Acta Universitatis Apulensis No. 76/2023
ISSN: 1582-5329 pp. 1-11
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.76.01

INVARIANT $(\alpha, \beta)-$METRICS ON REDUCED $\Sigma-$SPACES

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Abstract. In this paper we study the geometric properties of Finsler $\Sigma-$spaces. We prove that Matsumoto $\Sigma-$spaces, infinite series $\Sigma-$spaces and exponential $\Sigma-$spaces are Riemannian.

2010 *Mathematics Subject Classification*: 53C30, 53C60.

Keywords: Finsler metric, $(\alpha, \beta)-$metric, $\Sigma-$space, Matsumoto metric, infinite series metric, exponential metric.

1. Introduction

Let M be a C^∞ manifold and $\mu : M \times M \longrightarrow M$, $\mu(x, y) = x.y$ be a differentiable multiplication. The space M with the multiplication μ is said to be symmetric if the following conditions hold:

(1) $x.x = x$
(2) $x.(x.y) = y$
(3) $x.(y.z) = (x.y)(x.z)$
(4) every point x has a neighborhood U such that $x.y = y$ implies $y = x$, for all $y \in U$.

The notion of symmetric spaces is due to E. Cartan and reformulated by O. Loos as pair (M, μ) with conditions $(1)-(4)$ in [16]. A. J. Ledger [13, 14] initiated the study later, generalized symmetric spaces or regular $s-$spaces. Let M be a $C^\infty-$manifold with a family of maps $\{s_x\}_{x\in M}$. The space M is said to be a regular $s-$space if the following conditions hold:

(a) $s_x x = x$,
(b) s_x is a diffeomorphism,
(c) $s_x \circ s_y = s_{s_x y} \circ s_x$,
(d) $(s_x)_*$ has only one fixed vector, the zero vector.

$\Sigma-$spaces and reduced $\Sigma-$spaces where first introduced by O. Loos [16] as generalisation of reflection spaces and symmetric spaces [17]. They include also the class of regular $s-$manifolds [7].

The definition of symmetric Finsler space is a natural generalization of E. Cartan's definition of Riemannian symmetric spaces. We call a Finsler space (M, F) as a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p.
If we drop the involution property in the definition of symmetric Finsler space keeping the property $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$ we get a bigger class of Finsler manifolds as symmetric Finsler spaces [2, 5, 6, 8, 10, 11, 12, 19]. Finsler $\Sigma-$spaces were first proposed and studied by the second and third authors in [9].

2. Preliminaries

A Finsler metric on a C^∞ manifold of dimension n, is a function $F : TM \longrightarrow [0, \infty)$ which has the following properties:

(i) F is C^∞ on $TM_0 = TM - \{0\}$,

(ii) F is positively $1-$homogeneous on the fibers of tangent bundle TM,

(iii) For any non-zero $y \in T_xM$, the fundamental tensor $g_y : T_xM \times T_xM \longrightarrow R$ on T_xM is positive definite,

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]|_{s=t=0}, \quad u, v \in T_xM.$$

Then (M, F) is called $n-$dimensional Finsler manifold.

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$K(P, y) = \frac{g_y(R(u, y)y, u)}{g_y(y, y)g_y(u, u) - g_y^2(y, u)},$$

where $P = span\{u, y\}$ is a $2-$plane in T_xM, $R(u, y)y = \nabla_u \nabla_y y - \nabla_y \nabla_u y - \nabla_{[u,y]} y$ and ∇ is the Chern connection induced by F [18, 4]. For a Finsler metric F on $n-$dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 ... dx^n$ is defined by

$$\sigma_F(x) = \frac{Vol(B^n(1))}{Vol\{(y^i) \in R^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Let $G^i := \frac{1}{4}g^{il}[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}]$, denote the geodesic coefficients of F in the same local coordinate system. The $S-$curvature can be defined by

$$S(y) = \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}[ln\sigma_F(x)],$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ (see [4]). The Finsler metric F is said to be of isotropic $S-$curvature if

$$S = (n + 1)cF,$$

where $c = c(x)$ is a scalar function on M.

3. Invariant $(\alpha, \beta)-$metrics on $\Sigma-$spaces

We first recall the definition and some basic results concerning $\Sigma-$spaces [15, 20].

Definition 1. *Let M be a smooth connected manifold, Σ a Lie group, and $\mu : M \times \Sigma \times M \longrightarrow M$ a smooth map. Then the triple (M, Σ, μ) is a $\Sigma-$space if it satisfies*

(Σ_1) $\mu(x, \sigma, x) = x,$

(Σ_2) $\mu(x, e, y) = y,$

(Σ_3) $\mu(x, \sigma, \mu(x, \tau, y)) = \mu(x, \sigma\tau, y)$

(Σ_5) $\mu(x, \sigma, \mu(y, \tau, z)) = \mu(\mu(x, \sigma, y), \sigma\tau\sigma^{-1}, \mu(x, \sigma, z))$

where $x, y, z \in M$, $\sigma, \tau \in \Sigma$ and e is the identity element of Σ. The triple (M, Σ, μ) is usually dinoted by M.

For a fixed point $x \in M$ we define a map $\sigma_x : M \longrightarrow M$ by $\sigma_x(y) = \mu(x, \sigma, y)$ and a map $\sigma^x : M \longrightarrow M$ by $\sigma^x(y) = \sigma_y(x)$. With respect to these maps the above conditions became

(Σ_1') $\sigma_x(x) = x,$

(Σ_2') $e_x = id_M,$

(Σ_3') $\sigma_x\tau_x = (\sigma\tau)_x$

(Σ_4') $\sigma_x\tau_y\sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y).$

For each $x \in M$ by Σ_x we denote the image of Σ under the map $\Sigma \longrightarrow \Sigma_x$, $\sigma \longrightarrow \sigma_x$. For each $\sigma \in \Sigma$ we define (1,1) tensor field S^σ on the $\Sigma-$space M by

$$S^\sigma X_x = (\sigma_x)_* X_x \quad \forall x \in M, X_x \in T_x M.$$

Clearly S^σ is smooth.

Definition 2. *A $\Sigma-$space M is a reduced $\Sigma-$space if for each $x \in M$,*

1. *$T_x M$ is generated by the set of all $\sigma^x(X_x)$, that is*

$$T_x M = gen\{(I - S^\sigma)X_x | X_x \in T_x M, \sigma \in \Sigma\},$$

2. *If $X_x \in T_x M$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in $T_x M$ is fixed by all S^σ.*

Definition 3. *A Finsler $\Sigma-$space, denoted by (M, Σ, F) is a reduced $\Sigma-$space together with a Finsler metric F which is invariant under Σ_p for $p \in M$.*

Definition 4. *Let $\alpha = \sqrt{\tilde{a}_{ij}(x) y^i y^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x) y^i$ be a 1-form on an n-dimensional manifold M , and let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij} b_i(x) b_j(x)}. \tag{1}$$

Now , the function F is defined by ,

$$F := \alpha\phi(s) \qquad s = \frac{\beta}{\alpha}, \tag{2}$$

where $\phi = \phi(s)$ is a positive c^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi H(s) > 0, \qquad | s | \leq b < b_0. \tag{3}$$

Then by lemma 1.1.2 of [4], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2) is called an $(\alpha, \beta)-$ metric [1, 3, 4]. A Finsler space having the Finsler function :

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)}, \tag{4}$$

is called a Matsumoto space. A Finsler space having the Finsler function ,

$$F(x, y) = \frac{\beta^2(x, y)}{\beta(x, y) - \alpha(x, y)}, \tag{5}$$

is called a Finsler space with an infinite series $(\alpha, \beta)-$*metric. A Finsler space having the Finsler function:*

$$F(x, y) = \alpha(x, y) \exp(\frac{\beta(x, y)}{\alpha(x, y)}), \tag{6}$$

is called a Finsler space with an exponential metric $(\alpha, \beta)-$*metric.*

Now we present the main results.

Lemma 1. *Let* (M, Σ, F) *be a Matsumoto* $\Sigma-$*space with* F *defined by the Riemannian metric* $\tilde{a}$ *and the vector field* X*. Then* $(M, \Sigma, \tilde{a})$ *is a Riemannian* $\Sigma-$*space.*

Proof: Let σ_x be a diffeomorphism $\sigma_x : M \longrightarrow M$ defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then for $p \in M$ and for any $y \in T_pM$ we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)),$$

$$\frac{\tilde{a}(Y, Y)}{\sqrt{\tilde{a}(Y, Y)}} - \tilde{a}(X_p, Y) = \frac{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y) - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}}, \tag{7}$$

Substituting Y with $-Y$ in (7) we get:

$$\frac{\tilde{a}(Y, Y)}{\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_p, Y)} = \frac{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}, \tag{8}$$

Combining the equation (7) and (8), we get

$$\begin{aligned} \tilde{a}(Y, Y) &= \tilde{a}(d\sigma_x(Y), d\sigma_x(Y)), \\ \tilde{a}(X_p, Y) &= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(Y)). \end{aligned}$$

Thus σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$. □

Lemma 2. *Let* $(M, \Sigma, \tilde{a})$ *be a Riemannian* $\Sigma-$*space. Let* F *be a Matsumoto metric defined by the Riemannian metric* $\tilde{a}$ *and the vector field* X*. Then* (M, Σ, F) *is a Matsumoto* $\Sigma-$*space if and only if* X *is* σ_x-*invariant for all* $x \in M$*.*

Proof: Let X be σ_x-invariant. Then for any $p \in M$, we have $X_{\sigma_x(p)} = d\sigma_x X_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F(\sigma_x(p), d\sigma_x y_p) &= \frac{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}{\sqrt{\tilde{a}(d\sigma_x y, d\sigma_x y)} + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x y)} \\ &= \frac{\tilde{a}(y, y)}{\sqrt{\tilde{a}(y, y)} - \tilde{a}(d\sigma_x X_p, d\sigma_x y)} \\ &= \frac{\tilde{a}(y, y)}{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X_p, y)} \\ &= F(p, y). \end{aligned}$$

conversely, let F be a Σ_M- invariant, then for any $p \in M$ and $y \in T_pM$, we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y))$$

$$\frac{\tilde{a}(y, y)}{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X_p, y)} = \frac{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}{\sqrt{\tilde{a}(d\sigma_x y, d\sigma_x y)} - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x y)},$$

so we have

$$\tilde{a}(d\sigma_x X_p - X_{\sigma_x(p)}, d\sigma_x y_p) = 0,$$

Therefore $d\sigma_x X_p = X_{\sigma_x(p)}$. □

Theorem 3. *A Matsumoto $\Sigma-$space must be Riemannian.*

Proof: Let (M, Σ, F) be a Matsumoto $\Sigma-$space where F is a Riemannian metric $\tilde{a}$ and the vector field X. Let σ_x be a diffeomorphism defined by $\sigma_x(y) = \mu(x, \sigma, y)$. By lemma 1, $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x y) &= \frac{\tilde{a}(d\sigma_x y, d\sigma_x y)}{\sqrt{\tilde{a}(d\sigma_x y, d\sigma_x y)} - \tilde{a}(X_x, d\sigma_x y)} \\ &= \frac{\tilde{a}(y, y)}{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X_x, d\sigma_x y)} \\ &= F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of T_xM without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_xM$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

Lemma 4. *Let (M, Σ, F) be a infinite series $\Sigma-$ space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space.*

Proof: Let σ_x be a diffeomorphism $\sigma_x : M \longrightarrow M$ defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then for $p \in M$ and for any $Y \in T_pM$ we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)).$$

Applying equation (5) we get

$$\frac{\tilde{a}(X_p, Y)^2}{\tilde{a}(X_p, Y) - \sqrt{\tilde{a}(Y, Y)}} = \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)^2}{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y) - \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}},$$

which implies

$$\tilde{a}(X_p, Y)^2 \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y) - \tilde{a}(X_p, Y)^2 \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}$$

$$= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)^2 \tilde{a}(X_p, Y) - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)^2 \sqrt{\tilde{a}(Y, Y)}. \tag{9}$$

Applying the above equation to $-Y$, we get

$$\tilde{a}(X_p, Y)^2 \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y) + \tilde{a}(X_p, Y)^2 \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}$$
$$= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)^2 \tilde{a}(X_p, Y) + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)^2 \sqrt{\tilde{a}(Y, Y)}. \tag{10}$$

Applying the above equations we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y), \tag{11}$$

so we have

$$\tilde{a}(Y, Y) = \tilde{a}(d\sigma_x Y, d\sigma_x Y).$$

Thus σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$.□

Lemma 5. *Let $(M, \Sigma, \tilde{a})$ be a Riemannian $\Sigma-$space. Let F be an infinite series defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then (M, Σ, F) is an infinite series $\Sigma-$space if and only if X is σ_x-invariant for all $x \in M$.*

Proof: The proof is similar to the proof of lemma 2.□

Theorem 6. *An infinite series $\Sigma-$space must be Riemannian*

Proof: Let (M, Σ, F) be an infinet series $\Sigma-$space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Let σ_x be a diffeomorphism defined by $\sigma_x(y) = \mu(x, \sigma, y)$. by lemma 4, $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x y) &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}} \\ &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(y, y)}} \\ &= F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of T_xM without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_xM$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

Lemma 7. *Let (M, Σ, F) be Finsler $\Sigma-$space. with exponential metric $F = \alpha \exp(\dfrac{\beta}{\alpha})$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian Σ spaces.*

Proof: Let σ_x be a diffeomorphism $\sigma_x : M \longrightarrow M$ defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then for $p \in M$ and for any $y \in T_pM$ we have

$$F(p, y) = F(\sigma_x(p), d\sigma_x(y)),$$

Applying equation (6) we get

$$\sqrt{\tilde{a}(y,y)} \exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}), \tag{12}$$

Replacing Y by -Y in equation (12) we get

$$\sqrt{\tilde{a}(y,y)} \exp(\frac{-\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{-\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}), \tag{13}$$

combining the above equation (12) and (13) we have

$$\exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \exp(\frac{2\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(, d\sigma_x Y, , d\sigma_x Y)}}),$$

which implies

$$\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}} = \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}. \tag{14}$$

From equation (12) and (14), we have

$$\tilde{a}(Y, Y) = \tilde{a}(d\sigma_x Y, d\sigma_x Y).$$

Thus σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$. □

Lemma 8. *Let $(M, \Sigma, \tilde{a})$ be a Riemannian $\Sigma-$space. Let F be an exponential metric defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then (M, Σ, F) is an exponential $\Sigma-$space if and only if X is σ_x-invariant for all $x \in M$.*

Proof: Let X be σ_x-invariant. Then for any $p \in M$, we have $X_{\sigma_x(p)} = d\sigma_x X_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F(\sigma_x(p), d\sigma_x(Y_p)) &= \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}) \\ &= \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}) \\ &= \sqrt{\tilde{a}(Y, Y)} \exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y, Y)}}) \\ &= F(p, Y). \end{aligned}$$

Conversely, let F be a Σ_M- invariant, then for any $p \in M$ and $y \in T_pM$, we have

$$F(p, Y) = F(\sigma_x(p), d\sigma_x(Y)).$$

Applying the lemma 7, we get

$$\frac{\tilde{a}(X_p, Y)}{\tilde{a}(Y, Y)} = \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}, \tag{15}$$

which implies

$$\tilde{a}(Y, Y) = \tilde{a}(d\sigma_x Y), d\sigma_x Y). \tag{16}$$

From equation (15) and (16), we have

$$\tilde{a}(X_x, Y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y).$$

Therefore $(d\sigma_x)_p X_p = X_{\sigma_x(p)}$.□

Theorem 9. *An exponential Σ spaces must be Riemannian.*

Proof: The proof is similar to the above cases.□

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