with the following example.

Example 4.1. Let us consider the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}.$$

From here, we can write the skew-symmetric matrix A as follows

$$A = \begin{pmatrix} 0 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} & \frac{8}{3} \\ -\frac{1}{3} & 0 & \frac{7}{3} \\ -\frac{8}{3} & -\frac{7}{3} & 0 \end{pmatrix}.$$

Then,

$$B = \begin{pmatrix} 0 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{pmatrix}$$

matrix is the orthogonal complement of A. Therefore, we can obtain $B \in C_g^{\perp}$.

From [5], is known to be $\dim C_g^{\perp} = 1$ and $C_g^{\perp}$=$span\left\{ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \right\}$ for matrices of type (3×3). We will generalize this situation with the following theorem.

Teorem 4.1. Let $C_g^{\perp}$ be the complement part of the space of skew-symmetric matrices of g. Then,

$$C_g^{\perp} = span\{\psi_{ij}\}$$

for

$$\psi = \{\psi_{ij}(I_n) \mid \ \psi_{ij} = E_{ij} - E_{ji} - E_{in} + E_{ni} + E_{jn} - E_{nj}, \ 1 \leq i \leq j < n\}$$

where the $E = \{E_{ij} \in GL(n,\mathbb{R}) \mid E_{ij} = \frac{\partial}{\partial x_{ij}}, \ 1 \leq i \leq j < n\}$ set denotes the standard base of $\mathbb{R}_n^n$. Furthermore, $dim C_g^{\perp} = \frac{(n-1)(n-2)}{2}$.

Proof. From the proposition(4.1), we know that $C_g^{\perp} = B(n)$. Due to the theorem(3.2), the elements of the set ψ become a base of the $C_g^{\perp}$ space. Additionally, since $C_g^{\perp} = dim B_n$, we get

$$dim C_g^{\perp} = \frac{(n-1)(n-2)}{2}.$$

Example 4.2. According to theorem(4.1) for $n = 3$,we may write that $\dim C_g^{\perp} = 1$ and $C_g^{\perp} = span\{\psi_{12}\}$. Therefore, we have

$$\psi_{12} = E_{12} - E_{21} - E_{13} + E_{31} + E_{23} - E_{32}$$

and we obtained

$$\psi_{12} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

In addition, also for $n = 4$, we have $\dim C_g^{\perp} = 3$ and $C_g^{\perp} = span\{\psi_{12}, \psi_{13}, \psi_{23}\}$. Thus, we may write

$$\begin{aligned} \psi_{12} &= E_{12} - E_{21} - E_{14} + E_{41} + E_{24} - E_{42} \\ \psi_{13} &= E_{13} - E_{31} - E_{14} + E_{41} + E_{34} - E_{43} \\ \psi_{23} &= E_{23} - E_{32} - E_{24} + E_{42} + E_{34} - E_{43} \end{aligned}$$

We can therefore write them as

$$\psi_{12} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \psi_{13} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \psi_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Proposition 4.2. Let C_g, g be the space of additive consistent matrices of the space of skew-symmetric matrices. Then,

$$dimC_g = n - 1.$$

We will give the relationship between Lie groups with the following theorem.

Proof. Since it is $g = C_g \oplus C_g^{\perp}$, we write

$$dimg = dimC_g + dimC_g^{\perp}.$$

From here, considering $dimg = \frac{n(n-1)}{2}$ and theorem(4.1), we may write

$$\frac{n(n-1)}{2} = dimC_g + \frac{(n-1)(n-2)}{2}.$$

Consequently, we can easily calculate that it is $dimC_g = n - 1$.

Teorem 4.2. Let $H \subset G$ be the Lie subgroup and

$$H = \{N \in G \mid N \odot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\}$$

where $\odot$ is the Hadamart product. Then there is an isomorphism from the $A(n)$ Lie subgroup to the H Lie subgroup, so $A(n) \cong H$.

Proof. Let $\mathscr{C} : B(n) \longrightarrow A(n)$ is Cayley transformation and $e : B(n) \longrightarrow H$ is exponential transformation. Then,

$$\begin{tikzcd}
\boxed{A(n)} \arrow[r, "f"] & \boxed{H} \\
\boxed{B(n)} \arrow[u, "\mathscr{C}"] \arrow[ur, "e"'] &
\end{tikzcd}$$

considering the diagram above, we define

$$f : A(n) \longrightarrow H$$

where $f = e \circ \mathscr{C}^{-1}$. We aim to show that f is an isomorphism. For this, first of all, we can easily see that f is 1-1 and surjective, thanks to the transformations of $\mathscr{C}$ and e being 1-1 and surjective. Secondly, we say that f is a homomorphism since the transformations of $\mathscr{C}$ and e are homomorphisms. Consequently, f transform is an isomorphism, and $A(n) \cong H$.

Example 4.3. Let us consider the umbrella matrix

$$A = \frac{1}{21}\begin{pmatrix} 5 & 20 & -4 \\ -4 & 5 & 20 \\ 20 & -4 & 5 \end{pmatrix}.$$

If the inverse Cayley formula is applied to the A matrix, we get

$$B = \begin{pmatrix} 0 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{pmatrix}$$

matrix in example 4.1. Then, when we apply the exponential transformation to the B matrix, we may find that

$$N = \begin{pmatrix} 1 & e^{\frac{2}{3}} & e^{-\frac{2}{3}} \\ e^{-\frac{2}{3}} & 1 & e^{\frac{2}{3}} \\ e^{\frac{2}{3}} & -e^{\frac{2}{3}} & 1 \end{pmatrix}.$$

Hence, $N \in H$.

Likewise, we can correspond an element from Lie subgroup H with an element from Lie subgroup $A(n)$ using the transform $f = \mathscr{C} \circ e^{-1}$.

5. Conclusion

$A(n)$ is the Lie algebra of the umbrella matrix Lie group, while $B(n)$ is the space of skew-symmetric matrices whose rows sum to zero. The consistent additive PC is the orthogonal complement space of these matrices. This study has established a relationship between PC matrices and Umbrella matrices for the first time. Additionally, we have provided an isomorphism between the Lie group of PC matrices and the Lie group of $A(n)$ Umbrella matrices.

References

[1] R. Bala, B. Ram, *Trigonometric series with semi-convex coefficients*, Tamang J. Math. 18, 1 (1987), 75-84.

[2] J. Barzilai, *Consistency measures for pairwise comparison matrices*, J.Multi-Crit.Decis.Anal. **7** (1998), 75-174.

[3] M. Bekar and Y. Yayli, *Lie algebra of unit tangent bundle*, Advances in Appl. Clifford Algeb. **27** (2017), 965-975.

[4] F. K. Aksoyak and Y. Yayli, *Homothetic Motions and Lie Groups In*, J. of Dyna. Systems and Geo. Theories **11** (2013), 23-38.

[5] W. W. Koczkodaj, V. W. Marek, Y. Yayli, *Pairwise comparisons matrix decomposition into approximation and orthogonal component using lie theory*, Inter. J. of Appr. Reasoning **139** (2021), 201–210.

[6] W. W. Koczkodaj, F. Liu, V. W. Marek, J. Mazurek, M. Mazurek, L. Mikhailov, C. Ozel, W. Pedrycz, A. Przelaskowski, A. Schumann, et al., *On the use of group theory to generalize elements of pairwise comparisons matrix: a cautionary note*, Inter. J. of Appr. Reasoning **124** (2020), 59–65.

[7] I. Gok, O. Zeki Okuyucu, N. Ekmekci, Y. Yayli, *On mannheim partner curves in three dimensional lie groups*, Miskolc Mathem. Notes **15** (2014).

[8] E. Ozdamar, Ph.D. thesis, Ankara University, Ankara, Turkey, (1977).

[9] E. Esin, *Umbrella matrices and higher curvatures of a curve*, Commun. Fac. of Sci. Uni. of Ankara Seri. A1 Math. and Stat. **35** (1986), 28–34.

[10] N. Kuruoglu, *On the lie group of umbrella matrices*, Commun. Fac. of Sci. Uni. of Ankara Seri. A1 Math. and Stat. **32** (1983), 132–144.

[11] O. Bottema, B. Roth, *Theoretical kinematics*, Vol. **24**, Courier Corporation, 9-11, 1990.

[12] H.H. Hacısalıhoglu, *Differansiyel Geometri*, Vol. **1**, 62-63, 1998.

[13] P. M. Cohen, *Lie Groups*, Vol. **46**, Cambridge University Press, 68-69, 1968.

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