PRACTICE OF OPERATOR RELATIONSHIPS IN SYMBOLICAL CALCULUS

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Abstract.The new operator relationships and special operators are presented in this paper. The use of this theory in symbolical calculus appears to be a successful extension of numerical and approximation methods for solving of differential equations. In applying an operator method, the sought-for solutions are represented as operator series. Practical applicability of the operator relationships and some examples are described.

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1. Introduction

Perturbation methods theory is widely developed. The investigation of approximate solutions can be done using special mathematical software. Standard perturbation methods were realized in Maple software; see Naifeh and Chin [1]. Recent technological progress makes it possible to develop and to employ new powerful computational algorithms, associated with symbolical calculus. The special operator relationships allow to use this calculus in special approximate methods for solving of differential equations. In this paper one version of operator relationships suitable for symbolical calculus is presented. It is applied for approximate solving of ordinary differential equations.

2.Operator relationships

Let x, s, t be real variables. Then linear differential and integral operators D_x,D_s,...,L_s, L_t for following variables can be defined in this way:

$$D_x x^n := n x^{n-1}, L_x x^n := \frac{x^{n+1}}{n+1}, \ldots, n = 0, 1, 2 \ldots.$$

The examples of analytical interpretation can be written as

$$D_x(\sin(s+tx)) = t\cos(s+tx),$$

$$L_x(t\sin(s+tx)) = \int_0^t t\sin(s+tu)du = \cos s - \cos(s+tx),$$

etc.

Various special operators can be composed by using of these elementary operators. For expression of solution of nonlinear differential equations in operator form the special operator $A = L_x t^r D_s$ can be formed. Then

$$A\alpha x^m t^k s^l = \alpha(L_x x^r)(t^r t^k)(D_s s^l) = \frac{\alpha l}{m+1} x^{m+1} t^{k+r} s^{l-1},$$

if $\alpha \in \mathbf{R}$ is fixed and constants $r, m, k, l = 0, 1, 2, \ldots$. With this operator A the special operator

$$g(A) = \sum_{k=0}^{+\infty} A^k$$

was formed (see Navickas, [3]). The effect of this operator can be illustrated by following example:

$$g(L_x t D_s)s^n = s^n + \frac{n}{1!} xts^{n-1} + \frac{n(n-1)}{2!} x^2 t^2 s^{n-2} + \cdots + \frac{n!}{n!} x^n t^n = (s+tx)^n$$

Let $f_k(s,t), k = 0, 1, 2, \ldots$ be the functions of variables s,t and can be expressed by Maclaurin series. Then

$$g(L_x t D_s) f_k(s,t) = f_k(s+tx, t),$$

besides,

$$g(L_x t D_s) \sum_{k=0}^{+\infty} f_k(s,t) \frac{x^k}{k!} = \sum_{k=0}^{+\infty} g(L_x t D_s) L_x^k f_k(s,t) = \sum_{k=0}^{+\infty} L_x^k f_k(s+tx, t).$$

Let $A = L_x p(s,t) D_s$, $B = L_x q(s,t) D_t$, when $p(s,t)$ and $q(s,t)$ are functions of variables s and t and can be expressed by Maclaurin series. In this case such operator relationships holds true:

$$(g(A))^{-1} = 1 - A,$$

$$g(A+B) = g(g(A)B)g(A),$$

if $AB = BA$, then

$$(A-B)g(A)g(B) = g(A) - g(B),$$

etc. In practical calculations the identity

$$L_x(\alpha - \beta)D_s g(L_x \alpha D_s) g(L_x \beta D_s) = g(L_x \alpha D_s) - g(L_x \beta D_s)$$

is often used.

3.Operator algorithms in symbolical calculus

It is possible to write a solution of differential equation

$$y'' = P(y, y'), y(0, s, t) = s, y'(x, s, t)|_{x=v} = t$$

using the special operator $g(A)$, if the function $P(s,t)$ can be expressed by Maclaurin series, see [4]. Then solution $y(x, s, t)$ is described by using symbolical relationship

$$y = g(L_x t D_s + L_x P(s,t) D_t) s$$

The case of n-th order nonlinear differential equation was described in [5]. Using this method the solutions can be represented as operator series; where approximate solutions from polynomials of various degrees are obtained. Often in investigation of differential equations it is enough to find approximate expression of solution. Then a parameter of perturbation ε can be introduced and the solution of differential equation $z'' = \varepsilon P(z, z')$ can be written in this way:

$$z(x, s, t, \varepsilon) = \sum_{k=0}^{+\infty} z_k(x, s, t) \varepsilon^k$$

Whereas $y(x, s, t) = z(x, s, t, 1)$, then if n is quite small, the estimation of solution is limited by

$$y \approx z_0 + z_1 + \cdots + z_n.$$

Using introduced operators and its properties the following solution might be obtained

$$z = g(L_x t D_s + \varepsilon L_x P(s,t) D_t)s =$$

$$= g(g(L_x t D_s)\varepsilon L_x P(s,t) D_t) g(L_x t D_s)s =$$

$$= \sum_{k=0}^{+\infty} (L_x^k((g(L_x t D_s) P(s,t) D_t)^k (s+tx)))\varepsilon^k.$$

If

$$q_0(x,s,t) = s + tx,$$

$$q_{k+1}(x,s,t) = (g(L_x t D_s) P(s,t) D_t) q_k(x,s,t), k = 0,1,2,\ldots,$$

then the expression of solution is

$$z = \sum_{k=0}^{+\infty} (L_x^k q_k(x,s,t))\varepsilon^k$$

4. Application

The solution of differential equation

$$y'' = -\sin y, y(0,s,t) = s, y'(0,s,t) = t$$

in operator form is

$$y = \sum_{k=0}^{+\infty} (L_x t D_s - L_x \sin s D_t)^k s,$$

because $P(s,t) = -\sin s$.

Then the solution of equation $z'' = -\varepsilon \sin y$ can be written in this way:

$$z = \sum_{k=0}^{+\infty} (L_x(g(L_x t D_s)(-\sin s) D_t)^k (st + x))\varepsilon^k$$

From this relationship the coefficients

$$q_1 = -L_x \sin(s + tx),$$

$$q_2 = L_x^2 (2\sin(2s + 2tx) - \frac{3}{2}\sin(2s + tx) - \frac{tx}{2}\cos(2s + tx)) - L_x(\frac{x}{2}\sin tx).$$

The expression of further coefficients $q_3, q_4, \dots$ can be found much easier by using symbolical calculus of special mathematical software. For this calculation the mathematical software *Maple* was used. The large descriptions of used functions *diff*, *sum*, *collect* etc see in book [2].

5.Conclusions

The presented methodology can be developed for investigation of symbolical expressions new operators and practical application for approximate solutions of differential equation. The presented method can be a new version of well known perturbation method.

References

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