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# CHARACTERISTICS OF GENERALIZED WEAKLY $\phi$-SYMMETRIC KENMOTSU MANIFOLDS

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Abstract. The present paper attempt to introduce the notion of generalized weakly $\phi$-symmetric and generalized weakly Ricci $\phi$-symmetric Kenmotsu Manifolds. A generalized weakly $\phi$-symmetric and a generalized weakly Ricci $\phi$-symmetric Kenmotsu manifold is an $\eta$-Einstein manifolds. We also find out the curvature conditions for which the Riemann soliton is expanding, steady or shrinking. Finally the existence of generalized weakly $\phi$-symmetric Kenmotsu manifold is ensured by a suitable example.

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## 1. Introduction

During the last seven decades, the notion of locally symmetric manifolds by Cartan ([9]) continue to have been weakened by many authors in several ways such as recurrent manifold by Walker ([32]), semi-symmetric manifold by Szabò ([27]), Chaki's pseudo-symmetric manifold ([10]), Chaki's pseudo Ricci-symmetric manifold ([12]), Almost pseudo Ricci symmetric Manifolds *([11]),* Deszcz's pseudo-symmetric manifold ([14]), weakly symmetric manifold in the sense of Tamássy and Binh ([30]), generalized weakly symmetric manifold ([2]), Almost generalized Pseudo Ricci Symmetric Manifolds ([1]), hyper generalized weakly symmetric manifold ([7]) and so on.

The notion of locally $\phi$-symmetric Sasakian manifolds was introduced by Takahashi ([31]). Thereafter, the study of such notion has been weakened by the authors in ([22], [17], [23], [24], [25], [26], [28], [18], [22] and the references therein). Thereafter, Hui ([15]) studied $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric Kenmotsu manifolds (or $KM$). Recently, the authors ([6]) introduced and studied generalized weakly $\phi$-symmetric and generalized weakly Ricci $\phi$-symmetric $LP$ Sasakian manifolds.

**Definition 1.** *For the 1-forms* $A(X) = g(X, \pi_1)$, $B(X) = g(X, \pi_2)$, $D(X) = g(X, \varrho)$, $\alpha(X) = g(X, \delta_1)$, $\beta(X) = g(X, \delta_2)$ *and* $\gamma(X) = g(X, \sigma)$ *a* $KM$ *is said to be generalized weakly* $\phi$*-symmetric if* $R$ *admits the equation*

$$\begin{aligned} & \phi^2\left((\nabla_X R)(Y,U)V\right) \\ = \; & A(X)R(Y,U)V + \alpha(X)G(Y,U)V \\ & +B(Y)R(X,U)V + \beta(Y)G(X,U)V \\ & +B(U)\bar{R}(Y,X)V + \beta(U)\ G(Y,X)V \\ & +D(V)\bar{R}(Y,U)X + \gamma(V)\ G(Y,U)X \\ & +g(R(Y,U)V,\ X)\varrho + \ g(G(Y,U)V, X\ )\sigma. \end{aligned} \tag{1}$$

*The beauty of such* $(\phi GWS)_n$*-Manifold is that it has the flavour of*

(i) locally $\phi$-recurrent space ([32]) (for $A \neq 0$, $B = D = \alpha = \beta = \gamma = 0$),
(ii) locally $\phi$-symmetric space ([9]) (for $A = B = D = 0 = \alpha = \beta = \gamma,$),
(iii) quasi $\phi$-recurrent space ([21]) ( $A \neq 0$, $B{=}D = 0$, $\alpha \neq 0, \beta = \gamma = (\beta^* {-} \gamma^*)\alpha$),
(iv) pseudo $\phi$-symmetric space ([15]) (for $\frac{A}{2} = B = D = H \neq 0$, $\alpha = \beta = \gamma = 0$),
(v) generalized $\phi$-recurrent space ([19]) (for $A \neq 0$, $\alpha \neq 0, B = D = \beta = \gamma = 0$),
(vi) semi-pseudo $\phi$-symmetric space ([29]) ( $A = \alpha = \beta = \gamma = 0, B = D \neq 0$),
(vii) generalized pseudo $\phi$-symmetric space ([4]) (for $\frac{A}{2} = B = D = H_1 \neq 0, \frac{\alpha}{2} = \beta = \gamma = H_2 \neq 0$),
(viii) generalized semi-pseudo $\phi$-symmetric space ([5]) ( $A = 0 = \alpha$, $B = D \neq 0$, $\beta = \gamma \neq 0$),
(ix) almost generalized pseudo $\phi$-symmetric space ([3]) ( $A = H_1 + K_1, B = D = H_1 \neq 0$, $\alpha = H_2 + K_2$, $\beta = \gamma = H_2 \neq 0$),
(x) almost pseudo $\phi$-symmetric space ([10]) (for $A = H_1 + K_1, B = D = H_1 \neq 0$ and $\alpha = \beta = \gamma = 0$),
(xi) weakly $\phi$-symmetric space ([30]) ( for $A$, $B$, $D \neq 0$, $\alpha = \beta = \gamma = 0$).

**Definition 2.** *For the 1-forms* $A^*(X) = g(X, \pi_1^*)$, $B^*(X) = g(X, \pi_2^*)$, $D^*(X) = g(X, \varrho^*)$, $\alpha^*(X) = g(X, \delta_1^*)$, $\beta^*(X) = g(X, \delta_2^*)$ *and* $\gamma(X) = g(X, \sigma^*)$ *a* $KM$ *is said to be generalized weakly Ricci* $\phi$*-symmetric if* $Q$ *admits the following*

$$\begin{aligned} & \phi^2\left((\nabla_X Q)(U)\right) \\ = \; & A^*(X)QU + \alpha^*(X)U + B^*(U)QX \\ & +\beta^*(U)X + S(U,X)\varrho^* + g(U,X)\sigma^*. \end{aligned} \tag{2}$$

**Definition 3.** *A* $KM$ $(M^n, g)$ *is said to be* $\eta$*-Einstein if the Ricci tensor* $S$ *of type* $(0, 2)$ *takes the form*

$$S = \delta g + \kappa \eta \otimes \eta,$$

*where* $\delta$ *and* $\kappa$ *are smooth functions.*

We represent our paper as follows: Section 2, is concerned with some known results of $KM$. In section 3, we have studied generalized weakly $\phi$-symmetric $KM$. We have observed that generalized weakly $\phi$-symmetric $KM$ is $\eta$-Einstein whereas for each of (i) $\phi$-symmetric, (ii) $\phi$-recurrent and (iii) generalized $\phi$-recurrent is an Einstein manifold. Section 4 is dealt with generalized weakly Ricci $\phi$-symmetric $KM$ and we observed that generalized weakly Ricci $\phi$-symmetric $KM$ is $\eta$-Einstein. Section 5 is concerned with generalized weakly (Ricci) $\phi$-symmetric $KM$ and the case of Riemann soliton. We also find out the curvature conditions for which the Riemann soliton is expanding, steady or shrinking. Finally, we have constructed an example of generalized weakly $\phi$-symmetric $KM$.

## 2. Preliminaries

According to the definition of Blair ([8]), an *almost contact structure* $(\phi, \xi, \eta)$ on a $n = (2m+1)$-dimensional Riemannian manifold satisfies the following conditions

$$\phi^2 = -I + \eta \otimes \xi, \tag{3}$$

$$\eta(\xi) = 1, \tag{4}$$

$$\phi\xi = 0,\ \eta \circ \phi = 0,\ \text{rank}\ \phi = \frac{n-1}{2}. \tag{5}$$

Moreover, if $g$ is a Riemannian metric on $M^n$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

$$g(X, \xi) = \eta(X), \tag{7}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{8}$$

for any vector fields $X$, $Y$ on $M^n$, then the manifold $M^n$ ([8]) is said to admit an *almost contact metric structure* $(\phi, \xi, \eta, g)$.

**Definition 4.** *([20]) If in an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M^n$, the Riemann connection $\nabla$ of $g$ satisfies $(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields $X$, $Y$ on $M^n$, then the structure is called Kenmotsu.*

**Proposition 1.** *([20]) If $(M^n, \phi, \xi, \eta, g)$ is a $KM$, then for any vector fields $X$, $Y$, $Z$ on $M^n$, the following relations hold*

$$\nabla_X\xi = X - \eta(X)\xi, \tag{9}$$

$$(\nabla_X\eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{10}$$

$$S(X,\xi) = -(n-1)\eta(X), or\ Q\xi = -(n-1)\xi, \tag{11}$$
$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{12}$$
$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{13}$$
$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{14}$$
$$(\nabla_Z R)(X,Y)\xi = g(X,Z)Y - g(Y,Z)X - R(X,Y)Z. \tag{15}$$

## 3. Generalized weakly $\phi$-symmetric $KM$

We consider a generalized weakly $\phi$-symmetric $KM$. Then from (3), the equation (1) gives

$$\begin{aligned}
&-(\nabla_X R)(Y,U)V + \eta((\nabla_X R)(Y,U)V)\xi \\
= \ & A(X)R(Y,U)V + B(Y)R(X,U)V + B(U)R(Y,X)V \\
&+D(V)R(Y,U)X + g(R(Y,U)V,\ X)\varrho + \alpha(X)G(Y,U)V \\
&+\beta(Y)G(X,U)V + \beta(U)G(Y,X)V \\
&+\gamma(V)G(Y,U)X + g(G(Y,U)V,\ X)\sigma.
\end{aligned} \tag{16}$$

Introducing $V=\xi$ in the above equation and using (12) and (15), we obtain

$$\begin{aligned}
&\{1-D(\xi)\}R(X,Y,Z,U) \\
= \ &\{1-\gamma(\xi)\}[g(X,Z)g(U,Y) - g(X,U)g(Y,Z)] \\
&+\{A(X)-\alpha(X)\}[g(U,Y)\eta(Z) - g(Y,Z)\eta(U)] \\
&+\{D(Y)-\gamma(Y)\}[g(X,U)\eta(Z) - g(X,Z)\eta(U)] \\
&+\{B(Z)-\beta(Z)\}[g(U,Y)\eta(X) - g(X,Y)\eta(U)] \\
&+\{B(U)-\beta(U)\}[g(X,Y)\eta(Z) - g(Y,Z)\eta(X)].
\end{aligned} \tag{17}$$

This leads to the following:

**Theorem 1.** *The form of Riemann curvature tensor in a generalized weakly $\phi$-symmetric $KM$ is*

$$\begin{aligned}
&\{1-D(\xi)\}R(X,Y,Z,U) \\
= \ &\{1-\gamma(\xi)\}[g(X,Z)g(U,Y) - g(X,U)g(Y,Z)] \\
&+\hat{A}(X)[g(U,Y)\eta(Z) - g(Y,Z)\eta(U)] \\
&+\hat{D}(Y)[g(X,U)\eta(Z) - g(X,Z)\eta(U)] \\
&+\hat{B}(Z)[g(U,Y)\eta(X) - g(X,Y)\eta(U)] \\
&+\hat{B}(U)[g(X,Y)\eta(Z) - g(Y,Z)\eta(X)].
\end{aligned}$$

*for* $(A-\alpha)=\hat{A}$, $(D-\gamma)=\hat{D}$, $(B-\beta)=\hat{B}$.

However, for the choice of $\alpha=\beta=\gamma=0$ (17) yields

$$\begin{aligned}
& \{1-D(\xi)\}R(Y,U,X,Z) \\
= & \{g(X,Y)g(U,Z)-g(X,U)g(Y,Z)\} \\
& +A(X)\{g(U,Z)\eta(Y)-g(Y,Z)\eta(U)\} \\
& +B(Y)\{g(U,Z)\eta(X)-g(X,Z)\eta(U)\} \\
& +B(U)\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\} \\
& +D(Z)\{g(X,U)\eta(Y)-g(X,Y)\eta(U)\}.
\end{aligned} \tag{18}$$

**Corollary 2.** *For a weakly $\phi$-symmetric $KM$ the Riemann curvature tensor satisfied the relation (18).*

**Corollary 3.** *A $\phi$-symmetric $KM$ is of constant curvature $-1$.*

Again, from (16), we get

$$\begin{aligned}
& -g((\nabla_X R)(Y,U)V,W)+\eta((\nabla_X R)(Y,U)V)\eta(W) \\
= & A(X)g(R(Y,U)V,W)+B(Y)g(R(X,U)V,W) \\
& +B(U)g(R(Y,X)V,W)+D(V)g(R(Y,U)X,W) \\
& +D(W)g(R(Y,U)V,\ X)+\alpha(X)g(G(Y,U)V,W) \\
& +\beta(Y)g(G(X,U)V,W)+\beta(U)g(G(Y,X)V,W) \\
& +\gamma(V)g(G(Y,U)X,W)+\gamma(W)g(G(Y,U)V,\ X).
\end{aligned} \tag{19}$$

Now, taking an orthonormal frame field and then contracting (19) over $Y$ and $W$ we obtain

$$\begin{aligned}
& -(\nabla_X S)(U,V)+(\nabla_X \bar{R})(\xi,U,V,\xi) \\
= & A(X)S(U,V)+(n-1)\alpha(X)g(U,V) \\
& +B(U)S(X,V)+(n-1)\beta(U)g(X,V) \\
& +D(V)S(U,X)+(n-1)\gamma(V)g(U,X) \\
& +B(R(X,U)V)+\beta(G(X,U)V) \\
& +D(R(X,V)U)+\gamma(G(X,V)U).
\end{aligned} \tag{20}$$

Using (14) and (15), one can easily bring out

$$(\nabla_X R)(\xi,U,V,\xi)=0 \tag{21}$$

By virtue of (21), the equation (20) yields

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = & -A(X)S(U,V) - B(U)S(X,V) - D(V)S(U,X) \\ & -(n-1)\{\alpha(X)g(U,V) + \beta(U)g(X,V) + \gamma(V)g(U,X)\} \\ & -B(R(X,U)V) - D(R(X,V)U) - \beta(G(X,U)V) - \gamma(G(X,V)U). \end{aligned} \tag{22}$$

**Theorem 4.** *A generalized weakly $\phi$-symmetric $KM$ is a generalized weakly Ricci-symmetric provided $B(R(X,U)V)+D(R(X,V)U)+\beta(G(X,U)V)+\gamma(G(X,V)U) = 0$.*

**Corollary 5.** *A weakly $\phi$-symmetric $KM$ is a weakly Ricci-symmetric if $D(\xi) = 0$.*

**Corollary 6.** *A $\phi$-symmetric $KM$ is a Ricci-symmetric.*

Now, setting $V = \xi$ in (22) we get

$$\begin{aligned} & (\nabla_X S)(U,\xi) \\ = & -(n-1)[A(X) + \alpha(X)]\eta(U) \\ & -(n-2)[B(U) + \beta(U)]\eta(X) \\ & -(n-2)\gamma(\xi)g(U,X) \\ & +[B(X) - \beta(X)]\eta(U) \\ & +[D(X) - \gamma(X)]\eta(U) \\ & -D(\xi)[S(U,X) + g(U,X)]. \end{aligned} \tag{23}$$

Again, from the relation

$$(\nabla_X S)(U,V) = \nabla_X S(U,V) - S(\nabla_X U,V) - S(U,\nabla_X V).$$

we obtain

$$(\nabla_X S)(U,\xi) = -(n-1)g(X,U) - S(X,U). \tag{24}$$

for $V = \xi$. In view of (24), equation (23) becomes

$$\begin{aligned} & (n-1)g(X,U) + S(X,U) \\ = & (n-1)A(X)\eta(U) + (n-2)B(U)\eta(X) + D(\xi)S(U,X) \\ & +(n-1)\alpha(X)\eta(U) + (n-2)\beta(U)\eta(X) + (n-2)\gamma(\xi)g(U,X) \\ & -B(X)\eta(U) - D(X)\eta(U) + D(\xi)g(U,X) \\ & +\beta(X)\eta(U) + \gamma(X)\eta(U). \end{aligned} \tag{25}$$

Putting $U=\xi$ , $X=\xi$ and $U=X=\xi$ in (25) successively we obtain

$$\begin{aligned} & (n-1)A(X)+(n-1)\alpha(X)-B(X)-D(X)+\beta(X)+\gamma(X) \\ = \; & (n-2)[B(\xi)-D(\xi)+\beta(\xi)+\gamma(\xi)]\eta(X), \end{aligned} \tag{26}$$

$$\begin{aligned} & (n-2)[B(U)+\beta(U)] \\ = \; & \eta(U)[(n-1)\{A(\xi)-D(\xi)+\alpha(\xi)+\gamma(\xi)\}-B(\xi)+\beta(\xi)], \end{aligned} \tag{27}$$

and

$$(n-1)[A(\xi)+\alpha(\xi)-D(\xi)+\gamma(\xi)+\beta(\xi)]=(n-3)B(\xi). \tag{28}$$

respectively.

Now using (26), (27) and (28) in (25) we get

$$\begin{aligned} & S(X,U) \\ = \; & \frac{D(\xi)+(n-2)\gamma(\xi)-(n-1)}{1-D(\xi)}g(X,U) \\ & +\frac{2(n-3)B(\xi)+(n-2)\{\gamma(\xi)-D(\xi)\}}{1-D(\xi)}\eta(X)\eta(U). \end{aligned} \tag{29}$$

This leads to the following:

**Theorem 7.** *A generalized weakly $\phi$-symmetric $KM$ is an $\eta$-Einstein manifold.*

**Corollary 8.** *A $KM$ $(M^n,g)$ for each of (i) $\phi$-symmetric, (ii) $\phi$-recurrent and (iii) generalized $\phi$-recurrent is an Einstein manifold.*

## 4. Generalized weakly Ricci $\phi$-symmetric $KM$

In this section we consider a generalized weakly Ricci $\phi$-symmetric $KM$. Then by the virtue of (3), (2) yields

$$\begin{aligned} & -(\nabla_X Q)(U)+\eta((\nabla_X Q)(U))\xi \\ = \; & A^*(X)QU+\alpha^*(X)U \\ & +B^*(U)QX+\beta^*(U)X \\ & +S(U,X)\varrho^*+g(U,X)\sigma^*. \end{aligned} \tag{30}$$

from which it follows that

$$\begin{aligned} & -g(\nabla_X Q(U),V)+S(\nabla_X U,V)+\eta((\nabla_X Q)(U))\eta(V) \\ = \; & A^*(X)S(U,V)+B^*(U)S(V,X)+D^*(V)S(U,X) \\ & +\alpha^*(X)g(U,V)+\beta^*(U)g(V,X)+\gamma^*(V)g(U,X). \end{aligned} \tag{31}$$

Putting $U=\xi$ in (31) and using (9), (11) we get

$$\begin{aligned}&(n-1)g(X,V)+S(X,V)\\=\;&-(n-1)A^*(X)\eta(V)+B^*(\xi)S(V,X)-(n-1)D^*(V)\eta(X)\\&+\alpha^*(X)\eta(V)+\beta^*(\xi)g(V,X)+\gamma^*(V)\eta(X).\end{aligned}\tag{32}$$

Setting $X=V=\xi$, $X=\xi$ and $V=\xi$ successively in (32) we get

$$\gamma^*(\xi)-(n-1)D^*(\xi)=\alpha^*(\xi)+\beta^*(\xi)-(n-1)\{A^*(\xi)+B^*(\xi)\}.\tag{33}$$

and

$$\gamma^*(V)-(n-1)D^*(V)=[\alpha^*(\xi)+\beta^*(\xi)-(n-1)\{A^*(\xi)+B^*(\xi)\}]\eta(V).\tag{34}$$

and

$$\alpha^*(X)-(n-1)A^*(X)=[\beta^*(\xi)+\gamma^*(\xi)-(n-1)\{B^*(\xi)+D^*(\xi)\}]\eta(X).\tag{35}$$

respectively. Next using (33), (34) and (35) in (32), we get

$$\begin{aligned}&S(X,V)+\frac{(n-1)-\beta^*(\xi)}{1-B^*(\xi)}g(X,V)\\=\;&\frac{2\gamma^*(\xi)+\beta^*(\xi)-(n-1)[2D^*(\xi)+B^*(\xi)]}{1-B^*(\xi)}\eta(X)\eta(V).\end{aligned}\tag{36}$$

Thus we infer that

**Theorem 9.** *A generalized weakly Ricci $\phi$-symmetric $KM$ is an $\eta$-Einstein manifold.*

**Corollary 10.** *A weakly Ricci $\phi$-symmetric $KM$ is an $\eta$-Einstein manifold.*

**Corollary 11.** *A Ricci $\phi$-symmetric $KM$ is an Einstein manifold.*

## 5. Generalized weakly (Ricci) $\phi$-symmetric $KM$ and the Riemann solitons

In this section, we study the $KM$ with generalized weakly $\phi$-symmetric curvature conditions when the metric $g$ is Riemann soliton. Hirica and Udrisste [16] in 2016 introduced and studied Riemann soliton. A smooth manifold $M$ with Riemannian metric $g$ is called Riemann soliton if $g$ satisfies

$$2R+\lambda(g\wedge g)+(g\wedge \pounds_{\xi}g)=0\tag{37}$$

where $\xi$ is a potential vector field, $\pounds_\xi$ denotes the Lie-derivative and $\lambda$ is a constant and for $(0,2)$-tensors $\alpha$ and $\beta$, the Kulkarni-Nomizu product $(\alpha \wedge \beta)$ is given by

$$\begin{aligned} & (\alpha \wedge \beta)(Y,U,V,Z) \\ = \; & \alpha(Y,V)\beta(U,Z) + \alpha(U,Z)\beta(Y,V) \\ & -\alpha(Y,Z)\beta(U,V) - \alpha(U,V)\beta(Y,Z). \end{aligned} \tag{38}$$

A Riemann soliton is called expanding, steady and shrinking when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Now, using (38) in (37) and then contracting $U$ over $V$ we obtain

$$S(Y,Z) = \{(n-1)\lambda + (2n-3)\}g(Y,Z) - (n-2)\eta(Y)\eta(Z). \tag{39}$$

Comparing (29) and (39), we obtain

$$(n-1)\lambda = \frac{2(n-1)D(\xi) + (n-2)\gamma(\xi) - 3n + 4}{1 - D(\xi)}.$$

Thus we infer that

**Theorem 12.** *In a generalized weakly $\phi$-symmetric $KM$ the Riemann soliton is expanding, steady or shrinking according as* $2(n-1)D(\xi) + (n-2)\gamma(\xi) >=< 3n-4$.

**Corollary 13.** *In a weakly $\phi$-symmetric $KM$ the Riemann soliton is expanding, steady or shrinking according as* $2(n-1)D(\xi) >=< 3n-4$.

Again, comparing (36) and (39), we obtain

$$(n-1)\lambda = \frac{(2n-3)B^*(\xi) + \beta^*(\xi) - (3n-4)}{1 - B^*(\xi)}.$$

This leads to the following:

**Theorem 14.** *In a generalized weakly Ricci $\phi$-symmetric $KM$ the Riemann soliton is expanding, steady or shrinking according as* $(2n-3)B^*(\xi) + \beta^*(\xi) >=< (3n-4)$.

**Corollary 15.** *In a weakly Ricci $\phi$-symmetric $KM$ the Riemann soliton is expanding, steady or shrinking according as* $(2n-3)B^*(\xi) >=< (3n-4)$.

## 6. Example of a generalized weakly $\phi$-symmetric $KM$

(See [13]) Let $M^3(\phi,\xi,\eta,g)$ be a Kenmotsu manifold $(M^3,g)$ with a $\phi$-basis

$$e_1 = e^{-z}\frac{\partial}{\partial x},\; e_2 = e^{-z}\frac{\partial}{\partial y},\; e_3 = \frac{\partial}{\partial z}.$$

Then from Koszul's formula for Riemannian metric $g$, we obtain the Levi-Civita connection as follows

$$\begin{aligned} \nabla_{e_1} e_3 &= e_1, & \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 &= e_2, & \nabla_{e_2} e_2 = -e_3, && \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 &= 0, & \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $R$ (up to symmetry and skew-symmetry)

$$\begin{aligned} R(e_1, e_3)e_1 &= e_3, & R(e_2, e_3)e_2 = e_3, R(e_1, e_2)e_1 = e_2 \\ R(e_1, e_3)e_3 &= -e_1, & R(e_2, e_3)e_3 = -e_2, R(e_1, e_2)e_2 = -e_1. \end{aligned}$$

Since $\{e_1, e_2, e_3\}$ is a basis, any vector field $X$, $Y$, $U$, $V \in \chi(M)$ can be written as

$$X = \sum_1^3 a_i e_i,\ Y = \sum_1^3 b_i e_i,\ Z = \sum_1^3 c_i e_i.$$

This implies that

$$R(X, Y)Z = le_1 + me_2 + ne_3,$$

where

$$\begin{aligned} l &= [(a_1b_3 - a_3b_1)c_3 - (a_1b_2 - a_2b_1)c_2], \\ m &= [(a_2b_3 - a_3b_2)c_3 + (a_1b_2 - a_2b_1)c_1], \\ n &= -[(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2] \end{aligned}$$

Also,

$$\begin{aligned} R(e_1, Y)Z &= \{b_3c_3 - b_2c_2\}e_1 + b_2 c_1e_2 - b_3c_1e_3, \\ R(e_2, Y)Z &= b_1c_2e_1 + \{b_3 c_3 - b_1c_1\}e_2 + b_3 c_2e_3, \\ R(e_3, Y)Z &= -b_1c_3e_1 - b_2c_3e_2 + (b_1c_1 + b_2c_2)e_3, \\ R(X, e_1)Z &= -\{a_3c_3 - a_2c_2\}e_1 + a_2 c_1e_2 - a_3c_1e_3, \\ R(X, e_2)Z &= a_1c_2e_1 + \{a_3c_3 - a_1c_1\}e_2 + a_3c_2e_3, \\ R(X, e_3)Z &= a_1c_3e_1 - a_2c_3e_2 + (a_1c_1 + a_2c_2)e_3, \\ R(X, Y)e_1 &= (a_1b_2 - a_2b_1)e_2 - (a_1b_3 - a_3b_1)e_3, \\ R(X, Y)e_2 &= -(a_1b_2 - a_2b_1)e_1 - (a_2b_3 - a_3b_2)e_3, \\ R(X, Y)e_3 &= (a_1b_3 - a_3b_1)e_1 + (a_2b_3 - a_3b_2)e_2, \end{aligned}$$

Again,

$$G(X,Y)Z = pe_1 + qe_2 + re_3,$$

where

$$\begin{aligned}
p &= (a_1b_2 - a_2b_1)c_2 + (a_3b_1 - a_1b_3)c_3,\\
q &= (a_2b_1 - a_1b_2)c_1 + (a_3b_2 - a_2b_3)c_3,\\
r &= (a_3b_1 - a_1b_3)c_1 + (a_3b_2 - a_2b_3)c_2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
G(e_1,Y)Z &= (b_2c_2 - b_3c_3)e_1 - b_2c_1e_2 - b_3c_1e_3,\\
G(e_2,Y)Z &= -b_1c_2e_1 + (b_1c_1 - b_3c_3)e_2 - b_3c_2e_3,\\
G(e_3,Y)Z &= b_1c_3e_1 + b_2c_3e_2 - (b_1c_1 + b_2c_2)e_3,\\
G(X,e_1)Z &= -(a_2c_2 - a_3c_3)e_1 + a_2c_1e_2 + a_3c_1e_3,\\
G(X,e_2)Z &= a_1c_2e_1 - (a_1c_1 - a_3c_3)e_2 + a_3c_2e_3,\\
G(X,e_3)Z &= a_1c_3e_1 + a_2c_3e_2 - (a_1c_1 + a_2c_2)e_3,\\
G(X,Y)e_1 &= -(a_1b_2 - a_2b_1\ )e_2 + (a_3b_1 - a_1b_3)e_3\\
G(X,Y)e_2 &= (a_1b_2 - a_2b_1\ )e_1 + (a_3b_2 - a_2b_3)e_3\\
G(X,Y)e_3 &= (a_3b_1 - a_1b_3\ )e_1 + (a_3b_2 - a_2b_3)e_2
\end{aligned}$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$
\begin{aligned}
& (\nabla_{e_1} R)(X,Y)Z \\
= \; & -le_3 + ne_1 + a_1 R(e_3,Y)Z - a_3 R(e_1,Y)Z + b_1 R(X,e_3)Z \\
& -b_3 R(X,e_1)Z + c_1 R(X,Y)e_3 - c_3 R(X,Y)e_1 \\
= \; & -le_3 + ne_1 + a_1[-b_1c_3e_1 - b_2c_3e_2 + (b_1c_1 + b_2c_2)e_3] \\
& -a_3[\{b_3c_3 - b_2c_2\}e_1 + b_2c_1e_2 - b_3c_1e_3] \\
& +b_1[a_1c_3e_1 - a_2c_3e_2 + (a_1c_1 + a_2c_2)e_3] \\
& -b_3[a_1c_2e_1 + \{a_3c_3 - a_1c_1\}e_2 + a_3c_2e_3] \\
& +c_1[(a_1b_3 - a_3b_1)e_1 + (a_2b_3 - a_3b_2)e_2] \\
& -c_3[-(a_1b_2 - a_2b_1)e_1 - (a_2b_3 - a_3b_2)e_3] \\
= \; & [n - a_1b_1c_3 - a_3(b_3c_3 - b_2c_2) + b_1a_1c_3 - b_3a_1c_2 \\
& +c_1(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)]e_1 \\
& +[-a_1b_2c_3 - a_3b_2c_1 - b_1a_2c_3 - b_3(a_3c_3 - a_1c_1) \\
& +c_1(a_2b_3 - a_3b_2)]e_2 + [-l + a_1(b_1c_1 + b_2c_2) + a_3b_3c_1 \\
& +b_1(a_1c_1 + a_2c_2) - b_3a_3c_2 + c_3(a_2b_3 - a_3b_2)]e_3.
\end{aligned}
$$

$$
\begin{aligned}
& (\nabla_{e_2} R)(X,Y)Z \\
= \; & -me_3 + ne_2 + a_2 R(e_3,Y)Z - a_3 R(e_2,Y)Z + b_2 R(X,e_3)Z \\
& -b_3 R(X,e_2)Z + c_2 R(X,Y)e_3 - c_3 R(X,Y)e_2 \\
= \; & -me_3 + ne_2 + a_2[-b_1c_3e_1 - b_2c_3e_2 + (b_1c_1 + b_2c_2)e_3] \\
& -a_3[b_1c_2e_1 + \{b_3\,c_3 - b_1c_1\}e_2 + b_3\,c_2e_3] \\
& +b_2[a_1c_3e_1 - a_2c_3e_2 + (a_1c_1 + a_2c_2)e_3] \\
& -b_3[-\{a_3c_3 - a_2c_2\}e_1 + a_2c_1e_2 - a_3c_1e_3] \\
& +c_2[(a_1b_3 - a_3b_1)e_1 + (a_2b_3 - a_3b_2)e_2] \\
& -c_3[(a_1b_2 - a_2b_1)e_2 - (a_1b_3 - a_3b_1)e_3] \\
= \; & [-a_2b_1c_3 - a_3(b_3c_3 - b_2c_2) + b_2a_1c_3 \\
& +b_3(a_3c_3 - a_2c_2) + c_2(a_1b_3 - a_3b_1)]e_1 \\
& +[n - a_2b_2c_3 - a_3b_2c_1 - b_2a_2c_3 - b_3a_2c_1 \\
& +c_2(a_2b_3 - a_3b_2) - c_3(a_1b_2 - a_2b_1)]e_2 \\
& +[n + a_2(b_1c_1 + b_2c_2) + a_3b_3c_1 + b_2(a_1c_1 + a_2c_2) \\
& +b_3a_3c_1 + c_3(a_1b_3 - a_3b_1)]e_3.
\end{aligned}
$$

$$
(\nabla_{e_3} R)(X,Y)Z = 0.
$$

With the help of the above relations one can easily bring out the followings

$$\begin{aligned}
&\phi^2\left((\nabla_{e_1}R)(X,Y)Z\right)\\
=\;& -[n - a_1b_1c_3 - a_3(b_3c_3 - b_2c_2) + b_1a_1c_3\\
&+b_3(a_3c_3 - a_2c_2) + c_1(a_1b_3 - a_3b_1)]e_1\\
&-[-a_1b_2c_3 - a_3b_2c_1 - b_1a_2c_3\\
&-c_3(a_1b_2 - a_2b_1) + c_1(a_2b_3 - a_3b_2)]e_2,
\end{aligned}$$

$$\begin{aligned}
&\phi^2\left((\nabla_{e_2}R)(X,Y)Z\right)\\
=\;& -[-a_2b_1c_3 - a_3(b_3c_3 - b_2c_2) + b_2a_1c_3\\
&+b_3(a_3c_3 - a_2c_2) + c_2(a_1b_3 - a_3b_1)]e_1\\
&-[n - a_2b_2c_3 - a_3b_2c_1 - b_2a_2c_3 - b_3a_2c_1\\
&+c_2(a_2b_3 - a_3b_2) - c_3(a_1b_2 - a_2b_1)]e_2,
\end{aligned}$$

$$\phi^2\left((\nabla_{e_3}R)(X,Y)Z\right) = 0.$$

For the following choice of the the 1-forms

$$\begin{aligned}
A(e_1) &= \frac{1}{a_1\ + c_1},\ A(e_2) = \frac{1}{a_2\ + c_2},\ A(e_3) = \frac{1}{a_3 + c_3},\\
\alpha(e_1) &= -\frac{1}{a_1\ + c_1}, \alpha(e_2) = -\frac{1}{a_2\ + c_2}, \alpha(e_3) = \frac{1}{a_3 + c_3}\\
B(e_1) &= \frac{1}{c_1\ + b_1}, B(e_2) = \frac{1}{c_2\ + b_2}, B(e_3) = \frac{1}{c_3 + b_3},\\
\beta(e_1) &= -\frac{1}{c_1\ + b_1}, \beta(e_2) = -\frac{1}{c_2\ + b_2}, \beta(e_3) = -\frac{1}{c_3 + b_3},\\
D(e_1) &= \gamma(e_1) = 0 = D(e_2) = \gamma(e_2) = D(e_3) = \gamma(e_3),
\end{aligned}$$

one can easily verify the relations

$$\begin{aligned}
&\phi^2((\nabla_{e_i}R)(X,Y)Z)\\
=\;& A(e_i)R(X,Y)Z + B(X)R(e_i,Y)Z\\
&+B(Y)R(X,e_i)Z + D(Z)R(X,Y)e_i\\
&+g(R(X,Y)Z,e_i)\rho + \alpha(e_i)G(X,Y)Z\\
&+\beta(X)G(e_i,Y)Z + \beta(Y)G(X,e_iZ)\\
&+\gamma(U)G(X,Y,e_i,V) + g(G(X,Y)Z,e_i)\sigma
\end{aligned}$$

From the above, we can state that

**Theorem 16.** *There exist a Kenmotsu manifold* $(M^3, g)$ *which is a generalized weakly* $\phi$*-symmetric provided* $a_i = kb_i = -2kc_i$ *for* $1, 2, 3$.

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