

**BOUNDEDNESS FOR MULTILINEAR COMMUTATOR OF
LITTLEWOOD-PALEY OPERATOR
ON HARDY AND HERZ-HARDY SPACES**

CHANGHONG WU, LANZHE LIU

ABSTRACT. In this paper, the (H_b^p, L^p) and $(HK_{q,b}^{\alpha,p}, \dot{K}_q^{\alpha,p})$ type boundedness for the multilinear commutator associated with the Littlewood-paley operator are obtained.

Keywords: Littlewood-paley operator; Multilinear commutator; BMO; Hardy space; Herz-Hardy space.

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1. INTRODUCTION AND DEFINITION

Let $L_{loc}^q(R^n) = \{f^q \text{ is locally integrable on } R^n\}$. Suppose $f \in L_{loc}^1(R^n)$, $B = B(x_0, r) = \{x \in R^n : |x - x_0| < r\}$ denotes a ball of R^n centered at x_0 and having radius r , write $f_B = |B|^{-1} \int_B f(x)dx$ and $f^\#(x) = \sup_{x \in B} |B|^{-1} \int_B |f(x) - f_B|dx < \infty$. f is said to belong to $BMO(R^n)$, if $f^\# \in L^\infty(R^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$.

Let T be a linear operator and K be a function on $R^n \times R^n$,

$$T(f)(x) = \int_{R^n} K(x, y)f(y)dy \text{ for } f \in C_0^\infty,$$

where K satisfies:

- (1) $|K(x, y+h) - K(x, y)| \leq C \cdot |h|^\alpha \cdot |x - y|^{-n-\alpha}$ for $2|h| < |x - y|$, $0 < \alpha \leq 1$;
- (2) $\|T(f)\|_{L^{p_0}} \leq C\|f\|_{L^{p_0}}$ for some $1 < p_0 \leq \infty$;

Then we call T is the Calderón-Zygmund singular integral operator.

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see[2]) proved that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$). However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. But if $H^p(\mathbb{R}^n)$ is replaced by a suitable atomic space $H_b^p(\mathbb{R}^n)$ and $HK_{q,\vec{b}}^{\alpha,p}(\mathbb{R}^n)$ (see[1], [7], [12]), then $[b, T]$ maps continuously $H_b^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and $HK_{q,\vec{b}}^{\alpha,p}(\mathbb{R}^n)$ into $\dot{K}_q^{\alpha,p}$. In addition we have easily known that $H_b^p(\mathbb{R}^n) \subset H^p(\mathbb{R}^n)$, $\dot{K}_{q,\vec{b}}^{\alpha,p}(\mathbb{R}^n) \subset HK_q^{\alpha,p}(\mathbb{R}^n)$. The main purpose of this paper is to consider the continuity of the multilinear commutators related to the Littlewood-Paley operators and $BMO(\mathbb{R}^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions(see [1], [3-10], [12], [13]).

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Definition 1. Let b_i ($i = 1, \dots, m$) be a locally integrable functions and $0 < p \leq 1$. A bounded measurable function a on \mathbb{R}^n is called a (p, \vec{b}) atom, if

- (1) $\text{supp } a \subset B = B(x_0, r)$
- (2) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p}$
- (3) $\int_B a(y) dy = \int_B a(y) \prod_{l \in \sigma} b_l(y) dy = 0$ for any $\sigma \in C_j^m$, $1 \leq j \leq m$.

A temperate distribution(see[14][15]) f is said to belong to $H_b^p(\mathbb{R}^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x).$$

where a_j 's are (p, \vec{b}) atoms, $\lambda_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $\|f\|_{H_b^p} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in \mathbb{R}$. For $k \in \mathbb{Z}$, set $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_{C_k} is the characteristic function of set C_k . Denote the characteristic function of B_0 by χ_0 .

- (1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty \right\}.$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q}^p \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L_{loc}^q(R^n) : \|f\|_{K_q^{\alpha,p}} < \infty \right\}.$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p + \|f\chi_0\|_{L^q}^p \right]^{1/p}.$$

Definition 3. Let $\alpha \in R^n$, $1 < q < \infty$, $\alpha \geq n(1 - 1/q)$, $b_i \in BMO(R^n)$, $1 \leq i \leq m$. A function $a(x)$ is called a central (α, q, \vec{b}) -atom (or a central (α, q, \vec{b}) -atom of restrict type), if

- (1) $\text{supp } a \subset B = B(x_0, r)$ (or for some $r \geq 1$),
- (2) $\|a\|_{L^q} \leq |B(x_0, r)|^{-\alpha/n}$,
- (3) $\int_B a(x)x^\beta dx = \int_B a(x)x^\beta \prod_{i \in \sigma} b_i(x) dx = 0$ for any $\sigma \in C_j^m$, $1 \leq j \leq m$, $0 \leq |\beta| \leq \alpha$, where $\beta = (\beta_1, \dots, \beta_n)$ is the multi-indices with $\beta_i \in N$ for $1 \leq i \leq n$ and $|\beta| = \sum_{i=1}^n \beta_i$.

A temperate distribution f is said to belong to $HK_{q,\vec{b}}^{\alpha,p}(R^n)$ (or $HK_{q,\vec{b}}^{\alpha,p}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the $S'(R^n)$ sense, where a_j is a central (α, q, \vec{b}) -atom (or a central (α, q, \vec{b}) -atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j| < \infty$). Moreover,

$$\|f\|_{HK_{q,\vec{b}}^{\alpha,p}} \text{ (or } \|f\|_{HK_{q,\vec{b}}^{\alpha,p}}) = \inf \left(\sum_j |\lambda_j|^p \right)^{1/p},$$

where the infimum are taken over all the decompositions of f as above.

Definition 4. Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

The Littlewood-Paley multilinear commutator is defined by

$$S_\psi^{\vec{b}}(f)(x) = \left[\int \int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y - z) f(z) dz,$$

$\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = \int_{\mathbb{R}^n} \psi_t(y - x)f(x)dx$. We also define that

$$S_\psi(f)(x) = \left(\int \int_{\Gamma(x)} |F_t(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [15]).

2. THEOREMS AND PROOFS

Theorem 1. *Let $\varepsilon > 0, b_i \in BMO, 1 \leq i \leq m, \vec{b} = (b_1, \dots, b_m), n/(n + \varepsilon) < p \leq 1$. Then the multilinear commutator $S_{\vec{b}}^{\vec{\psi}}$ is bounded from $H_{\vec{b}}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

Proof. It suffices to show that there exist a constant $C > 0$, such that for every (p, \vec{b}) atom a ,

$$\|S_{\vec{\psi}}^{\vec{b}}(a)\|_{L^p} \leq C.$$

Let a be a (p, \vec{b}) atom supported on a ball $B = B(x_0, r)$. When $m = 1$ see [7], and now we prove $m > 1$. Write

$$\int_{\mathbb{R}^n} |S_{\vec{\psi}}^{\vec{b}}(a)(x)|^p dx = \int_{|x-x_0| \leq 2r} |S_{\vec{\psi}}^{\vec{b}}(a)(x)|^p dx + \int_{|x-x_0| > 2r} |S_{\vec{\psi}}^{\vec{b}}(a)(x)|^p dx = I + II.$$

For I , taking $q > 1$, by Hölder's inequality and the L^q -boundedness of $S_{\vec{\psi}}^{\vec{b}}$, we have

$$\begin{aligned} I &\leq \left(\int_{|x-x_0| \leq 2r} |S_{\vec{\psi}}^{\vec{b}}(a)(x)|^q dx \right)^{p/q} \cdot |B(x_0, 2r)|^{1-p/q} \\ &\leq C \|S_{\vec{\psi}}^{\vec{b}}(a)(x)\|_{L^q}^p \cdot |B(x_0, 2r)|^{1-p/q} \\ &\leq C \|\vec{b}\|_{BMO}^p \|a\|_{L^q}^p |B|^{1-p/q} \\ &\leq C \|\vec{b}\|_{BMO}^p. \end{aligned}$$

For II , denoting $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i = (b_i)_B, 1 \leq i \leq m$, where $(b_i)_B = |B(x_0, r)|^{-1} \int_{B(x_0, r)} b_i(x)dx$, by Hölder's inequality and the vanishing moment of a , we get

$$\begin{aligned} II &= \sum_{k=1}^{\infty} \int_{2^{k+1}r \geq |x-x_0| > 2^k r} |S_{\vec{\psi}}^{\vec{b}}(a)(x)|^p dx \\ &\leq C \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \left(\int_{2^{k+1}r \geq |x-x_0| > 2^k r} |S_{\vec{\psi}}^{\vec{b}}(a)(x)| dx \right)^p \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \\
&\quad \cdot \left[\int_{2^{k+1}r \geq |x-x_0| > 2^k r} \left(\int_{\Gamma(x)} \left| \int_B \prod_{j=1}^m (b_j(x) - b_j(z)) \psi_t(y \right. \right. \right. \\
&\quad \left. \left. \left. - z \right) a(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dx \Big]^p \\
&\leq C \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \times \\
&\quad \left[\int_{2^{k+1}r \geq |x-x_0| > 2^k r} \left(\int_{\Gamma(x)} \left(\int_B |\psi_t(y-z) \right. \right. \right. \\
&\quad \left. \left. \left. - \psi_t(y-x_0) \right| \prod_{j=1}^m |b_j(x) - b_j(z)| |a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} dx \Big]^p ;
\end{aligned}$$

noting that $z \in B$, $x \in B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)$, then

$$\begin{aligned}
S_{\psi}^{\vec{b}}(a)(x) &= \left[\int_{\Gamma(x)} \left(\int_B |\psi_t(y-z) - \psi_t(y-x_0)| \right. \right. \\
&\quad \left. \left. \cdot \prod_{j=1}^m |b_j(x) - b_j(z)| |a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
&\leq C \left[\int_{\Gamma(x)} \left(\int_B t^{-n} |a(z)| \prod_{j=1}^m |b_j(x) - b_j(z)| \right. \right. \\
&\quad \left. \left. \cdot \frac{(|x_0 - z|/t)^\varepsilon}{(1 + |x_0 - y|/t)^{n+1+\varepsilon}} dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
&\leq C \left(\int_{\Gamma(x)} \frac{t^{1-n}}{(t + |x_0 - y|)^{2(n+1+\varepsilon)}} dydt \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
 & \cdot \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| |x_0 - z|^\varepsilon |a(z)| dz \\
 \leq & C \left(\int \int_{\Gamma(x)} \frac{t^{1-n} 2^{2(n+1+\varepsilon)}}{(2t + 2|x_0 - y|)^{2(n+1+\varepsilon)}} dy dt \right)^{1/2} \\
 & \cdot \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| |x_0 - z|^\varepsilon |a(z)| dz;
 \end{aligned}$$

Notice that

$$2t + |x_0 - y| > 2t + |x_0 - x| - |x - y| > t + |x_0 - x|$$

when $|x - y| < t$, and it is easy to calculate that

$$\int_0^\infty \frac{t dt}{(t + |x - x_0|)^{2(n+1+\varepsilon)}} = C |x - x_0|^{-2(n+\varepsilon)};$$

then, we deduce

$$\begin{aligned}
 S_{\psi}^{\vec{b}}(a)(x) & \leq C \left(\int \int_{\Gamma(x)} \frac{t^{1-n}}{(2t + |x_0 - y|)^{2(n+1+\varepsilon)}} dy dt \right)^{1/2} \\
 & \cdot \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| |x_0 - z|^\varepsilon |a(z)| dz \\
 & \leq C \left(\int \int_{\Gamma(x)} \frac{t^{1-n}}{(t + |x - x_0|)^{2(n+1+\varepsilon)}} dy dt \right)^{1/2} \\
 & \cdot \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| |x_0 - z|^\varepsilon |a(z)| dz \\
 & \leq C \left(\int_0^\infty \frac{t dt}{(t + |x - x_0|)^{2(n+1+\varepsilon)}} \right)^{1/2} \\
 & \cdot \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| |x_0 - z|^\varepsilon |a(z)| dz \\
 & \leq C |B|^{\varepsilon/n-1/p} |x - x_0|^{-(n+\varepsilon)} \int_B \prod_{j=1}^m |b_j(x) - b_j(z)| dz.
 \end{aligned}$$

So

$$\begin{aligned}
II &\leq C|B|^{\varepsilon/n-1/p} \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \\
&\quad \cdot \left[\int_{2^{k+1}r \geq |x-x_0| > 2^k r} |x-x_0|^{-(n+\varepsilon)} \left(\int_B \prod_{j=1}^m |b_j(x) - b_j(z)| dz \right) dx \right]^p \\
&\leq C|B|^{\varepsilon/n-1/p} \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \\
&\quad \cdot \left[\sum_{j=0}^m \sum_{\sigma \in C_j^m} \int_{2^{k+1}r \geq |x-x_0| > 2^k r} |x-x_0|^{-(n+\varepsilon)} |(\vec{b}(x) - \lambda)_{\sigma}| dx \right. \\
&\quad \left. \cdot \int_B |(\vec{b}(z) - \lambda)_{\sigma^c}| dz \right]^p \\
&\leq C|B|^{\varepsilon/n-1/p} \sum_{j=0}^m \sum_{\sigma \in C_j^m} \left(\int_B |(\vec{b}(z) - \lambda)_{\sigma^c}| dz \right)^p \\
&\quad \times \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-p} \left[\int_{2^{k+1}r \geq |x-x_0| > 2^k r} |x-x_0|^{-(n+\varepsilon)} |(\vec{b}(x) - \lambda)_{\sigma}| dx \right]^p \\
&\leq C \sum_{j=0}^m \sum_{\sigma \in C_j^m} \|\vec{b}_{\sigma^c}\|_{BMO}^p \cdot \|\vec{b}_{\sigma}\|_{BMO}^p \sum_{k=1}^{\infty} |B(x_0, 2^{k+1}r)|^{1-(n+\varepsilon)p/n} k^p |B|^{(1+\varepsilon/n-1/p)p} \\
&\leq C \|\vec{b}\|_{BMO}^p \sum_{k=1}^{\infty} k^p \cdot 2^{kn(1-p(n+\varepsilon)/n)} \\
&\leq C \|\vec{b}\|_{BMO}.
\end{aligned}$$

This finishes the proof of Theorem 1.

Theorem 2. Let $0 < p < \infty$, $1 < q < \infty$, $n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \varepsilon$, $\varepsilon > 0$, and $b_i \in BMO(\mathbb{R}^n)$, $1 \leq i \leq m$, $\vec{b} = (b_1, \dots, b_m)$. Then $S_{\vec{b}}$ is bounded from $HK_{q,\vec{b}}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Proof. Let $f \in HK_{q,\vec{b}}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decom-

position for f as in Definition 3, we write

$$\begin{aligned}
 \|S_{\psi}^{\vec{b}}(f)(x)\|_{\dot{K}_q^{\alpha,p}} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|S_{\psi}^{\vec{b}}(f)\chi_k\|_{L^q}^p \right)^{1/p} \\
 &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{\infty} |\lambda_j| \|S_{\psi}^{\vec{b}}(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
 &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|S_{\psi}^{\vec{b}}(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
 &\quad + C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|S_{\psi}^{\vec{b}}(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
 &= I + II.
 \end{aligned}$$

For II , by the boundedness of $S_{\psi}^{\vec{b}}$ on L^q and the Hölder's inequality, we have

$$\begin{aligned}
 II &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|S_{\psi}^{\vec{b}}(a_j)\chi_j\|_{L^q} \right)^p \right]^{1/p} \\
 &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^q} \right)^p \right]^{1/p} \\
 &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \cdot 2^{-j\alpha} \right)^p \right]^{1/p} \\
 &\leq C \begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right]^{1/p}, & 0 < p \leq 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)p/2} \right]^{1/p}, & 1 < p < \infty \end{cases} \\
 &\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
 &\leq C \|f\|_{H\dot{K}_{q,\vec{b}}^{\alpha,p}}.
 \end{aligned}$$

For I , when $m = 1$, let $b_j^1 = |B_j|^{-1} \int_{B_j} b_1(x) dx$. We have

$$\begin{aligned}
 S_\psi^{b_1}(a_j)(x) &= \left[\int \int_{\Gamma(x)} \left| \int_{B_j} (b_1(x) - b_1(z)) \psi_t(y-z) a_j(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
 &\leq \left[\int \int_{\Gamma(x)} \left(\int_{B_j} |\psi_t(y-z) - \psi_t(y-x_0)| \right. \right. \\
 &\quad \left. \left. \cdot |b_1(y) - b_1(z)| |a_j(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
 &\leq \left[\int \int_{\Gamma(x)} \left(\int_{B_j} t^{-n} |a_j(z)| |b_1(x) - b_1(z)| \right. \right. \\
 &\quad \left. \left. \cdot \frac{(|x_0 - z|/t)^\varepsilon}{(1 + |x_0 - y|/t)^{n+1+\varepsilon}} dy \right)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
 &\leq C \left(\int \int_{\Gamma(x)} \frac{t^{1-n}}{(t + |x_0 - y|)^{2(n+1+\varepsilon)}} dydt \right)^{1/2} \\
 &\quad \cdot \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| |b_1(x) - b_1(z)| dz \\
 &\leq C \left(\int \int_{\Gamma(x)} \frac{t^{1-n}}{(2t + |x_0 - y|)^{2(n+1+\varepsilon)}} dydt \right)^{1/2} \\
 &\quad \cdot \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| |b_1(x) - b_1(z)| dz \\
 &\leq C \left(\int \int_{\Gamma(x)} \frac{t^{1-n}}{(t + |x - x_0|)^{2(n+1+\varepsilon)}} dydt \right)^{1/2} \\
 &\quad \cdot \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| |b_1(x) - b_1(z)| dz \\
 &\leq C \left(\int_0^\infty \frac{tdt}{(t + |x - x_0|)^{2(n+1+\varepsilon)}} \right)^{1/2} \\
 &\quad \cdot \left(\int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| |b_1(x) - b_1(z)| dz \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq C|x-x_0|^{-(n+\varepsilon)} \int_{B_j} |x_0-z|^\varepsilon |a_j(z)| |b_1(x) - b_1(z)| dz \\
&\leq C|x-x_0|^{-(n+\varepsilon)} \int_{B_j} |x_0-z|^\varepsilon |a_j(z)| |b_1(x) - b_j^1| dz \\
&\quad + C|x-x_0|^{-(n+\varepsilon)} \int_{B_j} |x_0-z|^\varepsilon |a_j(z)| |b_1(z) - b_j^1| dz \\
&\leq C|x-x_0|^{-(n+\varepsilon)} \left(|b_1(x) - b_j^1| 2^{j(\varepsilon+n(1-1/q)-\alpha)} + 2^{j(\varepsilon+n(1-1/q)-\alpha)} \|b_1\|_{BMO} \right).
\end{aligned}$$

So

$$\begin{aligned}
&\|S_\psi^{b_1}(a_j)\chi_k\|_{L^q} \\
&\leq C2^{j(\varepsilon+n(1-1/q)-\alpha)} \left[\left(\int_{B_k} |b_1(x) - b_j^1| |x-x_0|^{-q(n+\varepsilon)} dx \right)^{1/q} + \right. \\
&\quad \left. \left(\int_{B_k} |x-x_0|^{-q(n+\varepsilon)} dx \right)^{1/q} \|b_1\|_{BMO} \right] \\
&\leq C2^{j(\varepsilon+n(1-1/q)-\alpha)} \left[2^{-k(n+\varepsilon)} \cdot |B_k|^{1/q} \|b_1\|_{BMO} + 2^{-k(n+\varepsilon)} \cdot |B_k|^{1/q} \|b_1\|_{BMO} \right] \\
&\leq C\|b_1\|_{BMO} 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]};
\end{aligned}$$

thus

$$\begin{aligned}
I &= C \left[\sum_{j=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|S_\psi^{b_1}(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
&\leq C\|b_1\|_{BMO} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]} \right)^p \right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
 & \leq C \|b_1\|_{BMO} \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p} \right]^{1/p}, \\ \quad 0 < p \leq 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2} \right) \right. \\ \quad \left. \times \left(\sum_{j=-\infty}^{k-3} 2^{p'[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2} \right)^{p/p'} \right]^{1/p}, \\ \quad 1 < p < \infty \end{cases} \\
 & \leq C \|b_1\|_{BMO} \begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p} \right]^{1/p}, \\ \quad 0 < p \leq 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2} \right]^{1/p}, \\ \quad 1 < p < \infty \end{cases} \\
 & \leq C \|b_1\|_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
 & \leq C \|f\|_{HK_{q,\vec{b}}^{\alpha,p}}.
 \end{aligned}$$

When $m > 1$, Let $b_j^i = |B_j|^{-1} \int_{B_j} b_i(x) dx$, $1 \leq i \leq m$, $\vec{b} = (b_j^1, \dots, b_j^m)$. We have

$$\begin{aligned}
 S_{\vec{\psi}}^{\vec{b}}(a_j)(x) &= \left[\int \int_{\Gamma(x)} \left| \int_{B_j} \prod_{i=1}^m (b_i(x) - b_i(z)) \psi_t(y-z) a_j(z) dz \right|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} \\
 &\leq \left[\int \int_{\Gamma(x)} \left(\int_{B_j} \prod_{i=1}^m |b_i(x) - b_i(z)| |\psi_t(y-z) \right. \right. \\
 &\quad \left. \left. - \psi_t(y-x_0) |a_j(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} \\
 &\leq C |x - x_0|^{-(n+\varepsilon)} \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(z)| dz \\
 &\leq C |x - x_0|^{-(n+\varepsilon)} \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| \prod_{i=1}^m |b_i(x) - b_i(y)| dz
 \end{aligned}$$

$$\begin{aligned}
&\leq C|x-x_0|^{-(n+\varepsilon)} \sum_{i=0}^m \sum_{\sigma \in C_i^m} |(\vec{b}(x) - \vec{b})_\sigma| \\
&\quad \cdot \int_{B_j} |x_0 - z|^\varepsilon |a_j(z)| |(\vec{b}(y) - \vec{b})_{\sigma^c}| dz \\
&\leq C|x-x_0|^{-(n+\varepsilon)} \sum_{i=0}^m \sum_{\sigma \in C_i^m} |(\vec{b}(x) - \vec{b})_\sigma| 2^{j\varepsilon} \cdot 2^{-j\alpha} \cdot 2^{jn(1-1/q)} \|\vec{b}_{\sigma^c}\|_{BMO} \\
&\leq C|x-x_0|^{-(n+\varepsilon)} \cdot 2^{j(\varepsilon+n(1-1/q)-\alpha)} \sum_{i=0}^m \sum_{\sigma \in C_i^m} |(\vec{b}(x) - \vec{b})_\sigma| \|\vec{b}_{\sigma^c}\|_{BMO};
\end{aligned}$$

So

$$\begin{aligned}
&\|S_\psi^{\vec{b}}(a_j)\chi_k\|_{L^q} \\
&\leq C2^{j(\varepsilon+n(1-1/q)-\alpha)} \|\vec{b}_{\sigma^c}\|_{BMO} \left[\int_{B_k} \left(|x|^{-(n+\varepsilon)} \sum_{i=0}^m \sum_{\sigma \in C_i^m} |(\vec{b}(x) - \vec{b})_\sigma| \right)^q dx \right]^{1/q} \\
&\leq C\|\vec{b}_{\sigma^c}\|_{BMO} 2^{j(\varepsilon+n(1-1/q)-\alpha)} \cdot 2^{-k(n+\varepsilon)+kn/q} \\
&\leq C\|\vec{b}\|_{BMO};
\end{aligned}$$

then

$$\begin{aligned}
I &= C \left[\sum_{j=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|S_\psi^{\vec{b}}(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
&\leq C\|\vec{b}\|_{BMO} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]} \right)^p \right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
 & \leq C \|\vec{b}\|_{BMO} \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p} \right]^{1/p}, \\ 0 < p \leq 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2} \right) \right. \\ \quad \times \left. \left(\sum_{j=-\infty}^{k-3} 2^{p'[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]} \right)^{p/p'} \right]^{1/p}, \\ 1 < p < \infty \end{cases} \\
 & \leq C \|\vec{b}\|_{BMO} \begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p} \right]^{1/p}, & 0 < p \leq 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} 2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2} \right]^{1/p}, & 1 < p < \infty \end{cases} \\
 & \leq C \|\vec{b}\|_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
 & \leq C \|f\|_{HK_{q,\vec{b}}^{\alpha,p}}.
 \end{aligned}$$

Remark. Theorem 2 also hold for nonhomogeneous Herz-type spaces, we omit the details.

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Changhong Wu
Orient Science and Technology College of HAU,
Changsha, 410128, China
E-mail: acrossbow@tom.com

Lanzhe Liu
Department of Mathematics,
Changsha University of Science and Technology,
Changsha, 410076 , China
E-mail: lanzhe.liu@163.com