CERTAIN RESULTS ON MULTIVALENT FUNCTIONS INVOLVING WRIGHT HYPERGEOMETRIC FUNCTIONS

Thomas Rosy, S. Kavitha,
G. Murugusundaramoorthy

Abstract. The purpose of this paper is to derive certain results on multivalent functions in the open unit disc involving Wright Hypergeometric function operator $\mathcal{W}^m_{p.l}f(z)$. As special cases of these results, sufficient conditions for analytic functions to have a positive real part is defined.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in

$$\Delta := \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

For any $n \in \mathbb{N}$ and $a \in \mathbb{C}$ let, $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \Delta) \tag{2}$$

and let $\mathcal{A}_1 := \mathcal{A}$. For two functions $f(z)$ given by (2) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{3}$$

Let f and g be functions analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ $(l, m \in \mathbb{N} = \{1, 2, 3, ...\})$ such that

$$1 + \sum_{n=1}^{m} B_n - \sum_{n=1}^{l} A_n \geq 0,$$

the Wright generalized hypergeometric function [22] is given by

$$\begin{aligned} {}_l\Psi_m[(\alpha_1, A_1), \cdots, (\alpha_l, A_l); (\beta_1, B_1), \cdots, (\beta_m, B_m); z] : \\ = {}_l\Psi_m[(\alpha_k, A_k)_{1,l}; (\beta_k, B_k)_{1,m}; z] \end{aligned}$$

is defined by

$${}_l\Psi_m[(\alpha_k, A_k)_{1,l}(\beta_k, B_k)_{1,m}; z] = \sum_{n=0}^{\infty} \{\prod_{k=0}^{l} \Gamma(\alpha_k + nA_k)\}\{\prod_{k=0}^{m} \Gamma(\beta_k + nB_k)\}^{-1} \frac{z^n}{n!}.$$

If $A_k = 1\,(k = 1, 2, ..., l)$ and $B_k = 1\,(k = 1, 2, ..., m)$ we have the relationship:

$$\Theta_l\Psi_m[(\alpha_k, 1)_{1,l}(\beta_k, 1)_{1,m}; z] \equiv_l F_m(\alpha_1, \cdots \alpha_l; \beta_1, \cdots, \beta_m; z) \tag{4}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \tag{5}$$

($l \leqq m+1$; $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \Delta$) is the generalized hypergeometric function(see for details [4]) where $(\alpha)_n$ is the Pochhammer symbol (or ascending factorial) defined by

$$(\alpha)_n = \begin{cases} 1 & \text{for } n = 0 \\ \alpha\ (\alpha+1)\ldots(\alpha+n-1) & \text{for } n = 1, 2, 3\ldots \end{cases} \tag{6}$$

and

$$\Theta = \left(\prod_{k=0}^{l} \Gamma(\alpha_k)\right)^{-1} \left(\prod_{k=0}^{m} \Gamma(\beta_k)\right). \tag{7}$$

Corresponding to the function

$$\begin{aligned} \mathcal{W}^l_{p,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; \quad z \quad) \\ := \quad z^p\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z), \end{aligned}$$

the linear operator $\mathcal{W}^m_{p,l}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z)$ is defined by the Hadamard product

$$\begin{aligned} \mathcal{W}^l_{p,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= \mathcal{W}^l_{p,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z^p + \sum_{n=p+1}^{\infty} \sigma_n(\alpha_1) a_n z^n. \end{aligned} \tag{8}$$

where Θ is given by (7) and $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1) = \frac{\Theta\Gamma(\alpha_1 + A_1(n-p)) \ldots \Gamma(\alpha_l + A_l(n-p))}{(n-p)!\Gamma(\beta_1 + B_1(n-p)) \ldots \Gamma(\beta_m + B_m(n-p))} \tag{9}$$

For convenience, we write

$$\mathcal{W}^l_{p,m}[(\alpha_k, A_k)_{1,l}(\beta_k, B_k)_{1,m}]f(z) = \mathcal{W}^l_{p,m}[\alpha_1]f(z). \tag{10}$$

so that we have the recurrence

$$z\left(\mathcal{W}^l_{p,m}(\alpha_1)f(z)\right)' = \alpha_1 \mathcal{W}^l_{p,m}(\alpha_1+1)f(z) - (\alpha_1 - 1)\mathcal{W}^l_{p,m}(\alpha_1)f(z). \tag{11}$$

This operator, for the univalent case is introduced by Dziok and Raina [6], which is called as the Wright's operator (see [6]). If $A_k = 1$ $(k = 1, ..., l)$, $B_k =$

1 $(k = 1, ..., m)$, we have the Dziok and Srivastava linear operator [4]. In this paper, we extend the Dziok- Srivastava linear operator by using Wright generalized hypergeometric function. We remark here that the special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [8], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [20], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [1], [10], [11]) and the Srivastava-Owa fractional derivative operators (*cf.* [15], [16]). Hence, each of our result will give five more results involving the earlier operators as listed here eventhough we omit the details of those.

Many properties of analytic functions defined by all the above linear operators were studied by (among others) Cho and Kim [3], Frasin [7], Kim and Srivastva [9], Liu and Owa [12], Liu and Srivastava [13], [14], Owa and Srivastava [16], Patel *et al.* [17], Saitoh [18] and also by Shanmugam *et al.* [19].

In this paper we shall derive some properties of multivalent functions defined by the Wright-hypergeometric function operator. In order to prove the main results, we need the following lemma due to Yang [23].

Lemma 1 *Let P_n be the class of functions*

$$q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \dots$$

be analytic in the unit disk Δ. Let $h(z)$ be analytic and starlike (with respect to the origin) univalent in U with $h(0) = 0$. If

$$zq'(z) \prec h(z), \tag{12}$$

then

$$q(z) \prec 1 + \frac{1}{n} \int_0^z \frac{h(u)}{u} du.$$

2. Main Results

We begin with the following

Theorem 1 *Let $\alpha_1 + 1 > 0$, $\dfrac{\mathcal{W}^l_{p,m}[\alpha_1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)} \neq 0$ for $z \in \Delta$. Suppose that*

$$\Re\left(1 + \frac{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{(\alpha_1+1)\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\Phi(l,m)(f)\right) < M \quad (z \in \Delta), \tag{13}$$

where

$$\Phi(l,m)(f) = \left(1 + \frac{\alpha_1 \mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)} - \frac{\mathcal{W}^l_{p,m}(\alpha_1+2)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\right) \tag{14}$$

and

$$1 < M \leqq 1 + \frac{n}{2(\alpha_1+1)p\log 2}. \tag{15}$$

Then,

$$\Re\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\right) > 1 - \frac{2p(\alpha_1+1)(M-1)}{n}\log 2 \ (z \in \Delta). \tag{16}$$

The result in (16) is best possible.

Proof. Define the function $q(z)$ by

$$q(z) = \frac{\mathcal{W}^l_{p,m}[\alpha_1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)}. \tag{17}$$

Then, clearly $q(z) = 1 + q_n z^n + q_{n+1}z^{n+1} + \ldots$ is analytic in Δ with $q(z) \neq 0$ for $z \in \Delta$. It follows from (17) that

$$\frac{zq'(z)}{q(z)} = \frac{z(\mathcal{W}^l_{p,m}[\alpha_1]f(z))'}{\mathcal{W}^l_{p,m}[\alpha_1]f(z)} - \frac{z(\mathcal{W}^l_{p,m}[\alpha_1+1]f(z))'}{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)}. \tag{18}$$

By making use of the familiar identity (11) in (18), we get

$$\frac{\mathcal{W}^l_{p,m}(\alpha_1+2)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)} = \frac{1}{\alpha_1+1} + \frac{\alpha_1}{(\alpha_1+1)q(z)} - \frac{zq'(z)}{(\alpha_1+1)q(z)}. \tag{19}$$

Hence,

$$\frac{1}{\alpha_1+1}\frac{zq'(z)}{q^2(z)} = \frac{1}{(\alpha_1+1)}\Phi_1(l,m)f(z), \tag{20}$$

where

$$\Phi_1(l,m)f(z) = \frac{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\left(1 + \frac{\alpha_1\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\right) - \frac{\mathcal{W}^l_{p,m}(\alpha_1+2)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}. \tag{21}$$

It follows from (13) that

$$1-\frac{1}{p(\alpha_1+1)(M-1)}\Phi_1(l,m)f(z)\prec\frac{1+z}{1-z}.$$

Equivalently,

$$z\left(\frac{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1]f(z)}\right)'\prec p(\alpha_1+1)(M-1)\frac{z}{1-z}.$$

Hence, by using Lemma 1, we have

$$\frac{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1]f(z)}\prec 1-\frac{2p(\alpha_1+1)(M-1)}{n}\log(1-z)\equiv Q(z)\quad(z\in\Delta).\quad(22)$$

Since, $Q(z)$ is convex univalent in Δ and

$$\Re(Q(z))>1-\frac{2p(\alpha_1+1)(M-1)}{n}\log 2\quad(z\in\Delta),$$

from (22), we get the desired inequality (16). To show that the bound in (16) cannot be increased, we consider

$$q(z)=\left[1-\frac{2p(\alpha_1+1)(M-1)}{n}\log(1-z^n)\right]^{-1},\quad(z\in\Delta)$$

where M satisfies (14). A simple computation shows that $q(z)$ is in P_n and satisfies the inequality (13). On the other hand, we have

$$\Re\left(\frac{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1]f(z)}\right)\to 1-\frac{2p(\alpha_1+1)(M-1)}{n}\log 2\quad(z\in\Delta)$$

as $z\to e^{\frac{i\pi}{n}}$. The proof of the theorem is now complete.
For $p=1,\ l=1,\ m=1,\ A_1=1,\ B_1=1,\ \alpha_1=1,$ and $\beta_1=1$, we have the following result.

Corollary 1 *Let $f\in\mathcal{A}$ with $\frac{f(z)}{zf'(z)}\neq 0$ for $|z|<1$. If*

$$\Re\left[1+\frac{zf'(z)}{2f(z)}\left(1+\frac{zf''(z)}{zf'(z)}\right)-\frac{zf'(z)+z^2f''(z)/2}{f'(z)}\right]<M,\quad(z\in\Delta)$$

where

$$1 < M \leqq 1 + \frac{n}{4\log 2},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 1 - \frac{4(M-1)}{n}\log 2, \quad (z \in \Delta). \tag{23}$$

The bound in (23) is best possible.

Letting $M = 1 + \frac{n}{4\log 2}$, in Corollary 1, we have the following result.

Corollary 2 *$f \in \mathcal{A}$ with $\frac{f(z)}{zf'(z)} \neq 0$ for $|z| < 1$. If*

$$\Re\left[1 + \frac{zf'(z)}{2f(z)}\left(1 + \frac{zf''(z)}{zf'(z)}\right) - \frac{zf'(z) + z^2 f''(z)/2}{f'(z)}\right] < 1 + \frac{n}{4\log 2}, \quad (z \in \Delta),$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0. \tag{24}$$

In other words, f is starlike in Δ. The result is best possible.

Theorem 2 *Let $\delta(\alpha_1 + 1) > 0$, $\frac{\mathcal{W}^l_{p,m}[\alpha_1 + 1]f(z)}{z} \neq 0$ for $z \in \Delta$ and suppose that*

$$\Re\left\{\left(\frac{z}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}\right)^{\delta}\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1+2)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}\right)\right\} < M \quad (z \in \Delta). \tag{25}$$

then where

$$1 < M \leqq 1 + \frac{n}{2p\delta(\alpha_1+1)\log 2}. \tag{26}$$

Then,

$$\Re\left(\frac{z}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}\right)^{\delta} > 1 - \frac{2p\delta(\alpha_1+1)(M-1)}{n}\log 2 \ (z \in \Delta). \tag{27}$$

The result in (27) is best possible.

Proof. Define the function $q(z)$ by

$$q(z) = \left(\frac{\mathcal{W}_{p,m}^{l}(\alpha_1+1)f(z)}{z}\right)^{\delta}. \tag{28}$$

Then, clearly $q(z) = 1 + q_n z^n + q_{n+1}z^{n+1} + \ldots$ is analytic in Δ with $q(z) \neq 0$ for $z \in \Delta$. It follows from (28) that

$$\frac{zq'(z)}{q(z)} \;=\; \delta\frac{z(\mathcal{W}_{p,m}^{l}[\alpha_1+1]f(z))'}{\mathcal{W}_{p,m}^{l}[\alpha_1+1]f(z)} - 1. \tag{29}$$

By making use of the familiar identity (11) in (29), we get

$$\frac{\mathcal{W}_{p,m}^{l}(\alpha_1+2)f(z)}{\mathcal{W}_{p,m}^{l}(\alpha_1+1)f(z)} - 1 = \frac{1}{\delta(\alpha_1+1)}\frac{zq'(z)}{q(z)}. \tag{30}$$

Hence,

$$1 + \frac{1}{\delta(\alpha_1+1)}\frac{zq'(z)}{(q(z))^2} = \left(\frac{z}{\mathcal{W}_{p,m}^{l}(\alpha_1+1)f(z)}\right)^{\delta}\frac{\mathcal{W}_{p,m}^{l}(\alpha_1+2)f(z)}{\mathcal{W}_{p,m}^{l}(\alpha_1+1)f(z)}. \tag{31}$$

The remaining part of the proof is much akin to that of Theorem 1 and hence we omit the details involved.
For $p = 1$, $l = 1$, $m = 1$, $A_1 = 1$, $B_1 = 1$, $\alpha_1 = 1$, and $\beta_1 = 1$, we have the following result.

Corollary 3 *Let $f \in \mathcal{A}$ with $f'(z) \neq 0$ for $|z| < 1$. If*

$$\Re\left[\left(\frac{1}{f'(z)}\right)^{\delta}\left(\frac{zf'(z) + \frac{z^2 f''(z)}{2}}{zf'(z)}\right)\right] < M, \quad (z \in \Delta),$$

where

$$1 < M \leqq 1 + \frac{n}{4\delta\log 2},$$

then

$$\Re\left(f'(z)\right)^{\delta} > 1 - \frac{4\delta(M-1)}{n}\log 2, \quad (z \in \Delta). \tag{32}$$

The bound (32) is best possible.

Letting $\delta = 1$, and $M = 1 + \dfrac{n}{8\log 2}$, in Corollary 3, we have the following result.

Corollary 4 *$f \in \mathcal{A}$ with $f'(z) \neq 0$ for $|z| < 1$. If*

$$\Re\left[\left(\frac{1}{f'(z)}\right)\left(\frac{zf'(z) + \dfrac{z^2 f''(z)}{2}}{zf'(z)}\right)\right] < 1 + \frac{n}{8\log 2}, \quad (z \in \Delta),$$

$$\Re\left(f'(z)\right) > 0. \tag{33}$$

The result is sharp.

Theorem 3 *Let $\beta > 0$, $\dfrac{\mathcal{W}^l_{p,m}[\alpha_1 + 1]f(z)}{\mathcal{W}^l_{p,m}[\alpha_1]f(z)} \neq 0$ for $z \in \Delta$. Suppose that for $z \in \Delta$, $q(z) \in P_n$ satisfy*

$$\Re\left\{1 + \left(\frac{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1 + 1)f(z)}\right)^{\beta} \Phi_2(l,m)f(z)\right\} < M \tag{34}$$

where

$$\Phi_2(l,m)f(z) = \left((\alpha_1 + 1)\frac{\mathcal{W}^l_{p,m}(\alpha_1 + 2)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1 + 1)f(z)} - \alpha_1\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1 + 1)f(z)}\right)^{\beta} - 1\right) \tag{35}$$

$$1 < M \leqq 1 + \frac{n}{2\beta p \log 2}. \tag{36}$$

Then,

$$\Re\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1 + 1)f(z)}\right)^{\beta} > 1 - \frac{2\beta p(M-1)}{n}\log 2 \ (z \in \Delta). \tag{37}$$

The bound in (37) is best possible.

Proof. Define the function $q(z)$ by

$$q(z) = \left(\frac{\mathcal{W}^l_{p,m}(\alpha_1 + 1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}\right)^{\beta}. \tag{38}$$

Then, clearly $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \ldots$ is analytic in Δ with $q(z) \neq 0$ for $z \in \Delta$. By making use of the identity (11), and using a simple computation, we get

$$\frac{1}{\beta}\frac{zq'(z)}{q^2(z)} = \left(\frac{\mathcal{W}^l_{p,m}(\alpha_1)f(z)}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}\right)^\beta \Phi_2(l,m)f(z), \tag{39}$$

where $\Phi_2(l,m)f(z)$ is as defined in (35). The assertion of the theorem follows by applying similar steps as in the proof of Theorem 1.

For $p = 1$, $l = 1$, $m = 1$, $A_1 = 1$, $B_1 = 1$, $\alpha_1 = 1$, and $\beta_1 = 1$, we have the following result.

Corollary 5 *Let $\beta > 0$. $f \in \mathcal{A}$ with $\frac{zf'(z)}{f(z)} \neq 0$ for $|z| < 1$. If*

$$\Re\left[1 + \left(\frac{f(z)}{zf'(z)}\right)^\beta \left(1 + \frac{zf''(z)}{zf'(z)} - \left(\frac{f(z)}{zf'(z)}\right)^\beta\right)\right] < M, \quad (z \in \Delta),$$

where

$$1 < M \leqq 1 + \frac{n}{2\beta\log 2},$$

then

$$\Re\left(\frac{f(z)}{zf'(z)}\right)^\beta > 1 - \frac{2\beta(M-1)}{n}\log 2, \quad (z \in \Delta). \tag{40}$$

The bound in (40) is best possible.

Letting $\beta = 1$ and $M = 1 + \frac{n}{2\beta\log 2}$, in Corollary 5, we have the following result.

Corollary 6 *$f \in \mathcal{A}$ with $\frac{zf'(z)}{f(z)} \neq 0$ for $|z| < 1$. If*

$$\Re\left[1 + \left(\frac{f(z)}{zf'(z)}\right)^\beta \left(1 + \frac{zf''(z)}{zf'(z)} - \left(\frac{f(z)}{zf'(z)}\right)^\beta\right)\right] < 1 + \frac{n}{2\beta\log 2}, \quad (z \in \Delta),$$

then

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > 0. \tag{41}$$

The result is sharp.

Theorem 4 *Let* $\lambda, \alpha_1+1>0$, $\dfrac{z}{\mathcal{W}^l_{p,m}[\alpha_1+1]f(z)} \neq 0$ *for* $z \in \Delta$. *Suppose that for* $z \in \Delta$, $q(z) \in P_n$ *satisfy*

$$\Re\left\{\lambda\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{z} - \frac{\mathcal{W}^l_{p,m}(\alpha_1+2)f(z)}{z}\right)+1\right\} < M \tag{42}$$

where

$$1 < M \leqq 1 + \frac{n\lambda}{2(\alpha_1+1)p\log 2}. \tag{43}$$

Then,

$$\Re\left(\frac{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)}{z}\right) > 1 - \frac{2p(\alpha_1+1)(M-1)}{n\lambda}\log 2 \ (z \in \Delta). \tag{44}$$

The bound in (44) is best possible.

Proof. Defining the function $q(z)$ by

$$q(z) = \frac{z}{\mathcal{W}^l_{p,m}(\alpha_1+1)f(z)} \tag{45}$$

and following the steps as in the proof of Theorem 1, we get the assertion of Theorem 4.
For $p=1,\ \ l=1,\ \ m=1,\ A_1=1,\ B_1=1,\ \alpha_1=1,$ and $\beta_1=1$, we have the following result.

Corollary 7 *Let* $\lambda > 0$, $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *for* $|z|<1$. *If*

$$\Re\left[1 - \frac{\lambda z f''(z)}{2}\right] < M, \quad (z \in \Delta),$$

where

$$1 < M \leqq 1 + \frac{n\lambda}{4\log 2},$$

then

$$\Re\left(f'(z)\right) > 1 - \frac{4(M-1)}{n\lambda}\log 2, \quad (z \in \Delta). \tag{46}$$

The bound in (46) is best possible.

Letting and $M = 1 + \dfrac{n\lambda}{4\log 2}$, in Corollary (7), we have the following result.

Corollary 8 *Let* $\lambda > 0$, $f \in \mathcal{A}$ *with* $f'(z) \neq 0$ *for* $|z| < 1$. *If*

$$\Re\left[1 - \frac{\lambda z f''(z)}{2}\right] < 1 + \frac{n\lambda}{4\log 2}, \quad (z \in \Delta),$$

then

$$\Re\left(f'(z)\right) > 0. \tag{47}$$

The result is sharp.

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Authors:

Thomas Rosy, S. Kavitha,
Department of Mathematics,
Madras Christian College
East Tambaram, Chennai-600 059, India
e-mail: *thomas.rosy@gmail.com*

S. Kavitha,
Department of Mathematics,
Madras Christian College
East Tambaram, Chennai-600 059, India
e-mail: *kavithass19@rediffmail.com*

G. Murugusundaramoorthy
School of Science and Humanities,
VIT University,
Vellore - 632014, India.
e-mail: *gmsmoorthy@yahoo.com*