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A NOVEL BIVARIATE MITTAG-LEFFLER-TYPE FUNCTION AND THEIR PROPERTIES

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Abstract. In this paper, we attempted to present and study a novel Mittag-Leffler-type function with two variables. Its numerous properties, including Euler, Mellin, Laplace, Whittaker, and Mellin-Barnes integral representations, as well as operational and integral relationships with other known Mittag-Leffler functions of one variable, are established.

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1. Introduction and definitions

The Mittag-Leffler function has gained importance and popularity due to its applications in the solution of fractional order differential equations and fractional order integral equations (see for examples [5-7, 9-12,15,17,19-25]). Also, the Mittag-Leffler function plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, and signal processing. Besides this, the Mittag-Leffler function of several variables appears in the solution of certain boundary value problems involving fractional integrodifferential equations of Volterra type [22], initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation [12], and initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients [11].

In the usual notation $\Gamma(x)$ for the Gamma function and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$, $n \geq 0$ $\gamma \neq 0, -1, -2, ...,$ the Pochhammer symbol. Mittag-Leffler introduced the function $E_\alpha(z)$ in form [15]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha > 0, z \in \mathbb{C}). \tag{1.1}$$

Note that, the Mittag-Leffler function is a direct generalization of the exponential e^x function to which it reduces for $\alpha = 1$.

The Mittag-Leffler function (1.1) has since been extended by several workers. Next, we mention some generalizations of the Mittag-Leffler function in (1.1). In [25] Wiman introduced a generalization of $E_\alpha(z)$ with two parameters in the form:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0). \tag{1.2}$$

Prabhakar [17] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ of their parameter

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \tag{1.3}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0).$$

Further, three interesting unifications and generalizations of the function $E_\alpha(z)$ were considered by Skukla and Prajapati [24]

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n + \beta) n!}, \tag{1.4}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0, 1)).$$

Salim[20]

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)(\delta)_n}, \tag{1.5}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \delta > 0).$$

and Salim and Faraj [21]

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq} z^n}{\Gamma(\alpha n + \beta)(\delta)_{np}}, \tag{1.6}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \delta > 0, (p, q) > 0, q \leq \Re(\alpha) + p).$$

For our present study, we recall the following Mellin-Barnes integral representation for the function $E^{\gamma}_{\alpha,\beta}(z)$ [17]

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{2\pi i}\frac{1}{\Gamma(\gamma)}\int_L \frac{\Gamma(s)\Gamma(\gamma - s)}{\Gamma(\beta - \alpha s)}(-z)^{-s}ds, \tag{1.7}$$

$$(|\arg z| < \pi, \alpha \in \mathbb{R}^+, \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-, \Re(\beta) > 0).$$

Mehmet Ali and Banu Yilmaz [14] we extend the Mittag-Leffler function as follows :

$$E^{(\gamma;c)}_{\alpha,\beta}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + n, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.8}$$

$$(p \geq 0; \Re(c) > \Re(\gamma) > 0).$$

where for $B_p(x, y)$ we have

$$B_p(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0).$$

the extended Euler's Beta function defined in [1](see also [2]).

Gauhar Rahman et all. [4] introduced the following further generalization of the Mittag-Leffler function

$$E^{\gamma;q,c}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)(c)_{nq}}{B(\gamma, c - \gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.9}$$

$$(z, \beta, \gamma \in \mathbb{C}).$$

Very recently Bin-Saad et all. [13] suggest investigating and studying the following Mittag-Leffler function of two variables:

$$E_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} \frac{z_1^m z_2^{n-m}}{\Gamma(\alpha m + (n - m)\gamma + \beta)}, \tag{1.10}$$

$$(\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0).$$

In this work, we aim to introduce and study the following Mittag-Leffler-type function of two variables , in the form:

$$M_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{z_1^m z_2^n}{\Gamma(\alpha m + \gamma n + \beta)}, \tag{1.11}$$

$$(\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0).$$

It may be of interest to point out that the series representation (1.11), in particular yields the following relationships:

(i) For $n \mapsto 0$,$\beta \mapsto 1$, we get

$$M_{\alpha,1,\gamma}(z,0) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+1)} = E_\alpha(z), \tag{1.12}$$

(ii) For $n \mapsto 0$, we get

$$M_{\alpha,\beta,\gamma}(z,0) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+\beta)} = E_{\alpha,\beta}(z), \tag{1.13}$$

(iii) For $m \mapsto 0$, we get

$$M_{\alpha,\beta,\gamma}(0,z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gamma n+\beta)} = E_{\gamma,\beta}(z), \tag{1.14}$$

we infer from (1.11) and (1.2) that

$$M_{\alpha,\beta,\gamma}(z_1,z_2) = \sum_{m=0}^{\infty} E_{\gamma,\alpha m+\beta}(z_2) z_1^m, \tag{1.15}$$

and

$$M_{\alpha,\beta,\gamma}(z_1,z_2) = \sum_{n=0}^{\infty} E_{\alpha,\gamma n+\beta}(z_1) z_2^n, \tag{1.16}$$

Formulas (1.15) and (1.16) are very useful in obtaining other needed properties for the function $M_{\alpha,\beta,\gamma}(z_1,z_2)$. Note that the function $E^{\gamma}_{\alpha,\beta}(z)$ is a special case of the Wright generalized hypergeometric function ${}_p\Psi_q$ (see [7]):

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1 \left[z| \begin{array}{c} (\gamma,1) \\ (\beta,\alpha) \end{array} \right], \tag{1.17}$$

where (see[24]):

$${}_p\Psi_q \left[\begin{array}{c} (\alpha_1,A_1),...,(\alpha_P,A_P); \\ (\beta_1,B_1),...,(\beta_q,B_q); \end{array} \right] = \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{p}\Gamma(\alpha_j+A_j n)}{\Pi_{j=0}\Gamma(\beta_j+B_j n)} \frac{z^n}{n!}, \tag{1.18}$$

and the coefficients $A_1, ..., A_p$ and $B_1, ..., B_q$ are positive real numbers such that

$$1+\sum_{j=0}^{q} B_j - \sum_{j=0}^{p} A_j \geq 0.$$

Hence the series expansions (1.15) and (1.16), can be rewritten in the forms

$$E_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{m=0}^{\infty} {}_1\Psi_1 \left[z_2 | \begin{array}{c} (1,1) \\ (\alpha m+\beta, \gamma) \end{array} \right] z_1^m, \tag{1.19}$$

and

$$E_{\alpha,\beta,\gamma}(z_1, z_2) = \sum_{n=0}^{\infty} {}_1\Psi_1 \left[z_1 | \begin{array}{c} (1,1) \\ (\gamma n+\beta, \alpha) \end{array} \right] z_1^m, \tag{1.20}$$

respectively[10pt]

This paper is rather technical and is devoted to some analytic and computational properties of the Mittag-Leffler function of two variables $M_{\alpha,\beta,\gamma}(z_1, z_2)$. Indeed, this work contains a variety of useful tools in analysis that are of interest to anyone working in this type of special function. The layout of the paper is as follows. In Section 2, we give several integral representations involving the bivariate Mittag-Leffler function $M_{\alpha,\beta,\gamma}(z_1, z_2)$. In Section 3, we establish fractional calculus relations and connections for $M_{\alpha,\beta,\gamma}(z_1, z_2)$ with other known Mittag-Leffler functions. In Section 4, we obtain some differential and pure recurrence relations. In Section 5, we discuss some useful integral transforms like Mellin transform, Laplace transform, Euler transform and Whittaker transform.

2. Integral representations

In many situations, an integral representation of the Mittag-Leffler function is more convenient to use than its series representation. First of all, we establish an integral representation for $M_{\alpha,\beta,\gamma}(z_1, z_2)$ that is derived directly from Hankle representation of Gamma function $\Gamma(z)$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_c e^t t^{-z} dt, \tag{2.1}$$

where the path of integration is a simple loop beginning and ending at $-\infty$ and encircling the origin in the positive direction.

Theorem 1. *:Let $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$, then*

$$M_{\alpha,\beta,\gamma}(z_1, z_2) = \frac{1}{2\pi i} \int_c e^t t^{-\beta} (1 - z_1 t^{\alpha})^{-1} (1 - z_2 t^{\gamma})^{-1} dt. \tag{2.2}$$

Proof. :By Letting $z = \alpha m + \gamma n + \beta$ in (2.1) and multiplying both sides by $z_1^m z_2^n$, we get

$$\frac{z_1^m z_2^n}{\Gamma(\alpha m + \gamma n + \beta)} = \frac{1}{2\pi i}\int_c e^t t^{-(\beta+\alpha m+\gamma n)} z_1^m z_2^n dt \tag{2.3}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m + n\gamma + \beta)} = \frac{1}{2\pi i}\int_c e^t t^{-\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(z_1 t^{\alpha})^m (z_2 t^{\gamma})^n dt$$

Hence

$$M_{\alpha,\beta,\gamma}(z_1, z_2) = \frac{1}{2\pi i}\int_c e^t t^{-\beta}(1 - z_1 t^{\alpha})^{-1}(1 - z_2 t^{\gamma})^{-1} dt,$$

we led finally to the desired result (2.2).

Next, we express $M_{\alpha,\beta,\gamma}(z_1, z_2)$ as the Mellin-Barnes type integral.

Theorem 2. *:Let $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$ be Satisfied. Then $M_{\alpha,\beta,\gamma}(z_1, z_2)$ is reper-sented via the double Mellin-Barnes-type integralas*

$$M_{\alpha,\beta,\gamma}(z_1, z_2) = \frac{1}{(2\pi i)^2}\int_{L_t}\int_{L_s}\frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-t)(-z_1)^{-t}(-z_2)^{-s}}{\Gamma(\beta - \gamma s - \alpha t)} dt ds, \tag{2.4}$$

where $\{|\arg(Z_1)|, |\arg(Z_2)|\} < \pi$; and we assume that the contour L_s is in the s-plane and runs from $c - i\infty$ to $c + i\infty$ and the contour L_t is in the t-plane and runs from $c - i\infty$ to $c + i\infty$

Proof. :Starting from the assertion (1.15) and replacing $E^{\gamma}_{\alpha,\beta}(z)$ by its Mellin-Barnes integral representation (1.7), in the when $\gamma = 1$ we get

$$\begin{aligned}
M_{\alpha,\beta,\gamma}(z_1, z_2) &= \sum_{m=0}^{\infty}\left[\frac{1}{2\pi i}\int_{L_s}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta + \alpha m - \gamma s)}(-z_2)^{-s} ds\right] z_1^m, \\
&= \frac{1}{2\pi i}\int_{L_s}\Gamma(s)\Gamma(1-s)(-z_2)^{-s}\left\{\sum_{m=0}^{\infty}\frac{z_1^m}{\Gamma(\beta + \alpha m - \gamma s)}\right\} ds, \\
&= \frac{1}{2\pi i}\int_{L_s}\Gamma(s)\Gamma(1-s)(-z_2)^{-s}\left\{E_{\alpha,\beta-\gamma s}(Z_1)\right\} ds, \\
&= \frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}\frac{\Gamma(s)\Gamma(1-s)(-z_2)^{-s}\Gamma(t)\Gamma(1-t)(-z_1)^{-t}}{\Gamma(\beta - \gamma s - \alpha t)} dt ds, \\
M_{\alpha,\beta,\gamma}(z_1, z_2) &= \frac{1}{(2\pi i)^2}\int_{L_s}\int_{L_t}\frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-t)(-z_1)^{-t}(-z_2)^{-s}}{\Gamma(\beta - \gamma s - \alpha t)} dt ds,
\end{aligned}$$

which is the desired result. Further, we derive several integral formulas involving $M_{\alpha,\beta,\gamma}(z_1,z_2)$.

Theorem 3. *:Let* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$*, then*

$$M_{\alpha,\beta+\lambda,\gamma}(z_1,z_2) = \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt, \tag{2.5}$$

$$M_{\alpha,\beta+\lambda,\gamma}(z_1,z_2) = \frac{1}{\Gamma(\gamma)}\int_0^1 t^{\lambda-1}(1-t)^{\beta-1}M_{\alpha,\beta,\gamma}(z_1(1-t)^{\alpha},z_2(1-t)^{\gamma})dt, \tag{2.6}$$

$$z^{\beta}M_{\alpha,\beta+1,\gamma}(w_1z^{\alpha},w_2z^{\gamma}) = \int_0^z t^{\beta-1}M_{\alpha,\beta,\gamma}(w_1t^{\alpha},w_2t^{\gamma})dt, \tag{2.7}$$

$$s^{-\beta}(1-\frac{z_1}{s^{\alpha}})^{-1}(1-\frac{z_2}{s^{\gamma}})^{-1} = \int_0^{\infty} e^{-st}t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt, \tag{2.8}$$

$$\begin{aligned}&(x-t)^{\beta+\delta-1}M_{\alpha,\beta+\delta,\gamma}(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma})\\ &\quad= \frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\beta-1}M_{\alpha,\beta,\gamma}(w_1(s-t)^{\alpha},w_2(s-t)^{\gamma})ds,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&x^{\lambda+\beta-1}M_{\alpha,\beta+\lambda,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\\ &= \frac{1}{\Gamma(\delta)}\int_0^x t^{\lambda-1}(x-t)^{\beta-1}\left\{M_{\alpha,\beta,\gamma}(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma})\times M_{\alpha,\beta+\lambda,\gamma}(w_1t^{\alpha},w_2t^{\gamma})\right\}dt,\end{aligned} \tag{2.10}$$

Proof. : we have

$$\begin{aligned}&\int_0^1 t^{\beta-1}(1-t)^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha},z_2t^{\gamma})dt\\ &\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\int_0^1 t^{\alpha m+\gamma n+\beta-1}(1-t)^{\lambda-1}dt\\ &\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\frac{\Gamma(\alpha m+\gamma n+\beta)\Gamma(\lambda)}{\Gamma(\alpha m+\gamma n+\beta+\lambda)}\\ &\qquad= \Gamma(\lambda)M_{\alpha,\beta+\lambda,\gamma}(z_1,z_2)\end{aligned}$$

which is the proof of (2.5).

we have

$$\int_0^1 t^{\lambda-1}(1-t)^{\beta-1}M_{\alpha,\beta,\gamma}(z_1(1-t)^{\alpha}, z_2(1-t)^{\gamma})dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\int_0^1 t^{\lambda-1}(1-t)^{\alpha m+\gamma n+\beta-1}dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\frac{\Gamma(\lambda)\Gamma(\alpha m+\gamma n+\beta)}{\Gamma(\alpha m+\gamma n+\beta+\lambda)}$$
$$= \Gamma(\lambda)M_{\alpha,\beta+\lambda,\gamma}(z_1, z_2).$$

which is the proof of (2.6).

we have

$$\int_0^z t^{\beta-1}M_{\alpha,\beta,\gamma}(w_1t^{\alpha}, w_2t^{\gamma})dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\int_0^z t^{\alpha m+\gamma n+\beta-1}dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\frac{z^{\alpha m+\gamma n+\beta}}{(\alpha m+\gamma n+\beta)}$$
$$= z^{\beta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(w_1z^{\alpha})^m(w_2z^{\gamma})^n}{\Gamma(\alpha m+\gamma n+\beta+1)}$$
$$= z^{\beta}M_{\alpha,\beta+1,\gamma}(w_1z^{\alpha}, w_2z^{\gamma}).$$

which is the proof of (2.7)

we have

$$\int_0^{\infty} e^{-st}t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha}, z_2t^{\gamma})dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\int_0^{\infty} e^{-st}t^{\alpha m+\gamma n+\beta-1}dt$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n s^{-(\alpha m+\gamma n+\beta)}$$
$$= s^{-\beta}(1-\frac{z_1}{s^{\alpha}})^{-1}(1-\frac{z_2}{s^{\gamma}})^{-1}.$$

which is the proof of (2.8).

Let $u = \frac{s-t}{x-t}$ in the left-hand side of the assertion (2.9), then

$$\frac{1}{\Gamma(\delta)}\int_t^x (x-s)^{\delta-1}(s-t)^{\beta-1}M_{\alpha,\beta,\gamma}(w_1(s-t)^{\alpha}, w_2(s-t)^{\gamma})ds$$
$$= \frac{1}{\Gamma(\delta)}\int_0^1 (x-t)^{\delta-1}(1-u)^{\delta-1}u^{\beta-1}(x-t)^{\beta-1}(x-t)\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m w_2^n u^{\alpha m+\gamma}(x-t)^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\beta)}du$$
$$= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m w_2^n (x-t)^{\alpha m+\gamma n+\delta+\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^1 (1-u)^{\delta-1}u^{\alpha m+\gamma n+\beta-1}du$$
$$= \frac{1}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w_1^m w_2^n (x-t)^{\alpha m+\gamma n+\delta+\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\times\frac{\Gamma(\delta)\Gamma(\alpha m+\gamma n+\beta)}{\Gamma(\alpha m+\gamma n+\beta+\delta)}$$
$$= (x-t)^{\beta+\delta-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(w_1(x-t)^{\alpha})^m(w_2(x-t)^{\gamma})^n}{\Gamma(\alpha m+\gamma n+\beta+\delta)}$$
$$= (x-t)^{\beta+\delta-1}M_{\alpha,\beta+\delta,\gamma}(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}).$$

which is the proof of (2.9).
Starting from the left-hand side of assertion (2.10), we have

$$\int_0^x t^{\lambda-1}(x-t)^{\beta-1}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(w_1(x-t)^{\alpha})^m(w_2(x-t)^{\gamma})^n}{\Gamma(\alpha m+\gamma n+\beta)}\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(w_1t^{\alpha})^p(w_2t^{\gamma})^q}{\Gamma(\alpha p+\gamma q+\lambda)}\right\}dt$$
$$= \sum_{m,n,p,q=0}^{\infty}\frac{w_1^{m+p}w_2^{n+q}}{\Gamma(\alpha m+\gamma n+\beta)\Gamma(\alpha p+\gamma q+\lambda)}\times\int_0^x t^{\alpha p+\gamma q+\lambda-1}(x-t)^{\alpha m+\gamma n+\beta-1}dt$$
$$= \sum_{m,n,p,q=0}^{\infty}\frac{w_1^{m+p}w_2^{n+q}}{\Gamma(\alpha m+\gamma n+\beta)\Gamma(\alpha p+\gamma q+\lambda)}$$
$$\times\frac{\Gamma(\alpha p+\gamma q+\lambda)\Gamma(\alpha m+\gamma n+\beta)x^{\alpha(p+m)+\gamma(q+n)+\lambda+\beta-1}}{\Gamma(\alpha(p+m)+\gamma(q+n)+\beta)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(w_1x^{\alpha})^m(w_2x^{\gamma})^n x^{\lambda+\beta-1}}{\Gamma(\alpha m+\gamma n+\beta+\lambda)}$$
$$= x^{\lambda+\beta-1}M_{\alpha,\beta+\lambda,\gamma}(w_1x^{\alpha}, w_2x^{\gamma}).$$

3. Fractional Calculus Images

In this section, we turn to some fractional calculus images for the Mittag-Leffler function of two variables $M_{\alpha,\beta,\gamma}(z_1, z_2)$. Let us note that in mathematical treatises on fractional differential equations, the Riemann-Liouville approach to the notion of the fractional derivative of order $\mu(\mu \geq 0)$ is normally used (see [22]):

$$(\hat{D}^v f)(x) := \left(\frac{d}{dx}\right)^m (J^{m-v} f)(x), \quad m-1 < v \leqslant m, m \in \mathbb{N}, x>0, \tag{3.1}$$

where

$$(J^v f)(x) := \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \quad v > 0, x > 0,$$
$$(J^0 f)(x) := f(x), \quad x > 0,$$

is the Riemann-Liouville fractional integral of order v. For the case when $f(x) = x^a$, $a > -1$, we get the results

$$\hat{D}_z^v z^a = \frac{\Gamma(a+1)}{\Gamma(a-v+1)} z^{a-v}, \tag{3.2}$$

where $\hat{D}_z^v = (\frac{d}{dz})^v, z > 0, a > -1, v \geq 0$ is not restricted to integer values. Also, we recall the following relations from [16, p.74-75]. For $m-1 \leq p < m, n-1 \leq q < n$, we have

$$\hat{D}_t^p \hat{D}_t^q f(t) = \hat{D}_t^q \hat{D}_t^p f(t) = \hat{D}_t^{p+q} f(t) \Longleftrightarrow f^{(j)}(0) = 0, \tag{3.3}$$

and

$$\hat{D}_t^p \hat{D}_t^q f(t) = \hat{D}_t^{p+q} - \sum_{j=1}^{n} \left[\hat{D}_t^{q-j} f(t)\right]_{t=0} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}, \tag{3.4}$$

By exploiting the results (3.1) to (3.4), we can derive the following fractional connections for the function $M_{\alpha,\beta,\gamma}(z_1, z_2)$ with the different Mittag-Leffler functions defined in (1.1) to (1.6) with one variable.

Theorem 4. *Let* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$, $0 < \beta < 1$,$0 < \delta < 1$, $\Re(\delta) > (\lambda)p > 0$, $q > 0$ *, then the following fractional relations hold :*

$t^{\beta-1} M_{\alpha,\beta,\gamma}(z_1 t^\alpha, z_2 t^\gamma)$

$$= \hat{D}_t^{1-\beta}(1 - z_1\hat{D}_t^{-\alpha})^{-1}E_\alpha(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha)), \tag{3.5}$$

$$t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha, z_2t^\gamma)$$
$$= (1 - z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha))\left\{t^{\beta-1}\right\}, \tag{3.6}$$

$$t^{\beta-1}u^{\delta-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha u, z_2t^\gamma u)$$
$$= \Gamma(\delta)\hat{D}_u^{1-\delta}\left\{(1 - z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\delta}(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha u))\right\}\left\{t^{\beta-1}\right\}, \tag{3.7}$$

$$t^{\beta-1}u^{\delta-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha u^q, z_2t^\gamma u^q)$$
$$= \Gamma(\delta)\hat{D}_u^{1-\delta}\left\{(1 - z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\delta,q}(z_2(\hat{D}_u^{1-q}\hat{D}_t^{\alpha-\gamma}t^\alpha u))\right\}\left\{t^{\beta-1}\right\}, \tag{3.8}$$

$$t^{\beta-1}u^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha u, z_2t^\gamma u)$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{(1 - z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\lambda,\delta}(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha u))\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}, \tag{3.9}$$

$$t^{\beta-1}u^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha u^p, z_2t^\gamma u^p)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{(1 - z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta,p}^{\lambda,\delta,q}(z_2(\hat{D}_u^{p-q}\hat{D}_t^{\alpha-\gamma}t^\alpha u^p))\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}, \tag{3.10}$$

Proof. :we have

$$\begin{aligned}
&\hat{D}_t^{1-\beta}(1-z_1\hat{D}_t^{-\alpha})^{-1}E_\alpha(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha))\\
&\qquad= \hat{D}_t^{1-\beta}\sum_{m=0}^{\infty} z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{z_2^n(\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n})}{\Gamma(\alpha n+1)}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha n+1)}\hat{D}_t^{-\alpha m+(\alpha-\gamma)n-\beta+1}t^{\alpha n}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^\alpha)^m(z_2t^\gamma)^n t^{\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad= t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha, z_2t^\gamma)
\end{aligned}$$

which is the proof of (3.5).
Next, we have

$$\begin{aligned}
&(1-z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}\left(z_2(\hat{D}_t^{\alpha-\gamma}t^\alpha)\right)\left\{t^{\beta-1}\right\}\\
&\qquad= \sum_{m=0}^{\infty} z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{z_2^n\left(\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n}\right)}{\Gamma(\alpha n+\beta)}\left\{t^{\beta-1}\right\}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n}{\Gamma(\alpha n+\beta)}\hat{D}_t^{-\alpha m+(\alpha-\gamma)n}t^{\alpha n+\beta-1}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^\alpha)^m(z_2t^\gamma)^n t^{\beta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad= t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1t^\alpha, z_2t^\gamma)
\end{aligned}$$

which is the proof of (3.6).

We have

$$\begin{aligned}
\Gamma(\delta)\hat{D}_u^{1-\delta}&\left\{(1-z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\delta}(z_2(\hat{D}_t^{\alpha-\gamma}t^{\alpha}u))\right\}\left\{t^{\beta-1}\right\}\\
&=\Gamma(\delta)\hat{D}_u^{1-\delta}\left\{\sum_{m=0}^{\infty}z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{(\delta)_n z_2^n(\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n}u^n)}{n!\Gamma(\alpha n+\beta)}\right\}\left\{t^{\beta-1}\right\}\\
&=\Gamma(\delta)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_n z_1^m z_2^n}{n!\Gamma(\alpha n+\beta)}(\hat{D}_t^{-\alpha m+(\alpha-\gamma)n}t^{\alpha n+\beta-1})(\hat{D}_u^{1-\delta}u^n)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}u^{n+\delta-1}}{\Gamma(\alpha m+\gamma(n-m)+\beta)}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^{\alpha}u)^m(z_2t^{\gamma}u)^n t^{\beta-1}u^{\delta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&=t^{\beta-1}u^{\delta-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha}u,z_2t^{\gamma}u)
\end{aligned}$$

which is the proof of (3.7).

We have

$$\begin{aligned}
\Gamma(\delta)\hat{D}_u^{1-\delta}&\left\{(1-z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\delta,q}(z_2(\hat{D}_u^{1-q}\hat{D}_t^{\alpha-\gamma}t^{\alpha}u)\right\}\left\{t^{\beta-1}\right\}\\
&=\Gamma(\delta)\hat{D}_u^{1-\delta}\left\{\sum_{m=0}^{\infty}z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{(\delta)_{nq}z_2^n\hat{D}_u^{n-qn}(\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n}u^n)}{n!\Gamma(\alpha n+\beta)}\right\}\left\{t^{\beta-1}\right\}\\
&=\Gamma(\delta)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{nq}z_1^m z_2^n}{n!\Gamma(\alpha n+\beta)}(\hat{D}_t^{-\alpha m+(\alpha-\gamma)n}t^{\alpha n+\beta-1})(\hat{D}_u^{1-\delta+(1-q)n}u^n)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}u^{qn+\delta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^{\alpha}u^q)^m(z_2t^{\gamma}u^q)^n t^{\beta-1}u^{\delta-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&=t^{\beta-1}u^{\delta-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha}u^q,z_2t^{\gamma}u^q)
\end{aligned}$$

which is the proof of (3.8).

We have

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{(1-z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta}^{\lambda,\delta}(z_2(\hat{D}_t^{\alpha-\gamma}t^{\alpha}u)\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}\\
&\qquad=\frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{\sum_{m=0}^{\infty}z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{(\lambda)_n z_2^n(\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n}u^n)}{(\delta)_n\Gamma(\alpha n+\beta)}\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}\\
&\qquad=\frac{\Gamma(\lambda)}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_n z_1^m z_2^n}{(\delta)_n\Gamma(\alpha n+\beta)}(\hat{D}_t^{-\alpha m+(\alpha-\gamma)n}t^{\alpha n+\beta-1})(\hat{D}_u^{\delta-\lambda}u^{n+\lambda-1})\\
&\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}u^{n+\lambda-1}}{\Gamma(\alpha m+\gamma(n-m)+\beta)}\\
&\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^{\alpha}u)^m(z_2t^{\gamma}u)^n t^{\beta-1}u^{\lambda-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad=t^{\beta-1}u^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha}u,z_2t^{\gamma}u)
\end{aligned}$$

which is the proof of (3.9).

We have

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{(1-z_1\hat{D}_t^{-\alpha})^{-1}E_{\alpha,\beta,p}^{\lambda,\delta,q}\left(z_2\hat{D}_u^{p-q}\hat{D}_t^{\alpha-\gamma}t^{\alpha}u^p\right)\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}\\
&\qquad=\frac{\Gamma(\lambda)}{\Gamma(\delta)}\hat{D}_u^{\delta-\lambda}\left\{\sum_{m=0}^{\infty}z_1^m\hat{D}_t^{-\alpha m}\sum_{n=0}^{\infty}\frac{(\lambda)_{nq}z_2^n\hat{D}_u^{np-nq}\hat{D}_t^{\alpha n-\gamma n}t^{\alpha n}u^{pn}}{(\delta)_{np}\Gamma(\alpha n+\beta)}\right\}\left\{t^{\beta-1}u^{\delta-1}\right\}\\
&\qquad=\frac{\Gamma(\lambda)}{\Gamma(\delta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{nq}z_1^m z_2^n}{(\delta)_{np}\Gamma(\alpha n+\beta)}(\hat{D}_t^{-\alpha m+(\alpha-\gamma)n}t^{\alpha n+\beta-1})(\hat{D}_u^{\delta-\lambda+np-nq}u^{np+\delta-1})\\
&\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m z_2^n t^{\alpha m+\gamma n+\beta-1}u^{np+\lambda-1}}{\Gamma(\alpha m+\gamma(n-m)+\beta)}\\
&\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z_1t^{\alpha}u^p)^m(z_2t^{\gamma}u^p)^n t^{\beta-1}u^{\lambda-1}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&\qquad=t^{\beta-1}u^{\lambda-1}M_{\alpha,\beta,\gamma}(z_1t^{\alpha}u^p,z_2t^{\gamma}u^p)
\end{aligned}$$

which is the proof of (3.10).

4. Differential and Pure Recurrence relations

By the definitions (1.11) and by employing the operator (3.2), we derive the following interesting differential recurrence relations.

Theorem 5. *: If* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$*,* $p \in \mathbb{N}$ *, then*

$$\hat{D}^p_{z_1}\left\{M_{\alpha,\beta,\gamma}(z_1, z_2)\right\} = \Gamma(P+1)\sum_{n=0}^{\infty}\left\{E^{p+1}_{\alpha,\beta+\gamma n+\alpha p}(z_1)\right\} z_2^n, \tag{4.1}$$

$$\hat{D}^p_{z_2}\left\{M_{\alpha,\beta,\gamma}(z_1, z_2)\right\} = \Gamma(P+1)\sum_{n=0}^{\infty}\left\{E^{p+1}_{\gamma,\beta+\gamma p+\alpha m}(z_2)\right\} z_1^m, \tag{4.2}$$

$$\hat{D}^p_{t}\left\{t^{\beta-1}M_{\alpha,\beta,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma})\right\} = t^{\beta-p-1}M_{\alpha,\beta-p,\gamma}(z_1 t^{\alpha}, z_2 t^{\gamma}), \tag{4.3}$$

Proof. : From definitions (1.11) and (3.2) , we find that

$$\begin{aligned}\hat{D}^p_{z_1}\left\{M_{\alpha,\beta,\gamma}(z_1, z_2)\right\} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m! z_1^{m-p} z_2^n}{(m-p)!\Gamma(\alpha m+\gamma n+\beta)}, \quad p \le m\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+p)! z_1^{m} z_2^n}{m!\Gamma(\alpha(m+p)+\gamma n+\beta)}\\ &= \Gamma(P+1)\sum_{n=0}^{\infty}\left\{E^{p+1}_{\alpha,\beta+\gamma n+\alpha p}(z_1)\right\} z_2^n,\end{aligned}$$

which is the result (4.1).

$$\begin{aligned}\hat{D}^p_{z_2}\left\{M_{\alpha,\beta,\gamma}(z_1, z_2)\right\} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n! z_1^{m} z_2^{n-p}}{(n-p)!\Gamma(\alpha m+\gamma n+\beta)}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+p)! z_1^{m} z_2^n}{n!\Gamma(\alpha m+\gamma(n+p)+\beta)}\\ &= \Gamma(P+1)\sum_{n=0}^{\infty}\left\{E^{p+1}_{\gamma,\beta+\gamma p+\alpha m}(z_2)\right\} z_1^m,\end{aligned}$$

which is the result (4.2).

Finally, from(1.11), we have

$$\begin{aligned}
\hat{D}_t^p \left\{ t^{\beta-1} M_{\alpha,\beta,\gamma}(z_1 t^\alpha, z_2 t^\gamma) \right\} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z_1^m z_2^n}{\Gamma(\alpha m + \gamma n + \beta)} \hat{D}_t^p t^{\alpha m + \gamma n + \beta - 1} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z_1^m z_2^n t^{\alpha m + \gamma n + \beta - p - 1}}{\Gamma(\alpha m + \gamma n + \beta - p)} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(z_1 t^\alpha)^m (z_2 t^\gamma)^n t^{\beta - p - 1}}{\Gamma(\alpha m + \gamma n + \beta - p)} \\
&= t^{\beta-p-1} M_{\alpha,\beta-p,\gamma}(z_1 t^\alpha, z_2 t^\gamma).
\end{aligned}$$

Now, we derive anotherdifferential recurrence relations for the Mittag-Leffler function $M_{\alpha,\beta,\gamma}(z_1, z_2)$.

Theorem 6. *If* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$ *, then*

$$M_{\alpha,\beta,\gamma}(z_1^\alpha, z_2^\gamma) = \beta M_{\alpha,\beta+1,\gamma}(z_1^\alpha, z_2^\gamma) + z_1 \frac{d}{dz_1} M_{\alpha,\beta+1,\gamma}(z_1^\alpha, z_2^\gamma) + z_2 \frac{d}{dz_2} M_{\alpha,\beta+1,\gamma}(z_1^\alpha, z_2^\gamma), \tag{4.4}$$

$$M_{\alpha,\beta,\gamma}(z_1, z_2) = \beta M_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_1 \alpha \frac{d}{dz_1} M_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_2 \gamma \frac{d}{dz_2} M_{\alpha,\beta+1,\gamma}(z_1, z_2), \tag{4.5}$$

$$M_{\alpha,\beta,\gamma}(z_1 t^\alpha, z_2 t^\gamma) = \beta M_{\alpha,\beta+1,\gamma}(z_1 t^\alpha, z_2 t^\gamma) + t \frac{d}{dt} M_{\alpha,\beta+1,\gamma}(z_1 t^\alpha, z_2 t^\gamma), \tag{4.6}$$

Proof. : From the right-hand side of formula (4.4), we have

$$\begin{aligned}
&\beta M_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_1\frac{d}{dz_1}M_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma}) + z_2\frac{d}{dz_2}M_{\alpha,\beta+1,\gamma}(z_1^{\alpha}, z_2^{\gamma})\\
&= \beta\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{\alpha m}z_2^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&+ z_1\frac{d}{dz_1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{\alpha m}z_2^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)} + z_2\frac{d}{dz_2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{\alpha m}z_2^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{\alpha m}z_2^{\gamma n}(\beta+\alpha m+\gamma n)}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{\alpha m}z_2^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&= M_{\alpha,\beta,\gamma}(z_1^{\alpha}, z_2^{\gamma})
\end{aligned}$$

which is the left-hand side of formula (4.4) and then the proof is completed. Next, we have

$$\begin{aligned}
&\beta M_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_1\alpha\frac{d}{dz_1}M_{\alpha,\beta+1,\gamma}(z_1, z_2) + z_2\gamma\frac{d}{dz_2}M_{\alpha,\beta+1,\gamma}(z_1, z_2)\\
&= \beta\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{m}z_2^{n}}{\Gamma(\alpha m+\gamma n+\beta+1)} + z_1\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{mz_1^{m-1}z_2^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&+ z_2\gamma\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{nz_1^{m}z_2^{n-1}}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{m}z_2^{n}(\beta+\alpha m+\gamma n)}{\Gamma(\alpha m+\gamma n+\beta+1)}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^{m}z_2^{n}}{\Gamma(\alpha m+\gamma n+\beta)}\\
&= M_{\alpha,\beta,\gamma}(z_1, z_2)
\end{aligned}$$

which is the left-hand side of formula (4.5) and then the proof is completed. Finally, consider the right-hand side of formula (4.6), we have

$$\beta M_{\alpha,\beta+1,\gamma}(z_1t^\alpha, z_2t^\gamma) + t\frac{d}{dt}M_{\alpha,\beta+1,\gamma}(z_1t^\alpha, z_2t^\gamma)$$
$$= \beta\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m t^{\alpha m} z_2^n t^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m t^{\alpha m} z_2^n t^{\gamma n}(\alpha m+\gamma n)}{\Gamma(\alpha m+\gamma n+\beta+1)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha m+\gamma n+\beta) z_1^m t^{\alpha m} z_2^n t^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta+1)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z_1^m t^{\alpha m} z_2^n t^{\gamma n}}{\Gamma(\alpha m+\gamma n+\beta)}$$
$$= M_{\alpha,\beta,\gamma}(z_1t^\alpha, z_2t^\gamma)$$

which is the left-hand side of formula (4.6) and then the proof is completed.

5. Integral transforms

In this section, we establish some useful integral transforms like Euler transform, Laplace transform, Mellin transform and Whittaker transform. First, we introduce a Mellin integral type for the function $M_{\alpha,\beta,\gamma}(z_1, z_2)$. Recall that, the double Mellin transform of the function $f(x, y)$ is defined by [8]:

$$M = \{f(x,y) : s, t\}$$

$$= \int_0^\infty\int_0^\infty f(x,y)x^{s-1}y^{t-1}dsdt = f^*(s,t), \quad \Re(s) > 0, \Re(t) > 0, \qquad (5.1).$$

then

$$f(x,y) = \frac{1}{2\pi i}\int_L\int_L f^*(s,t)x^{-s}y^{-t}dsdt. \qquad (5.2).$$

By using the above definition, we prove the following result.

Theorem 7 (Double Mellin transform)**.** *: If* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$ *,then*

$$\int_0^\infty\int_0^\infty u_1^{s-1}u_2^{t-1}M_{\alpha,\beta,\gamma}(-w_1u_1, -w_2u_2)dsdt = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-t)}{w_1^t w_2^s\Gamma(\beta-\gamma s-\alpha t)}, \quad (5.3).$$

Proof. : In the Mellin-Barnes integral representation (2.4), let $z_1 = -w_1u_1$ and $z_2 = -w_2u_2$, we get

$$M_{\alpha,\beta,\gamma}(-w_1u_1, -w_2u_2)$$
$$= \frac{1}{(2\pi i)^2}\int_L\int_L \frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-t)(w_1u_1)^{-t}(w_2u_2)^{-s}}{\Gamma(\beta-\gamma s-\alpha t)}dsdt$$
$$= \frac{1}{(2\pi i)^2}\int_L\int_L f^*(s,t)u_1^{-t}u_2^{-s}dsdt,$$

where

$$f^*(s,t) = \frac{\Gamma(s)\Gamma(t)\Gamma(1-s)\Gamma(1-t)}{w_1^t\ w_2^s\ \Gamma(\beta-\gamma s-\alpha t)}.$$

Next, we aim to obtain Whittaker for the function $M_{\alpha,\beta,\gamma}(z_1, z_2)$.
First, we recall the definition of Whittaker function $W_{k,\mu}(z)$ (see[24, p.39(24)]):

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right)\ {}_1F_1\left(\mu-k+\frac{1}{2}; 2\mu+1; z\right),$$

where ${}_1F_1$ is Kummer's function [24] , and the integral formula:

$$\int_0^\infty e^{\frac{-t}{2}}t^{v-1}W_{k,\mu}(t)dt = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},\quad \Re(v\pm\mu) > -\frac{1}{2}.$$

Theorem 8 (Whittaker transform)**.** *If* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$*, then*

$$\int_0^\infty e^{\frac{-pt}{2}}\ t^{p-1}\ W_{k,\mu}(pt)\times M_{\alpha,\beta,\gamma}(w_1t^s, w_2t^\sigma)dt$$
$$= p^\rho\sum_{m=0}^\infty {}_2\Psi_2\left[\begin{array}{c} (\frac{1}{2}+\delta m+\rho+\mu,\sigma), (\frac{1}{2}+\delta m+\rho-\mu,\sigma); \\ (\beta+\alpha m,\gamma), (1+\delta m+\rho-\lambda,\sigma); \end{array} \frac{w_2}{p^\sigma}\right]\left(\frac{w_1}{p^\delta}\right)^m, \quad (5.4)$$

Proof. : Let $pt = u$ in the left-hand side of assertion (5.4), we get

$$\frac{1}{p}\int_0^\infty e^{\frac{-u}{2}}\left(\frac{u}{p}\right)^{\rho-1} W_{k,\mu}(u)\times M_{\alpha,\beta,\gamma}\left(w_1\left(\frac{u}{p}\right)^s, w_2\left(\frac{u}{p}\right)^\sigma\right)du$$

$$= p^{-\rho}\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{\left(\frac{w_1}{p^\delta}\right)^m\left(\frac{w_2}{p^\sigma}\right)^n}{\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^\infty e^{\frac{-u}{2}}u^{\delta m+\sigma n+\rho-1}W_{k,\mu}(u)du$$

$$= p^{-\rho}\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{\left(\frac{w_1}{p^\delta}\right)^m\left(\frac{w_2}{p^\sigma}\right)^n}{\Gamma(\alpha m+\gamma n+\beta)}\times\frac{\Gamma(\frac{1}{2}+\delta m+\sigma n+\rho+\mu)\Gamma(\frac{1}{2}+\delta m+\sigma n+\rho-\mu)}{\Gamma(1+\delta m+\sigma n+\rho-\lambda)}$$

$$= p^{\rho}\sum_{m=0}^\infty {}_2\Psi_2\left[\begin{array}{c} (\frac{1}{2}+\delta m+\rho+\mu,\sigma),(\frac{1}{2}+\delta m+\rho-\mu,\sigma); \\ \\ (\beta+\alpha m,\gamma),(1+\delta m+\rho-\lambda,\sigma); \end{array} \frac{w_2}{p^\sigma}\right]\left(\frac{w_1}{p^\delta}\right)^m$$

which in view of (1.18), yields the assertion (5.4).

Theorem 9 (Euler transform)**.** *If* $\Re(\beta)>\Re(\gamma)>\Re(\alpha)>0$ *, then*

$$\int_0^1\int_0^1 z_1^{a_1-1}(1-z_1)^{b_1-1}z_2^{a_2-1}(1-z_2)^{b_2-1}M_{\alpha,\beta,\gamma}(x_1z_1^\sigma,x_2z_2^\delta)dz_1dz_2$$

$$= \Gamma(b_1)\Gamma(b_2)\sum_{m=0}^\infty {}_1\Psi_2\left[\begin{array}{c} (a_2,\delta); \\ \\ (\beta+\alpha m,\gamma),(a_2+b_2,\delta); \end{array} x_2\right]\frac{\Gamma(a_1+\sigma m)\ x_1^m}{\Gamma(a_1+b_1+\sigma m)}, \quad (5.5)$$

Proof. : Denote the left-hand side of equation (5.5) by I, then from the definition(1.11), we get

$$I=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{x_1^mx_2^n}{\Gamma(\alpha m+\gamma n+\beta)}\times\int_0^\infty z_1^{\sigma m+a_1-1}(1-z_1)^{b_1-1}dz_1\times\int_0^\infty z_2^{\delta n+a_2-1}(1-z_2)^{b_2-1}dz_2.$$

Now, by using the Beta function (see, e.g.[18]):

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we obtain

$$I=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\Gamma(\sigma m+a_1)\Gamma(b_1)\Gamma(\delta n+a_2)\Gamma(b_2)\ x_1^m\ x_2^n}{\Gamma(\alpha m+\gamma n+\beta)\Gamma(\sigma m+a_1+b_1)\Gamma(\delta n+a_2+b_2)},$$

which because of (1.18), yields the assertion (5.5).

Theorem 10 (Laplace transform)**.** *If* $\Re(\beta) > \Re(\gamma) > \Re(\alpha) > 0$ *, then*

$$\int_0^\infty \int_0^\infty z_1^{a_1-1}\ e^{-s_1 z_1}\ z_2^{a_2-1}\ e^{-s_2 z_2}\ M_{\alpha,\beta,\gamma}(x_1 z_1^{\sigma_1}, x_2 z_2^{\sigma_2}) dz_1 dz_2$$

$$= s_1^{-a_1} s_2^{-a_2} \sum_{m=0}^\infty {}_1\Psi_1 \left[\begin{array}{c} (a_2, \sigma_2); \\ \\ (\beta + \alpha m, \gamma); \end{array} x_2\ s_2^{-\sigma_2} \right] \times \Gamma(a_1 + \sigma_1 m)\ x_1^m\ s_1^{-\sigma_1 m}, \quad (5.6)$$

Proof. : Denote the left-hand side of equation (5.6) by me, then from the definition(1.11), we get

$$I = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{x_1^m x_2^n}{\Gamma(\alpha m + \gamma n + \beta)} \times \int_0^\infty z_1^{\sigma_1 m + a_1 - 1} e^{-s_1 z_1} dz_1 \times \int_0^\infty z_2^{\sigma_2 n + a_2 - 1} e^{-s_2 z_2} dz_2, \tag{5.7}$$

Now, by using the integral representation of the Gamma function [18]

$$a^{-z}\ \Gamma(z) = \int_0^\infty t^{z-1}\ e^{-at}\ dt,$$

we obtain

$$= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\Gamma(\sigma_1 m + a_1)\Gamma(\sigma_2 n + a_2)\ x_1^m\ x_2^n\ s_1^{-(\sigma_1 m + a_1)}}{\Gamma(\alpha m + \gamma n + \beta)},$$

$$= s_1^{-a_1} s_2^{-a_2} \sum_{m=0}^\infty {}_1\Psi_1 \left[\begin{array}{c} (a_2, \sigma_2); \\ \\ (\beta + \alpha m, \gamma); \end{array} x_2 s_2^{-\sigma_2} \right] \times \Gamma(a_1 + \sigma_1 m)\ x_1^m\ s_1^{-\sigma_1 m}$$

which because of (1.18), yields the assertion (5.6).

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