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# CYCLIC AND $\lambda$ -CONSTACYCLIC CODES OVER THE RING $\mathbb{Z}_5[u,v]/\langle u^2-u, v^2, uv, vu\rangle$

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Abstract. In this study, unitary elements and related elements are determined on two variable rings with coefficients of $Z_5$. For the $u^2=u, v^2=0$ and $u\cdot v = v\cdot u = 0$ states, $\lambda$ -constacyclic codes and the types of codes with their gray images were determined over $\lambda = (1+3u), (2+4u)$ and 4 unitary elements on the $\mathbb{Z}_5[u,v]$ / $\langle u^2-u, v^2, uv, vu\rangle$ ring. It has been shown that codes with $[5n, k, d_H]$ parameter are obtained on the $Z_5$ object.

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## 1. Introduction

Cyclic codes, constacyclic codes, quasi cyclic codes, negacyclic codes and skew cyclic codes were studied in one-variable and two-variable rings with coefficients of $\mathbb{Z}_{p^k}$ field being $p$ a prime number and $k$ an integer. Most of these studies have been codes in the literature corresponding to $\mathbb{Z}_p$ prime fields for $k=1$ . Basic information in coding theory, parameters of codes and code definitions are given in the book of Steven Roman [1], which is a general reference. In one-variable rings; Constacyclic codes for unitary element $1+u$ in ring $\mathbb{F}_2+u\mathbb{F}_2$ with 4 elements whose coefficients are in binary field Qian J. and his team in [2]. Study in [2], it has been shown that $(1+u)$-constacyclic codes correspond to cyclic codes on the field, thanks to the Gray transform between the four elements ring and the binary body. In [3], with a different method, the work in this four elements ring was transferred to the eight elements $\mathbb{F}_2+u\mathbb{F}_2+u^2\mathbb{F}_2$ ring and similar results were obtained. Previous studies have addressed constacyclic strains in bivariate rings, similar to studies in univariate rings. The study titled "On some special codes over $\mathbb{F}_3+v\mathbb{F}_3+u\mathbb{F}_3+u^2\mathbb{F}_3$" written by M.Özkan, which we frequently use in this article, and the constacyclic codes on the rings with the coefficients in the ternary field and their images in the $\mathbb{F}_3$ field are presented in [4]. In [5], a class of constacyclic codes in the bivariate ring with coefficients of $\mathbb{Z}_4$ for $p = 2$ and $k = 2$ cases is given by H. Islam. In another article, Gray images of constacyclic codes for ring $\mathbb{F}_2 + u_1\mathbb{F}_2 + u_2\mathbb{F}_2$ for bivariate variables $u_1$ and $u_2$ and which codes they are have been studied in [6]. In [7], the images of the codes under Gray transform on bivariate rings with $\mathbb{Z}_3$ coefficient published by Timothy Kom and his team are given. In this study, a new Gray transform is defined using the ring presented in [7]. A different perspective has been gained for the codes under the Gray transformation and new codes have been written.

## 2. Preliminaries

Let $S = \mathbb{Z}_5[u,v]/\langle u^2 - u, v^2, uv, vu\rangle$ and $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Then $S = \{a + ub + vc : a, b, c \in \mathbb{Z}_5\}$ is a commutative ring with cardinality 125 and characteristic 5. The set of units of the ring is $vI = \{1, 4, 1 + 3u, 2 + 4u\} = \{\lambda \in I : \lambda^2 = 1\}$ . In this study, 4 of the unitary elements of the ring were examined. The ring S contains more than one maximal ideal. Hence, it is a finite non-chain ring. Also, $S = \mathbb{Z}_5[u,v]/\langle u^2 - u, v^2, uv, vu\rangle \cong \mathbb{Z}_5 + u\mathbb{Z}_5 + v\mathbb{Z}_5$ with $u^2 = u, v^2 = 0$ and $u \cdot v = v \cdot u = 0$. The definitions to be used in this study are given below.

**Definition 1.** *A linear code $C$ over $S$ of length $n$ is a $S$-submodule of $S^n$ . An element of $C$ is named a codeword. A cyclic code $C$ of length $n$ over $S$ is a linear code with the characteristic that if $c = (c_0, c_1, c_2, c_3, \ldots, c_{n-1}) \in ]$, then $\sigma(c) = \sigma(c_{n-1}, c_0, c_1, c_2 \ldots, c_{n-2}) \in C$. $\sigma$ is named cyclic shift operator from $S^n$ to $S^n$.*

**Definition 2.** *A linear code $C$ of length $n$ over $S$ is $\lambda$ -constacyclic code if $c = (c_0, c_1, c_2, c_3, \ldots, c_{n-1}) \in C$, then $\gamma_{(\lambda)}(c) = (\lambda c_{n-1}, c_0, c_1, c_2, c_3, \ldots, c_{n-2}) \in C$, where $\lambda$ is a unit in $S$. $\gamma_{(\lambda)}$ is called $\lambda$ -constacyclic shift operator from $S^n$ to $S^n$.*

**Definition 3.** *Let $a \in \mathbb{Z}_5^{3n}$ with $a = (a_0, a_1, a_2, \ldots, a_n, \ldots, a_{2n}, \ldots, a_{3n-1}) = (a^{(0)}|a^{(1)}|a^{(2)})$, where $a^{(i)} \in \mathbb{Z}_5^n$ for $i = 0, 1, 2$ and $|$ is the usual vector concatenation. Let $p$ be a map from $\mathbb{Z}_5^{3n}$ to $\mathbb{Z}_5^{3n}$ defined by $\rho(a) = (\sigma(a^{(0)})|\sigma(a^{(1)})|\sigma(a^{(2)}))$ where $\sigma$ is a cyclic shift operator from $\mathbb{Z}_5^n$ to $\mathbb{Z}_5^n$ . A code $C$ of length $3n$ over $\mathbb{Z}_5$ is called a quasi-cyclic code of index 3 if $\rho(C) = C$.*

**Proposition 1.** *A subset $C$ of $S^n$ is a [n,d]-cyclic code if and only if its polynomial representation is an ideal of $S_n = S[x]/\langle x^n - 1\rangle$ .*

**Proposition 2.** *A subset $C$ of $S^n$ is a constacyclic code of length $n$ if and only if its polynomial representation is an ideal of $S_{n,\lambda} = S[x]/\langle x^n - \lambda\rangle$ .*

## 3. Gray Map and Cyclic Codes Over $S$

In this section, we introduce a Gray map $\Gamma$ on the ring $S$ and consider the algebraic structures of cyclic codes over the ring $S$.

In order to connect the structure of the ring $S$ with $\mathbb{Z}_5^3$ . We define the Gray map $\Gamma$;

$$\Gamma : S \to \mathbb{Z}_5^3$$

$$a + ub + vc \to \Gamma(a + ub + vc) = (a + 4b, b, c)$$

where $a + ub + vc \in S$ and $a, b, c \in \mathbb{Z}_5$. From the definition, we observe that

$$\Gamma(0) = (0, 0, 0), \Gamma(1) = (1, 0, 0), \Gamma(2) = (2, 0, 0), \Gamma(3) = (3, 0, 0), \Gamma(4) = (4, 0, 0),$$

$$\Gamma(u) = (4, 1, 0), \Gamma(2u) = (3, 2, 0), \Gamma(3u) = (2, 3, 0), \Gamma(4u) = (1, 4, 0),$$

$$\Gamma(v) = (0, 0, 1), \Gamma(2v) = (0, 0, 2), \Gamma(3v) = (0, 0, 3), \Gamma(4v) = (0, 0, 4),$$

$$\Gamma(1 + u) = (0, 1, 0), \Gamma(1 + 2u) = (4, 2, 0), \Gamma(1 + 3u) = (3, 3, 0), \Gamma(1 + 4u) = (2, 4, 0),$$

$$\Gamma(2 + u) = (1, 1, 0), \Gamma(2 + 2u) = (0, 2, 0), \Gamma(2 + 3u) = (4, 3, 0), \Gamma(2 + 4u) = (3, 4, 0),$$

$$\Gamma(3 + u) = (2, 1, 0), \Gamma(3 + 2u) = (1, 2, 0), \Gamma(3 + 3u) = (0, 3, 0), \Gamma(3 + 4u) = (4, 4, 0),$$

$$\Gamma(4 + u) = (3, 1, 0), \Gamma(4 + 2u) = (2, 2, 0), \Gamma(4 + 3u) = (1, 3, 0), \Gamma(4 + 4u) = (0, 4, 0),$$

$$\Gamma(1 + v) = (1, 0, 1), \Gamma(1 + 2v) = (1, 0, 2), \Gamma(1 + 3v) = (1, 0, 3), \Gamma(1 + 4v) = (1, 0, 4),$$

$$\Gamma(2 + v) = (2, 0, 1), \Gamma(2 + 2v) = (2, 0, 2), \Gamma(2 + 3v) = (2, 0, 3), \Gamma(2 + 4v) = (2, 0, 4),$$

$$\Gamma(3 + v) = (3, 0, 1), \Gamma(3 + 2v) = (3, 0, 2), \Gamma(3 + 3v) = (3, 0, 3), \Gamma(3 + 4v) = (3, 0, 4),$$

$$\Gamma(4 + v) = (4, 0, 1), \Gamma(4 + 2v) = (4, 0, 2), \Gamma(4 + 3v) = (4, 0, 3), \Gamma(4 + 4v) = (4, 0, 4),$$

$$\Gamma(u + v) = (4, 1, 1), \Gamma(u + 2v) = (4, 1, 2), \Gamma(u + 3v) = (4, 1, 3), \Gamma(u + 4v) = (4, 1, 4),$$

$$\Gamma(2u{+}v) = (3, 2, 1), \Gamma(2u{+}2v) = (3, 2, 2), \Gamma(2u{+}3v) = (3, 2, 3), \Gamma(2u{+}4v) = (3, 2, 4),$$

$$\Gamma(3u{+}v) = (2, 3, 1), \Gamma(3u{+}2v) = (2, 3, 2), \Gamma(3u{+}3v) = (2, 3, 3), \Gamma(3u{+}4v) = (2, 3, 4),$$

$$\Gamma(4u{+}v) = (1, 4, 1), \Gamma(4u{+}2v) = (1, 4, 2), \Gamma(4u{+}3v) = (1, 4, 3), \Gamma(4u{+}4v) = (1, 4, 4),$$

$$\Gamma(1{+}u{+}v) = (0, 1, 1), \Gamma(1{+}u{+}2v) = (0, 1, 2), \Gamma(1{+}u{+}3v) = (0, 1, 3), \Gamma(1{+}u{+}4v) = (0, 1, 4),$$

$$\Gamma(1{+}2u{+}v) = (4, 2, 1), \Gamma(1{+}2u{+}2v) = (4, 2, 2), \Gamma(1{+}2u{+}3v) = (4, 2, 3), \Gamma(1{+}2u{+}4v) = (4, 2, 4),$$

$$\Gamma(1{+}3u{+}v) = (3, 3, 1), \Gamma(1{+}3u{+}2v) = (3, 3, 2), \Gamma(1{+}3u{+}3v) = (3, 3, 3), \Gamma(1{+}3u{+}4v) = (3, 3, 4),$$

$$\Gamma(1{+}4u{+}v) = (2, 4, 1), \Gamma(1{+}4u{+}2v) = (2, 4, 2), \Gamma(1{+}4u{+}3v) = (2, 4, 3), \Gamma(1{+}4u{+}4v) = (2, 4, 4),$$

$\Gamma(2{+}u{+}v)=(1,1,1), \Gamma(2{+}u{+}2v)=(1,1,2), \Gamma(2{+}u{+}3v)=(1,1,3), \Gamma(2{+}u{+}4v)=(1,1,4),$

$\Gamma(2{+}2u{+}v)=(0,2,1), \Gamma(2{+}2u{+}2v)=(0,2,2), \Gamma(2{+}2u{+}3v)=(0,2,3), \Gamma(2{+}2u{+}4v)=(0,2,4),$

$\Gamma(2{+}3u{+}v)=(4,3,1), \Gamma(2{+}3u{+}2v)=(4,3,2), \Gamma(2{+}3u{+}3v)=(4,3,3), \Gamma(2{+}3u{+}4v)=(4,3,4),$

$\Gamma(2{+}4u{+}v)=(3,4,1), \Gamma(2{+}4u{+}2v)=(3,4,2), \Gamma(2{+}4u{+}3v)=(3,4,3), \Gamma(2{+}4u{+}4v)=(3,4,4),$

$\Gamma(3{+}u{+}v)=(2,1,1), \Gamma(3{+}u{+}2v)=(2,1,2), \Gamma(3{+}u{+}3v)=(2,1,3), \Gamma(3{+}u{+}4v)=(2,1,4),$

$\Gamma(3{+}2u{+}v)=(1,2,1), \Gamma(3{+}2u{+}2v)=(1,2,2), \Gamma(3{+}2u{+}3v)=(1,2,3), \Gamma(3{+}2u{+}4v)=(1,2,4),$

$\Gamma(3{+}3u{+}v)=(0,3,1), \Gamma(3{+}3u{+}2v)=(0,3,2), \Gamma(3{+}3u{+}3v)=(0,3,3), \Gamma(3{+}3u{+}4v)=(0,3,4),$

$\Gamma(3{+}4u{+}v)=(4,4,1), \Gamma(3{+}4u{+}2v)=(4,4,2), \Gamma(3{+}4u{+}3v)=(4,4,3), \Gamma(3{+}4u{+}4v)=(4,4,4),$

$\Gamma(4{+}u{+}v)=(3,1,1), \Gamma(4{+}u{+}2v)=(3,1,2), \Gamma(4{+}u{+}3v)=(3,1,3), \Gamma(4{+}u{+}4v)=(3,1,4),$

$\Gamma(4{+}2u{+}v)=(2,2,1), \Gamma(4{+}2u{+}2v)=(2,2,2), \Gamma(4{+}2u{+}3v)=(2,2,3), \Gamma(4{+}2u{+}4v)=(2,2,4),$

$\Gamma(4{+}3u{+}v)=(1,3,1), \Gamma(4{+}3u{+}2v)=(1,3,2), \Gamma(4{+}3u{+}3v)=(1,3,3), \Gamma(4{+}3u{+}4v)=(1,3,4),$

$\Gamma(4{+}4u{+}v)=(0,4,1), \Gamma(4{+}4u{+}2v)=(0,4,2), \Gamma(4{+}4u{+}3v)=(0,4,3), \Gamma(4{+}4u{+}4v)=(0,4,4).$

It can be easily checked that $\Gamma$ is bijective. The map $\Gamma$ can be extended in a natural way to $S^n$ component-wise. For $q=(q_0,q_1,\dots,q_{n-1})\in S^n$ , $\Gamma$ can be defined as follows:

$$\Gamma : S^n \to \mathbb{Z}_5^{3n}$$

$$\Gamma(q_0,q_1,\dots,q_{n-1})=(a_0{+}4b_0,a_1{+}4b_1,\dots,a_{n-1}{+}4b_{n-1},b_0,b_1,\dots,b_{n-1},c_0,c_1,\dots,c_{n-1})$$

where $q_i=a_i+ub_i+vc_i\in S$ and $a_i,b_i,c_i\in\mathbb{Z}_5$ for $i=0,1,\dots,n-1$.

Let $C$ be a linear code of length $n$ over $S$ . For any $r=(r_0,r_1,\dots,r_{n-1})\in C$ the Hamming weight $w_H(C)$ of a code $C$ is the smallest weight among all its non-zero codewords. For $r=(r_0,r_1,\dots,r_{n-1})$ and $r'=(r_0{}',r_1{}',\dots,r_{n-1}')$ in $C$, the Hamming distance between $r$ and $r'$ is defined by $d_H(r,r')=w_H(r-r')$ and the Hamming distance for a code $C$ is defined by $d_H(C)=\min\{d_H(r,r')|r,r'\in C\}$ .

The Lee weight of any element $r=(r_0,r_1,\dots,r_{n-1})\in S^n$ is defined by $w_L(r)=\sum_{i=0}^{n-1} w_L(r_i)$, where $w_L(r_i)=w_H(a_i+4b_i,b_i,c_i)$ for $r_i=a_i+ub_i+vc_i\in S, i=0,1,\dots,n-1$. The Lee distance for the code $C$ is defined by

$d_L(C)=\min\{d_L(r,r')|r\neq r',\forall r,r'\in C\}$, where $d_L(r,r')$ is the Lee distance between $r$ and r$'$ defined by $d_L(r,r')=w_L(r-r')$ .

**Theorem 1.** *The Gray map $\Gamma : S^n\to\mathbb{Z}_5^{3n}$ is a distance preserving $\mathbb{Z}_5$ -linear map from $S^n$ (Lee distance, $d_L$ ) to $\mathbb{Z}_5^{3n}$ (Hamming distance, $d_H$ ).*

*Proof.* Let $q = (q_0, q_1, \ldots, q_{n-1}), k = (k_0, k_1, \ldots, k_{n-1}) \in S^n$ , where $q_i = a_i + ub_i + vc_i$ , $k_i = e_i + uf_i + vg_i \in S$ for $i = 0, 1, \ldots, n-1$ and $\beta \in \mathbb{Z}_5$. Then $\Gamma(q+k) = \Gamma(q_0+k_0, q_1+k_1, \ldots, q_{n-1}+k_{n-1}) = (a_0+e_0+4(b_0+f_0), \ldots, a_{n-1}+e_{n-1}+4(b_{n-1}+f_{n-1}), b_0+f_0, \ldots, b_{n-1}+f_{n-1}, c_0+g_0, \ldots, c_{n-1}+g_{n-1}) = (a_0+4b_0, \ldots, a_{n-1}+4b_{n-1}, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1})+(e_0+4f_0, \ldots, e_{n-1}+4f_{n-1}, f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}) = \Gamma(q)+\Gamma(k)$. And, $\beta\,\Gamma(q) = \beta(a_0+4b_0, \ldots, a_{n-1}+4b_{n-1}, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1})$
$= (\beta\, a_0+4\beta\, b_0, \ldots, \beta\, a_{n-1}+4\beta\, b_{n-1}, \beta\, b_0, \ldots, \beta\, b_{n-1}, \beta\, c_0, \ldots, \beta\, c_{n-1}) = \Gamma(\beta\, q)$ .

Hence, $\Gamma$ is a $\mathbb{Z}_5$ -linear map. Since $\Gamma$ is a linear map, we have $\Gamma(q-k) = \Gamma(q) - \Gamma(k)$, for any $q, k \in S^n$. By the definition of the Lee distance, we have $d_L(q,k) = w_L(q-k) = w_H(\Gamma(q-k)) = w_H(\Gamma(q) - \Gamma(k)) = d_H(\Gamma(q), \Gamma(k))$. This shows that $\Gamma$ is a distance preserving $\mathbb{Z}_5$ -linear map.

**Theorem 2.** *If $C$ is a linear code of length $n$ over $S$ with cardinality $|C| = 5^k$ and Lee distance $d_L$ , then the Gray image $\Gamma(C)$ is a $[5n, k, d_H]$ linear code over $\mathbb{Z}_5$.*

*Proof.* The proof is given in article [7].

**Example 1.** *$C_1 = \{0, u, 2u, 3u, 4u\}$ and $C_2 = \{0, v, 2v, 3v, 4v\}$ codes are linear codes of length 1 over $S$ ring. Transforms $\Gamma(C_1)$ and $\Gamma(C_2)$ are linear codes $[5,1,2]$ and $[5,1,1]$ over $\mathbb{Z}_5$, respectively.*

**Example 2.** $C = \{(0,0,0,0,0)\ , (v,v,v,v,v), (2v,2v,2v,2v,2v), (3v,3v,3v,3v,3v),$

$(4v,4v,4v,4v,4v), (u,0,0,0,0), (2u,0,0,0,0), (3u,0,0,0,0), (4u,0,0,0,0), (u+v,v,v,v,v),$

$(u+2v,2v,2v,2v,2v), (u+3v,3v,3v,3v,3v), (u+4v,4v,4v,4v,4v), (2u+v,v,v,v,v),$

$(2u+2v,2v,2v,2v,2v), (2u+3v,3v,3v,3v,3v), (2u+4v,4v,4v,4v,4v), (3u+v,v,v,v,v),$

$(3u+2v,2v,2v,2v,2v), (3u+3v,3v,3v,3v,3v), (3u+4v,4v,4v,4v,4v), (4u+v,v,v,v,v),$

$(4u+2v,2v,2v,2v,2v), (4u+3v,3v,3v,3v,3v)\ , (4u+4v,4v,4v,4v,4v)\}$

code is linear code of length 5 over $S$ ring. Transform $\Gamma(C)$ is a $[25,2,2]$ linear code over $\mathbb{Z}_5$.

**Theorem 3.** *Let $\Gamma$ be the Gray map from $S^n$ to $\mathbb{Z}_5^{3n}$ . Let $\sigma$ be the cyclic shift operator and $\rho$ be the quasi-cyclic shift operatör as defined in the preliminaries. Then $\Gamma\sigma = \rho\,\Gamma$.*

*Proof.* Let $q = (q_0, q_1, \ldots, q_{n-1}) \in S^n$, where $q_i = a_i+ub_i+vc_i \in S$ and $a_i, b_i, c_i \in \mathbb{Z}_5$, for $i = 0, 1, \ldots, n-1$.

Now $\Gamma(q) = (a_0+4b_0, a_1+4b_1, \ldots, a_{n-1}+4b_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1})$. Applying $\rho$ on both sides, we get

$\rho\,\Gamma(q) = \rho(a_0 + 4b_0, a_1 + 4b_1, \ldots, a_{n-1} + 4b_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1})$ $= (a_{n-1} + 4b_{n-1}, a_0 + 4b_0, \ldots, a_{n-2} + 4b_{n-2}, b_{n-1}, b_0, \ldots, b_{n-2}, c_{n-1}, c_0, \ldots, c_{n-2}) \ldots$ (1). On the other hand, we have $\Gamma\,\sigma(q) = \Gamma(q_{n-1}, q_0, \ldots, q_{n-2}) = (a_{n-1} + 4b_{n-1}, a_0 + 4b_0, \ldots, a_{n-2} + 4b_{n-2}, b_{n-1}, b_0, \ldots, b_{n-2}, c_{n-1}, c_0, \ldots, c_{n-2}) \ldots$ (2).

Equality is obtained from (1) and (2).

**Corollary 4.** *Let $C$ be a subset of $S^n$ . Then $C$ is a cyclic code of length $n$ over $S$ if and only if the Gray image $\Gamma(C)$ is a quasi-cyclic code of index 3 over $\mathbb{Z}_5$ with length $3n$.*

*Proof.* The proof is given in article [7].

**Theorem 5.** *Let $\Gamma$ be the Gray map from $S^n$ to $\mathbb{Z}_5^{3n}$ , $\sigma$ be the cyclic shift operator and $\Gamma_\pi$ be the permutation version of the Gray map $\Gamma$ as given before. Then $\Gamma_\pi\sigma = \sigma^3\,\Gamma_\pi$.*

*Proof.* For any $q = (q_0, q_1, \ldots, q_{n-1}) \in S^n$, where $q_i = a_i + ub_i + vc_i \in S$ and $a_i, b_i, c_i \in \mathbb{Z}_5$ for $i = 0, 1, \ldots, n-1$. We have, $\sigma(q) = (q_{n-1}, q_0, \ldots, q_{n-2})$ . Applying $\Gamma_\pi$ , we get $\Gamma_\pi\,\sigma(q) = \Gamma_\pi(q_{n-1}, q_0, q_1, \ldots, q_{n-2}) = (\,\Gamma_\pi(q_{n-1}), \Gamma_\pi(q_0), \ldots, \Gamma_\pi(q_{n-2})\,)$ $= (a_{n-1} + 4b_{n-1}, b_{n-1}, c_{n-1}, a_0 + 4b_0, b_0, c_0, \ldots, a_{n-2} + 4b_{n-2}, b_{n-2}, c_{n-2}) \ldots$ (1) On the other hand, we have $\Gamma_\pi(q) = (a_0 + 4b_0, b_0, c_0, a_1 + 4b_1, b_1, c_1, \ldots, a_{n-1} + 4b_{n-1}, b_{n-1}, c_{n-1})$ $\sigma\,\Gamma_\pi(q) = (c_{n-1}, a_0+4b_0, b_0, c_0, a_1+4b_1, b_1, c_1, \ldots, a_{n-1}+4b_{n-1}, b_{n-1})$ $\sigma^2\,\Gamma_\pi(q) = (b_{n-1}, c_{n-1}, a_0 + 4b_0, b_0, c_0, a_1 + 4b_1, b_1, c_1, \ldots, a_{n-1} + 4b_{n-1})$ $\sigma^3\,\Gamma_\pi(q) = (a_{n-1}+4b_{n-1}, b_{n-1}, c_{n-1}, a_0+4b_0, b_0, c_0, a_1+4b_1, b_1, c_1, \ldots, a_{n-2}+4b_{n-2}, b_{n-2}, c_{n-2}) \ldots$ (2)

Equality is obtained from (1) and (2).

**Corollary 6.** *Let $C$ be a subset of $S^n$ . Then $C$ is a cyclic code of length $n$ over $S$ if and only if $\Gamma_\pi(C)$ is equivalent to a 3-quasi-cyclic code of length $3n$ over $\mathbb{Z}_5$.*

*Proof.* The proof is given in article [7].

## 4. Constacyclic Codes Over $S$

Here, $n$-length $\lambda$-constacyclic codes on the $S$ ring with $\lambda = (1 + 3u), (2 + 4u)$ and 4 unitary elements are examined. But in this part, $(1 + 3u)$ and $(2 + 4u)$ elements aren't provided transformations. Transform is provided for only 4 unitary elements.

**Definition 4.** *For $a \in \mathbb{Z}_5^{3n}$ with $a(a_0, a_1, \ldots, a_{n-1}, a_n, \ldots, a_{2n}, \ldots, a_{3n-1}) = (a^{(0)}|a^{(1)}|a^{(2)})$, where $a^{(i)} \in \mathbb{Z}_5^n$ for $i = 0, 1, 2$, quasi-twisted shift operator on $\mathbb{Z}_5^{3n}$ is defined by $\upsilon(a) = (\gamma_{(4)}(a^{(0)})|\gamma_{(4)}(a^{(1)})|\gamma_{(4)}(a^{(2)}))$ , where $\gamma_{(4)}$ is a 4-constacyclic shift operator* *from $\mathbb{Z}_5^n$ to $\mathbb{Z}_5^n$. A linear code $C$ of length $3n$ over $\mathbb{Z}_5$ is called a quasi-twisted code of index 3 if $\upsilon(C) = C$.*

**Theorem 7.** *Let $\gamma_{(4)}$ be 4-constacyclic shift operator, $\Gamma$ be the Gray map and $\upsilon$ be the quasi-twisted shift operator as given before. Then $\Gamma\,\gamma_{(4)} = \upsilon\,\Gamma$.*

*Proof.* Let $q = (q_0, q_1, \ldots, q_{n-1}) \in S^n$, where $q_i = a_i + ub_i + vc_i \in S$ and $a_i, b_i, c_i \in \mathbb{Z}_5$, for $i = 0, 1, \ldots, n-1$ . Then $\Gamma\,\gamma_{(4)}(q) = \Gamma(4q_{n-1}, q_0, \ldots, q_{n-2}) = \Gamma(4a_{n-1} + u(4b_{n-1}) + v(4c_{n-1}), a_0 + ub_0 + vc_0, \ldots, a_{n-2} + ub_{n-2} + vc_{n-2}) = (4a_{n-1} + b_{n-1}, a_0 + 4b_0, \ldots, a_{n-2} + 4b_{n-2}, 4b_{n-1}, b_0, \ldots, b_{n-2}, 4c_{n-1}, c_0, \ldots, c_{n-2})$ ... (1) On the other hand, we have $\upsilon\,\Gamma(q) = \upsilon(a_0+4b_0, a_1+4b_1, \ldots, a_{n-1}+4b_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_0, c_1, \ldots, c_{n-1})$ $= (4(a_{n-1}+4b_{n-1}), a_0+4b_0, \ldots, a_{n-2}+4b_{n-2}, 4b_{n-1}, b_0, \ldots, b_{n-2}, 4c_{n-1}, c_0, \ldots, c_{n-2})$ $= (4a_{n-1} + b_{n-1}, a_0 + 4b_0, \ldots, a_{n-2} + 4b_{n-2}, 4b_{n-1}, b_0, \ldots, b_{n-2}, 4c_{n-1}, c_0, \ldots, c_{n-2})$ ... (2) Equality is obtained from (1) and (2).

**Corollary 8.** *A code $C$ is a 4-constacyclic code over $S$ if and only if $\Gamma(C)$ is a quasi-twisted code of index 3 over $\mathbb{Z}_5$ with length $3n$.*

*Proof.* The proof is given in article [7].

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