MULTIVALUED STARLIKE FUNCTIONS OF COMPLEX ORDER

M. Çağlar, Y. Polatoğlu, A. Şen,
E. Yavuz and S. Owa

Abstract. Let $\mathcal{A}_\alpha$ be the class of functions $f(z) = z^\alpha(z + a_2z^2 + \cdots)$ which are analytic in the open unit disc $\mathbb{U}$. For $f(z) \in \mathcal{A}_\alpha$ using the fractional calculus, a subclass $\mathcal{S}^*_\alpha(1-b)$ which is the class of starlike functions of complex order $(1-b)$ is introduced. The object of the present paper is to discuss some properties for $f(z)$ belonging to the class $\mathcal{S}^*_\alpha(1-b)$.

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1. Introduction

Let $\mathcal{A}_\alpha$ denote the class of functions $f(z)$ of the form

$$f(z) = z^\alpha \left(z + \sum_{n=2}^{\infty} a_n z^n \right) \quad (0 < \alpha < 1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} | |z| < 1\}$. Let Ω be the class of analytic functions $w(z)$ in $\mathbb{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$. Also, let $\mathcal{P}$ denote the class of functions $p(z)$ which are analytic in $\mathbb{U}$ with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ $(z \in \mathbb{U})$.

For analytic functions $g(z)$ in $\mathbb{U}$, we introduce the definitions of fractional calculus (fractional integrals and fractional derivatives) given by Owa [4], [5], also by Srivastava and Owa [6].

Definition 1 *The fractional integral of order λ for an analytic function $g(z)$ in $\mathbb{U}$ is defined by*

$$\mathrm{D}_z^{-\lambda} g(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{g(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0),$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 2 *The fractional derivative of order λ for an analytic function $g(z)$ in $\mathbb{U}$ is defined by*

$$\mathrm{D}_z^{\lambda} g(z) = \frac{d}{dz}(\mathrm{D}_z^{\lambda-1} g(z)) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{g(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 3 *Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ for an analytic function $g(z)$ in $\mathbb{U}$ is defined by*

$$\mathrm{D}_z^{\lambda+n} g(z) = \frac{d^n}{dz^n}(\mathrm{D}_z^{\lambda} g(z)) \quad (0 \leq \lambda < 1, n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}).$$

Remark 4 *From the definitions of the fractional calculus, we see that*

$$\mathrm{D}_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \quad (\lambda > 0, k > 0),$$

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (0 \leq \lambda < 1, k > 0)$$

$$\mathrm{D}_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \ (0 \leq \lambda < 1, k > 0, n \in \mathbb{N}_0, \ k-n \neq -1, -2, -3, \cdots).$$

and

$$\mathrm{D}_z^{\lambda}(\mathrm{D}_z^{n} z^k) = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \quad (0 \leq \lambda < 1, k > 0, n \in \mathbb{N}_0, \ k-n \neq -1, -2, -3, \cdots).$$

Therefore, we say that, for any real λ,

$$\mathrm{D}_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \quad (k > 0, k-\lambda \neq -1, -2, -3, \cdots).$$

Applying the fractional calculus, we introduce the subclass $\mathcal{S}_{\alpha}^{*}(1-b)$ of $\mathcal{A}_{\alpha}$.

Definition 5 *A function $f \in \mathcal{A}_{\alpha}$ is said to be starlike of complex order $(1-b)$ ($b \in \mathbb{C}$ and $b \neq 0$) if $f(z)$ satisfies*

$$1+\frac{1}{b}\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}-1\right)=p(z) \quad (z \in \mathbb{U})$$

for some $p(z) \in \mathcal{P}$. The subclass of $\mathcal{A}_{\alpha}$ consisting of such functions is denoted by $\mathcal{S}_{\alpha}^{*}(1-b)$.

Let $h(z)$ and $s(z)$ be analytic in $\mathbb{U}$. Then $h(z)$ is said to be subordinate to $s(z)$, written by $h(z) \prec s(z)$, if there exists some function $w(z) \in \Omega$ such that $h(z) = s(w(z))$ in $\mathbb{U}$. In particular, if $s(z)$ is univalent in $\mathbb{U}$, then the subordination $h(z) \prec s(z)$ is equivalent to $h(0) = s(0)$ and $h(\mathbb{U}) \subset s(\mathbb{U})$ (see [1]).

2. Main Results

In order to consider some properties for the class $\mathcal{S}_{\alpha}^{*}(1-b)$, we need the following lemma due to Jack [2], or due to Miller and Mocanu [3].

Lemma 6 *Let $w(z)$ be a non-constant and analytic function in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_1 \in \mathbb{U}$, then we have*

$$z_1 w'(z_1) = k w(z_1),$$

where k is real and $k \geq 1$.

Now, we derive the following.

Theorem 7 *If $f(z) \in \mathcal{A}_{\alpha}$ and satisfies the condition*

$$\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}-1\right) \prec \frac{2bz}{1-z}=F(z), \tag{1}$$

then $f(z) \in S_{\alpha}^{*}(1-b)$. This result is sharp since the function $f(z)$ satisfies the fractional differential equation $\mathrm{D}_z^{\alpha}f(z)=\frac{\Gamma(\alpha+2)z}{(1-z)^{2b}}$.

Proof. For $f(z) \in \mathcal{A}_\alpha$, it is easy to see that

$$\begin{aligned} \mathrm{D}_z^\alpha f(z) &= \mathrm{D}_z^\alpha (z^{\alpha+1} + a_2 z^{\alpha+2} + a_3 z^{\alpha+3} + \cdots + a_n z^{\alpha+n} + \cdots) \\ &= \frac{\Gamma(\alpha+2)}{1!} z + a_2 \frac{\Gamma(\alpha+3)}{2!} z^2 + a_3 \frac{\Gamma(\alpha+4)}{3!} z^3 + \cdots \\ &+ a_n \frac{\Gamma(\alpha+n+1)}{n!} z^n + \cdots . \end{aligned}$$

On the other hand, we define the function $w(z)$ by

$$\frac{\mathrm{D}_z^\alpha f(z)}{\Gamma(\alpha+2) z} = (1 - w(z))^{-2b} \quad (w(z) \neq 1),$$

where the value of $(1 - w(z))^{-2b}$ is 1 at $z = 0$ (i.e, we consider the corresponding Riemann branch), then $w(z)$ is analytic in $\mathbb{U}$, $w(0) = 0$, and

$$\left(z \frac{\mathrm{D}_z^{\alpha+1} f(z)}{\mathrm{D}_z^\alpha f(z)} - 1 \right) = \frac{2bzw'(z)}{1 - w(z)}.$$

Now, it is easy to realize that the subordination (1) is equivalent to $|w(z)| < 1$ for all $z \in \mathbb{U}$. Indeed, assume the contrary: then, there exists a $z_1 \in \mathbb{U}$ such that $|w(z_1)| = 1$. Then, by Lemma 6, $z_1 w'(z_1) = k w(z_1)$ for some real $k \geq 1$. For such z_1 we have

$$\left(z_1 \frac{\mathrm{D}_z^{\alpha+1} f(z_1)}{\mathrm{D}_z^\alpha f(z_1)} - 1 \right) = \frac{2kbw(z_1)}{1 - w(z_1)} = F(w(z_1)) \notin F(\mathbb{D}),$$

because $|w(z_1)| = 1$ and $k \geq 1$. But this contradicts (1), so the assumption is wrong, i.e, $|w(z)| < 1$ for every $z \in \mathbb{U}$.

The sharpness of this result follows from the fact that

$$\mathrm{D}_z^\alpha f(z) = \frac{\Gamma(\alpha+2) z}{(1-z)^{2b}} \Rightarrow \left(z \frac{\mathrm{D}_z^{\alpha+1} f(z)}{\mathrm{D}_z^\alpha f(z)} - 1 \right) = \frac{2bz}{1-z}.$$

Theorem 8 *If $f(z) \in \mathcal{S}_\alpha^*(1-b)$, then*

$$\frac{\Gamma(\alpha+2)(1-r)^{|b| - \mathrm{Re} b}}{(1+r)^{|b| + \mathrm{Re} b}} \leq |\mathrm{D}_z^\alpha f(z)| \leq \frac{\Gamma(\alpha+2)(1+r)^{|b| - \mathrm{Re} b}}{(1-r)^{|b| + \mathrm{Re} b}}.$$

This result is sharp since the function satisfies the fractional differential equation $\mathrm{D}_z^\alpha f(z) = \frac{\Gamma(\alpha+2) z}{(1-z)^{2b}}$.

Proof. If $p(z) \in \mathcal{P}$, then we have

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}.$$

Using the definition of the class $\mathcal{S}_\alpha^*(1-b)$, then we can write

$$\left|\left[1 + \frac{1}{b}\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)} - 1\right)\right] - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}. \tag{2}$$

After the simple calculations from the (2) we get

$$\left|z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)} - \frac{1-(1-2b)r^2}{1-r^2}\right| \leq \frac{2|b|r}{1-r^2}. \tag{3}$$

The inequality (3) can be written in the form

$$\frac{1-2|b|r-(1-2\mathrm{Re}b)r^2}{1-r^2} \leq \mathrm{Re}\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}\right) \leq \frac{1+2|b|r-(1-2\mathrm{Re}b)r^2}{1-r^2}. \tag{4}$$

On the other hand, since

$$\mathrm{Re}\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}\right) = r\frac{\partial}{\partial r}\log|\mathrm{D}_z^{\alpha}f(z)|,$$

then, the inequality (4) can be written in the form

$$\frac{1-2|b|r-(1-2\mathrm{Re}b)r^2}{r(1-r)(1+r)} \leq \frac{\partial}{\partial r}\log|\mathrm{D}_z^{\alpha}f(z)| \leq \frac{1+2|b|r-(1-2\mathrm{Re}b)r^2}{r(1-r)(1+r)}. \tag{5}$$

Integrating both sides of the inequality (5) from 0 to r, and using the normalization $\left(\frac{\mathrm{D}_z^{\alpha}f(z)}{\Gamma(\alpha+2)}\right)$, we complete the proof of the theorem.

Theorem 9 *If $f(z) \in \mathcal{S}_\alpha^*(1-b)$, then*

$$|a_n| \leq \frac{n\Gamma(\alpha+2)}{\Gamma(\alpha+n+1)}\prod_{k=0}^{n-2}(k+2|b|). \tag{6}$$

This inequality is sharp because the extremal function satisfies the fractional differential equation $\mathrm{D}_z^{\alpha}f(z) = \frac{\Gamma(\alpha+2)z}{(1-z)^{2b}}$.

Proof. Using the definition of the class $\mathcal{S}^*_\alpha(1-b)$, we can write

$$1+\frac{1}{b}\left(z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}-1\right)=p(z)\Leftrightarrow z\frac{\mathrm{D}_z^{\alpha+1}f(z)}{\mathrm{D}_z^{\alpha}f(z)}=b(p(z)-1)+1\Leftrightarrow$$

$$\Gamma(\alpha+2)z+\Gamma(\alpha+3)a_2z^2+\frac{1}{2!}\Gamma(\alpha+4)a_3z^3+\cdots+\frac{1}{(n-1)!}\Gamma(\alpha+n+1)a_nz^n+\cdots$$
$$=\left(\Gamma(\alpha+2)z+\frac{1}{2!}\Gamma(\alpha+3)a_2z^2+\frac{1}{3!}\Gamma(\alpha+4)a_3z^3+\cdots+\frac{1}{n!}\Gamma(\alpha+n+1)a_nz^n+\cdots\right)$$
$$\cdot\left(1+bp_1z+bp_2z^2+\cdots+bp_nz^n+\cdots\right) \tag{7}$$

Equaling the coefficient of z^n in both sides (7), we get $a_1\equiv 1$ and

$$|a_n|\leq\frac{n!2|b|}{(n-1)\Gamma(\alpha+n+1)}\sum_{k=1}^{n-1}\frac{1}{k!}\Gamma(\alpha+k+1)|a_k|,\ \ |a_1|=1, \tag{8}$$

and (6) is obtained by induction making use of (8) and fact that $|p_n|\leq 2$ for all $n\geq 2$ whenever $p(z)\in\mathcal{P}$.

Remark 10 *Let us consider a function $f(z)$ defined by*

$$\mathrm{D}_z^{\alpha}f(z)=\frac{\Gamma(\alpha+2)z}{(1-z)^{2b}}$$

which was given in the theorems. Note that

$$\begin{aligned}\frac{z}{(1-z)^{2b}}&=z\left(\sum_{n=0}^{\infty}\binom{-2b}{n}(-z)^n\right)\\&=z+\sum_{n=1}^{\infty}\binom{-2b}{n}(-1)^n(z)^{n+1}\\&=z+\sum_{n=1}^{\infty}\frac{2b(2b+1)(2b+2)\cdots(2b+n-1)}{n!}z^{n+1}\\&=z+\sum_{n=2}^{\infty}\frac{2b(2b+1)(2b+2)\cdots(2b+n-2)}{(n-1)!}z^n\\&=z+\sum_{n=2}^{\infty}\frac{(2b)_{n-1}}{(1)_{n-1}}z^n,\end{aligned}$$

where $(a)_n$ *in the Pochhammer symbol defined by*

$$(a)_n = \begin{cases} 1 \quad (n=0,\ a \neq 0) \\ a(a+1)(a+2)\cdots(a+n-1) \quad (n=1,2,3,\cdots). \end{cases}$$

Therefore, we obtain that

$$\begin{aligned} f(z) &= \mathrm{D}_z^{-\alpha}(\mathrm{D}_z^{\alpha} f(z)) \\ &= \mathrm{D}_z^{-\alpha}\left(\frac{\Gamma(\alpha+2)z}{(1-z)^{2b}}\right) \\ &= \Gamma(\alpha+2)\mathrm{D}_z^{-\alpha}\left(z + \sum_{n=2}^{\infty}\frac{(2b)_{n-1}}{(1)_{n-1}}z^n\right) \\ &= z^{\alpha+1} + \sum_{n=2}^{\infty}\frac{(2b)_{n-1}\Gamma(\alpha+2)\Gamma(n+1)}{(1)_{n-1}\Gamma(n+1+\alpha)}z^{\alpha+n} \\ &= z^{\alpha}\left(z + \sum_{n=2}^{\infty}\frac{n(2b)_{n-1}}{(\alpha+2)_{n-1}}z^n\right) \end{aligned}$$

because

$$\frac{\Gamma(n+1)}{(1)_{n-1}} = \frac{n!}{(n-1)!} = n$$

and

$$\begin{aligned} \frac{\Gamma(\alpha+2)}{\Gamma(n+1+\alpha)} &= \frac{1}{(n+\alpha)(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+2)} \\ &= \frac{1}{(\alpha+2)_{n-1}}. \end{aligned}$$

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Authors:

Mert Çağlar
Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: m.caglar@iku.edu.tr

Yaşar Polatoğlu
Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: y.polatoglu@iku.edu.tr

Arzu Şen
Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: a.sen@iku.edu.tr

Emel Yavuz
Department of Mathematics and Computer Science,
TC İstanbul Kültür University, 34156 İstanbul, Turkey
e-mail: e.yavuz@iku.edu.tr

Shigeyoshi Owa
Department of Mathematics,
Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
e-mail: owa@math.kindai.ac.jp