DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS FOR ANALYTIC FUNCTIONS DEFINED BY THE GENERALIZED SĂLĂGEAN DERIVATIVE

Veronica Oana Nechita

Abstract. Let q_1 and q_2 be univalent in the unit disk U, with $q_1(0) = q_2(0) = 1$. We give applications of first order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions $f \in \mathcal{A}$ to satisfy one of the conditions $q_1(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \prec q_2(z)$ or $q_1(z) \prec z\frac{D_\lambda^{m+1} f(z)}{\{D_\lambda^m f(z)\}^2} \prec q_2(z)$, where $D_\lambda^m f(z)$ is the generalized Sălăgean differential operator.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

We also consider the class

$$\mathcal{A} = \left\{f \in \mathcal{H} : f(z) = z + a_2 z^2 + ...\right\}.$$

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in \mathcal{Q}$. In [3], the authors considered the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \tag{1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1). Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a, n]$. Recently, in [4], the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \tag{2}$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination. Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superodination (2).

For two functions $f(z) = z + \sum \_n = 2^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\lambda > 0$. The generalized Sălăgean derivative of a function f is defined in [1] by

$$D_\lambda^o f(z) = f(z), D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z), D_\lambda^m f(z) = D_\lambda^1 \left(D_\lambda^{m-1} f(z) \right).$$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we write the generalized Sălăgean derivative as a Hadamard product

$$D_\lambda^m f(z) = f(z) * \left\{ z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m z^n \right\} = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^m a_n z^n.$$

When $\lambda = 1$, we get the classic Sălăgean derivative [7], denoted by $D^m f(z)$.

In this paper we will determine some properties on admissible functions defined with the generalized Sălăgean derivative.

2. Preliminaries

In our present investigation we shall need the folllowing results.

Theorem 1 *[[3], Theorem 3.4h., p.132] Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z) \cdot \phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) h is convex or

(ii) Q is starlike.

In addition, assume that

(iii) $Re \dfrac{zh'(z)}{Q(z)} > 0$.

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \prec \theta[q(z)] + zp'(z) \cdot \phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 1, we get

Corollary 2 *Let q be univalent in U, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If p is analytic in U, with $p(0) = q(0)$ and

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z),$$

then $p \prec q$, and q is the best dominant.

Theorem 3 *[5] Let θ and ϕ be analytic in a domain D and let q be univalent in U, with $q(0) = a$, $q(U) \subset D$. Set $Q(z) = zq'(z) \cdot \phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that*

(i) $Re\left[\frac{\theta'[q(z)]}{\phi[q(z)]}\right] > 0$ *and*

(ii) $Q(z)$ *is starlike.*

If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \phi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zp'(z) \cdot \phi[q(z)] \prec \theta[p(z)] + zp'(z) \cdot \phi[p(z)] \Rightarrow q \prec p$$

and q is the best subordinant.

By taking $\theta(w) := w$ and $\phi(w) := \gamma$ in Theorem 3, we get

Corollary 4 *[2] Let q be convex in U, $q(0) = a$ and $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $p(z) + \gamma zp'(z)$ is univalent in U, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \Rightarrow q \prec p$$

and q is the best subordinant.

3. Main results

Theorem 5 *Let q be univalent in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^{m+2} f(z) \cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1} f(z)\right]^2}\right\} \prec q(z) + \gamma z q'(z) \qquad (3)$$

then

$$\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Let

$$p(z) := \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)}.$$

By a simple computation we get

$$\frac{zp'(z)}{p(z)} = \frac{z\{D_\lambda^m f(z)\}'}{D_\lambda^m f(z)} - \frac{z\left\{D_\lambda^{m+1} f(z)\right\}'}{D_\lambda^{m+1} f(z)}. \qquad (4)$$

By using the identity

$$z\{D_\lambda^m f(z)\}' = \frac{1}{\lambda} D_\lambda^{m+1} f(z) + \left(1 - \frac{1}{\lambda}\right) D_\lambda^m f(z), \qquad (5)$$

we obtain from (4) that

$$\frac{zp'(z)}{p(z)} = \frac{1}{\lambda}\left[\frac{1}{p(z)} - \frac{D_\lambda^{m+2} f(z)}{D_\lambda^{m+1} f(z)}\right]$$

and

$$p(z) + \gamma z p'(z) = \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^{m+2} f(z) \cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1} f(z)\right]^2}\right\}$$

The subordination (3) from hypothesis becomes

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

We obtain the conclusion of our theorem by simply applying Corrolary 2. □

For $m = 0$, we have the following result.

Corollary 6 *Let q be univalent in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\left[1-\frac{\gamma}{\lambda}(2-\lambda)\right]\frac{f(z)}{(1-\lambda)f(z)+\lambda z f'(z)}+$$

$$\frac{\gamma}{\lambda}\left\{1-\frac{\lambda^2 z^2 f''(z) f(z)-(1-\lambda)f^2(z)}{\left[(1-\lambda)f(z)+\lambda z f'(z)\right]^2}\right\} \prec q(z)+\gamma z q'(z)$$

then

$$\frac{f(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \prec q(z)$$

and q is the best dominant.

For $\lambda=1$, the generalized Sălăgean derivative becomes the classic Sălăgean derivative, denoted by $D^m f(z)$. In this case we have the following consequence of Theorem 5.

Corollary 7 *[6] Let q be univalent in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{D^m f(z)}{D^{m+1} f(z)}+\gamma\left\{1-\frac{D^{m+2} f(z)\cdot D^m f(z)}{\left[D^{m+1} f(z)\right]^2}\right\} \prec q(z)+\gamma z q'(z)$$

then

$$\frac{D^m f(z)}{D^{m+1} f(z)} \prec q(z)$$

and q is the best dominant.

We next consider the case when $\lambda=1$ and $m=0$.

Corollary 8 *[6] Let q be univalent in U with $q(0)=1$, $\gamma\in\mathbb{C}^*$ and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-Re\frac{1}{\gamma}\right\}.$$

If $f\in\mathcal{A}$ and

$$(1-\gamma)\frac{f(z)}{zf'(z)}+\gamma\left\{1-\frac{f''(z)\cdot f(z)}{[f'(z)]^2}\right\}\prec q(z)+\gamma zq'(z)$$

then

$$\frac{f(z)}{zf'(z)}\prec q(z)$$

and q is the best dominant.

For our next application, we select in Theorem 5 a particular dominant q.

Corollary 9 *Let $A,B,\gamma\in\mathbb{C}$, $A\neq B$ such that $|B|\leq 1$ and $Re\gamma>0$. If $f\in\mathcal{A}$ satisfies the subordination*

$$\frac{D_\lambda^m f(z)}{D_\lambda^{m+1}f(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^{m+2}f(z)\cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1}f(z)\right]^2}\right\}\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ is the best dominant.

We apply Corrolary 4 to obtain the following result.

Theorem 10 *Let q be convex in U, $q(0)=1$ and $\gamma\in\mathbb{C}$, $Re\gamma>0$. If $f\in\mathcal{A}$, $\dfrac{D_\lambda^m f(z)}{D_\lambda^{m+1}f(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$, $\dfrac{D_\lambda^m f(z)}{D_\lambda^{m+1}f(z)}+\dfrac{\gamma}{\lambda}\left\{1-\dfrac{D_\lambda^{m+2}f(z)\cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1}f(z)\right]^2}\right\}$ is univalent in U and*

$$q(z)+\gamma zq'(z)\prec\frac{D_\lambda^m f(z)}{D_\lambda^{m+1}f(z)}+\frac{\gamma}{\lambda}\left\{1-\frac{D_\lambda^{m+2}f(z)\cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1}f(z)\right]^2}\right\},$$

then

$$q(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)}$$

and q is the best subordinant.

Corollary 11 *Let q be convex in U, $q(0) = 1$ and $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $f \in \mathcal{A}$, $\dfrac{f(z)}{(1-\lambda) f(z) + \lambda z f'(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left[1 - \dfrac{\gamma}{\lambda}(2-\lambda)\right] \dfrac{f(z)}{(1-\lambda) f(z) + \lambda z f'(z)} +$ $\dfrac{\gamma}{\lambda}\left\{1 - \dfrac{\lambda^2 z^2 f''(z) f(z) - (1-\lambda) f^2(z)}{[(1-\lambda) f(z) + \lambda z f'(z)]^2}\right\}$ is univalent in U and*

$$q(z) + \gamma z q'(z) \prec \left[1 - \frac{\gamma}{\lambda}(2-\lambda)\right] \frac{f(z)}{(1-\lambda) f(z) + \lambda z f'(z)} +$$

$$\frac{\gamma}{\lambda}\left\{1 - \frac{\lambda^2 z^2 f''(z) f(z) - (1-\lambda) f^2(z)}{[(1-\lambda) f(z) + \lambda z f'(z)]^2}\right\},$$

then

$$q(z) \prec \frac{f(z)}{(1-\lambda) f(z) + \lambda z f'(z)}$$

and q is the best subordinant.

Proof. The conclusion follows from Theorem 10 for $m = 0$.□

If we take $\lambda = 1$ in Theorem 10, we have the next corollary.

Corollary 12 *[6] Let q be convex in U, $q(0) = 1$ and $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $f \in \mathcal{A}$, $\dfrac{D^m f(z)}{D^{m+1} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\dfrac{D^m f(z)}{D^{m+1} f(z)} + \gamma\left\{1 - \dfrac{D^{m+2} f(z) \cdot D^m f(z)}{[D^{m+1} f(z)]^2}\right\}$ is univalent in U and*

$$q(z) + \gamma z q'(z) \prec \frac{D^m f(z)}{D^{m+1} f(z)} + \gamma\left\{1 - \frac{D^{m+2} f(z) \cdot D^m f(z)}{[D^{m+1} f(z)]^2}\right\},$$

then

$$q(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)}$$

and q is the best subordinant.

We combine the results of Theorem 5 and Theorem 10 to obtain the following "sandwich theorem".

Corollary 13 *Let q_1, q_2 be convex in U, $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $f \in \mathcal{A}$, $\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^{m+2} f(z) \cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1} f(z)\right]^2}\right\}$ is univalent in U and*

$$q_1(z) + \gamma z q_1'(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} + \frac{\gamma}{\lambda}\left\{1 - \frac{D_\lambda^{m+2} f(z) \cdot D_\lambda^m f(z)}{\left[D_\lambda^{m+1} f(z)\right]^2}\right\} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{D_\lambda^m f(z)}{D_\lambda^{m+1} f(z)} \prec q_2(z)$$

and the functions q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 14 *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\left(1 + \frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} + \frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} - \frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1} f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3} \prec q(z) + \gamma z q'(z) \tag{6}$$

then

$$z\frac{D_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} \prec q(z)$$

and q is the best dominant.

Proof. Let

$$p(z) := z\frac{D_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2}.$$

By calculating the logarithmic derivative of p, we get

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z\left\{D_\lambda^{m+1} f(z)\right\}'}{D_\lambda^{m+1} f(z)} - 2\frac{z\left\{D_\lambda^m f(z)\right\}'}{D_\lambda^m f(z)}. \tag{7}$$

We use the identity (5) in (7) to obtain

$$\frac{zp'(z)}{p(z)} = \frac{1}{\lambda}\left[1 + \frac{D_\lambda^{m+2} f(z)}{D_\lambda^{m+1} f(z)} - 2\frac{D^{m+1} f(z)}{D^m f(z)}\right]$$

and

$$p(z) + \gamma z p'(z) = \left(1 + \frac{\gamma}{\lambda}\right)\frac{z D_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} + \frac{\gamma}{\lambda}\frac{z D_\lambda^{m+2} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} - \frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1} f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3}.$$

The subordination (6) becomes

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).$$

We obtain the conclusion of our theorem by simply applying Corrolary 2.□

In Theorem 14 we let $m = 0$ and obtain the following result.

Corollary 15 *Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1+\gamma)\frac{(1-\lambda) z}{f(z)} + [\lambda + (3\lambda - 1)\gamma]\frac{z^2 f'(z)}{[f(z)]^2} + \gamma\lambda\frac{z^3 f''(z)}{[f(z)]^2} - 2\gamma\lambda\frac{z^3 [f'(z)]^2}{[f(z)]^3}$$
$$\prec q(z) + \gamma z q'(z)$$

then

$$(1-\lambda)\frac{z}{f(z)} + \lambda\frac{z^2 f'(z)}{f^2(z)} \prec q(z)$$

and q is the best dominant.

We consider $\lambda = 1$ in Theorem 14 to get the following corrolary.

Corollary 16 *[6] Let q be univalent in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose*

$$Re\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ *and*

$$(1+\gamma)\frac{zD^{m+1}f(z)}{\{D^m f(z)\}^2}+\gamma\frac{zD^{m+2}f(z)}{\{D^m f(z)\}^2}-2\gamma\frac{z\{D^{m+1}f(z)\}^2}{\{D^m f(z)\}^3}\prec q(z)+\gamma z q'(z)$$

then

$$z\frac{D^{m+1}f(z)}{\{D^m f(z)\}^2}\prec q(z)$$

and q *is the best dominant.*

For $m=0$ and $\lambda=1$ in Theorem 10, we obtain the following result.

Corollary 17 *[6] Let* q *be univalent in* U *with* $q(0)=1$, $\gamma\in\mathbb{C}^*$ *and suppose*

$$Re\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ *and*

$$\frac{zf'(z)}{\{f(z)\}^2}-\gamma z^2\left(\frac{z}{f(z)}\right)''\prec q(z)+\gamma z q'(z)$$

then

$$\frac{z^2 f'(z)}{[f(z)]^2}\prec q(z)$$

and q *is the best dominant.*

We take a particular dominant q in Theorem 10 to get the next corolarry.

Corollary 18 *Let* $A, B, \gamma\in\mathbb{C}$, $A\neq B$ *such that* $|B|\leq 1$ *and* $Re\gamma>0$. *If* $f\in\mathcal{A}$ *satisfies the subordination*

$$\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1}f(z)}{\{D_\lambda^m f(z)\}^2}+\frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2}f(z)}{\{D_\lambda^m f(z)\}^2}-\frac{2\gamma}{\lambda}\frac{z\{D_\lambda^{m+1}f(z)\}^2}{\{D_\lambda^m f(z)\}^3}$$
$$\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2},$$

then

$$z\frac{D_\lambda^{m+1}f(z)}{\{D_\lambda^m f(z)\}^2}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Next, applying Corrolary 4 we have the following theorem.

Theorem 19 *Let q be convex in U, $q(0)=1$ and $\gamma\in\mathbb{C}$, $Re\gamma>0$. If $f\in\mathcal{A}$, $z\frac{D_\lambda^{m+1}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}\in\mathcal{H}[1,1]\cap\mathcal{Q}$, $\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}+\frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}-\frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1}f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3}$ is univalent in U and*

$$q(z)+\gamma zq'(z)\prec\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}+\frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}-\frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1}f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3},$$

then

$$q(z)\prec z\frac{D_\lambda^{m+1}f(z)}{\left\{D_\lambda^m f(z)\right\}^2}$$

and q is the best subordinant.

Corollary 20 *Let q be convex in U, $q(0)=1$ and $\gamma\in\mathbb{C}$, $Re\gamma>0$. If $f\in\mathcal{A}$, $(1-\lambda)\frac{z}{f(z)}+\lambda\frac{z^2f'(z)}{f^2(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$, $(1+\gamma)\frac{(1-\lambda)z}{f(z)}+[\lambda+(3\lambda-1)\gamma]\frac{z^2f'(z)}{[f(z)]^2}+\gamma\lambda\frac{z^3f''(z)}{[f(z)]^2}-2\gamma\lambda\frac{z^3[f'(z)]^2}{[f(z)]^3}$ is univalent in U and*

$$q(z)+\gamma zq'(z)\prec(1+\gamma)\frac{(1-\lambda)z}{f(z)}+[\lambda+(3\lambda-1)\gamma]\frac{z^2f'(z)}{[f(z)]^2}$$

$$+\gamma\lambda\frac{z^3f''(z)}{[f(z)]^2}-2\gamma\lambda\frac{z^3[f'(z)]^2}{[f(z)]^3},$$

then

$$q(z)\prec(1-\lambda)\frac{z}{f(z)}+\lambda\frac{z^2f'(z)}{f^2(z)}$$

and q is the best subordinant.

Proof. The conclusion follows from Theorem 14 by taking $m=0$.□

We write Theorem 14 and Theorem 19 together and obtain the following "sandwich theorem".

Corollary 21 *Let q_1, q_2 be convex in U, $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, $Re\gamma > 0$. If $f \in \mathcal{A}$, $z\frac{D_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} \in \mathcal{H}[1,1] \cap \mathcal{Q}$, $\left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} + \frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} - \frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1} f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3}$ is univalent in U and*

$$\begin{aligned} q_1(z) + \gamma z q_1'(z) &\prec \left(1+\frac{\gamma}{\lambda}\right)\frac{zD_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} + \frac{\gamma}{\lambda}\frac{zD_\lambda^{m+2} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} - \frac{2\gamma}{\lambda}\frac{z\left\{D_\lambda^{m+1} f(z)\right\}^2}{\left\{D_\lambda^m f(z)\right\}^3} \\ &\prec q_2(z) + \gamma z q_2'(z), \end{aligned}$$

then

$$q_1(z) \prec z\frac{D_\lambda^{m+1} f(z)}{\left\{D_\lambda^m f(z)\right\}^2} \prec q_2(z)$$

and the functions q_1 and q_2 are respectively the best subordinant and the best dominant.

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Author:

Veronica Oana Nechita
Department of Mathematics,
Babeş-Bolyai University,
Cluj-Napoca,
Romania
email: *vnechita@math.ubbcluj.ro*