COMMON FIXED POINTS INVOLVING ALTERING DISTANCES

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Abstract. Fixed points involving altering distances have been studied by many authors, one cite here Sastry and Babu in 1998 [Sastry, K. P. R.; Babu, G. V. R. Fixed point theorems in metric spaces by altering distances. Bull. Calcutta Math. Soc. 90 (1998), no. 3, 175–182] and Bebu in 2001 [Bebu, I. A new proof for a fixed point theorem in compact metric spaces. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 63 (2001), no. 2, 43–46]. The purpose of this contribution is to give some common fixed point theorems for occasionally weakly compatible mappings satisfying an altering distance in a metric space. These results improve and extend previous ones especially the results cited above.

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1. Introduction

Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mappings of a metric space $(\mathcal{X}, d)$. $\mathcal{S}$ and $\mathcal{T}$ are commuting if for all $x \in \mathcal{X}$

$$\mathcal{S}\mathcal{T}x = \mathcal{T}\mathcal{S}x.$$

In 1982, Sessa [11] defines $\mathcal{S}$ and $\mathcal{T}$ to be weakly commuting if for all $x \in \mathcal{X}$

$$d(\mathcal{S}\mathcal{T}x, \mathcal{T}\mathcal{S}x) \leq d(\mathcal{T}x, \mathcal{S}x).$$

After, in 1986, Jungck [3] introduced the notion of compatible mappings as follows: $\mathcal{S}$ and $\mathcal{T}$ are compatible if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0 \qquad (1)$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that for some $t \in \mathcal{X}$

$$\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t.$$

Later on, in 1993, the same author with Murthy and Cho [5] gave the concept of compatibility of type (A) which says that $\mathcal{S}$ and $\mathcal{T}$ are compatible of type (A) if in lieu of equality (1) we have the two equalities

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) = 0.$$

In their paper [9], Pathak and Khan defined a new concept of the compatibility called compatibility of type (B). $\mathcal{S}$ and $\mathcal{T}$ are compatible of type (B) if instead of (1) we have the two inequalities

$$\begin{aligned}
\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right] \\
&\quad \text{and} \\
\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].
\end{aligned}$$

It is very clear to see that compatible mappings of type (A) are compatible of type (B), but the converse is false in general (see [9]).

In 1998, Pathak et al. [8] gave another type of the compatibility called compatibility of type (C). To be compatible of type (C), $\mathcal{S}$ and $\mathcal{T}$ must satisfy the next conditions

$$\begin{aligned}
\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) &\leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n)\right. \\
&\quad \left. + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right] \\
&\quad \text{and} \\
\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) &\leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{S}^2x_n)\right. \\
&\quad \left. + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].
\end{aligned}$$

Clearly, compatible mappings of type (A) are compatible of type (C), but the converse is not true (see [8]).

In papers [6] and [7] the notion of compatible mappings of type (P) was introduced which consists that $\mathcal{S}$ and $\mathcal{T}$ verified the below equality

$$\lim_{n\to\infty} d(\mathcal{S}^2 x_n, \mathcal{T}^2 x_n) = 0.$$

In his paper [4], Jungck weakens all the above notions by giving the concept of weak compatible mappings. $\mathcal{S}$ and $\mathcal{T}$ above are weakly compatible if they commute at their coincidence points.

Recently, in 2008, Al-Thagafi and Shahzad [1] gave a proper generalization of nontrivial weakly compatible mappings which do have a coincidence point.

Definition 1.1. **([1])** *Two self-mappings $\mathcal{S}$ and $\mathcal{T}$ of a set $\mathcal{X}$ are* **occasionally weakly compatible** *shortly* **(owc)** *iff there is a point z in $\mathcal{X}$ which is a coincidence point of $\mathcal{S}$ and $\mathcal{T}$ at which $\mathcal{S}$ and $\mathcal{T}$ commute.*

2. Preliminaries

For existence and uniqueness of common fixed points, some authors used the so-called **altering distance**.

An altering distance is a mapping $\psi : [0, \infty) \to [0, \infty)$ which satisfies the two conditions:
(a) ψ is increasing and continuous,
(b) $\psi(t) = 0$ iff $t = 0$.

In 1998, Sastry and Babu [10] proved the existence and uniqueness of a fixed point for a mapping in a complete metric space by using an altering distance.

Theorem 2.1. **([10])** *Let $\mathcal{T}$ be a continuous mapping of the complete metric space into itself and $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ an altering distance. Suppose that*

$$(2)\quad \psi\left(d\left(\mathcal{T}x, \mathcal{T}y\right)\right) \leq a\left[\psi\left(d\left(x, \mathcal{T}x\right)\right) + \psi\left(d\left(y, \mathcal{T}y\right)\right)\right] + b\psi\left(d\left(x, y\right)\right) + c\left[\psi\left(d\left(x, \mathcal{T}y\right)\right)\psi\left(d\left(y, \mathcal{T}x\right)\right)\right]^{\frac{1}{2}}$$

for all x, y in $\mathcal{X}$, where $a, b \geq 0$ and $a^2 + b^2 \neq 0$.
(i) $\mathcal{T}$ has a fixed point if $2a + b < 1$;
(ii) $\mathcal{T}$ has at most one fixed point if $a + b < 1$.

Afterwards, Bebu [2] removed the condition of monotonicity from the definition of altering distance for proving the existence of a fixed point for a mapping in a compact metric space.

Theorem 2.2. **([2])** *Let $\mathcal{T}$ be a continuous mapping of the compact metric space $(\mathcal{X}, d)$ into itself and $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous function satisfying $\psi(t) = 0$ iff $t = 0$. Suppose that (2) holds for all x, y in $\mathcal{X}$, where $a, b \geq 0$, $a^2 + b^2 \neq 0$. Then*
(i) $\mathcal{T}$ has a fixed point if $2a + b < 1$;
(ii) $\mathcal{T}$ has at most one fixed point if $b + c < 1$.

The objective of this paper is to extend these theorems to four occasionally weakly compatible mappings, to improve the same results by removing the completion and the compacity of the space and the continuity imposed on $\mathcal{T}$, also to extend constantes a, b and c to functions.

3. Main results

Theorem 3.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be four mappings of a metric space $(\mathcal{X}, d)$ into itself such that the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are owc and satisfy the inequality*

$$\begin{aligned}\psi\left(d\left(\mathcal{A}x, \mathcal{B}y\right)\right) \leq\ & a\left(d\left(\mathcal{S}x, \mathcal{T}y\right)\right)\left[\psi\left(d\left(\mathcal{A}x, \mathcal{S}x\right)\right) + \psi\left(d\left(\mathcal{B}y, \mathcal{T}y\right)\right)\right] \\ & + b\left(d\left(\mathcal{S}x, \mathcal{T}y\right)\right)\psi\left(d\left(\mathcal{S}x, \mathcal{T}y\right)\right) \\ & + c\left(d\left(\mathcal{S}x, \mathcal{T}y\right)\right)\left[\psi\left(d\left(\mathcal{S}x, \mathcal{B}y\right)\right)\psi\left(d\left(\mathcal{T}y, \mathcal{A}x\right)\right)\right]^{\frac{1}{2}} \qquad (3.1)\end{aligned}$$

for all x, y in $\mathcal{X}$, where $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = 0$ iff $t = 0$ and $a, b, c : [0, \infty) \to [0, 1)$ satisfying the inequality $b(t) + c(t) < 1$ for $t > 0$.
Then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point $z \in \mathcal{X}$.

Proof. Since the pair $(\mathcal{A}, \mathcal{S})$ is owc as well as the pair $(\mathcal{B}, \mathcal{T})$, then, there exist two elements u and v in $\mathcal{X}$ such that

$$\begin{aligned}\mathcal{A}u &= \mathcal{S}u \text{ and } \mathcal{A}\mathcal{S}u = \mathcal{S}\mathcal{A}u; \\ \mathcal{B}v &= \mathcal{T}v \text{ and } \mathcal{B}\mathcal{T}v = \mathcal{T}\mathcal{B}v.\end{aligned}$$

First, we prove that $\mathcal{A}u = \mathcal{B}v$. Indeed by inequality (3.1) we have

$$\psi\left(d\left(\mathcal{A}u, \mathcal{B}v\right)\right) \leq a\left(d\left(\mathcal{S}u, \mathcal{T}v\right)\right)\left[\psi\left(d\left(\mathcal{A}u, \mathcal{S}u\right)\right) + \psi\left(d\left(\mathcal{B}v, \mathcal{T}v\right)\right)\right]$$

$$\begin{aligned} & +b\left(d\left(\mathcal{S}u,\mathcal{T}v\right)\right)\psi\left(d\left(\mathcal{S}u,\mathcal{T}v\right)\right) \\ & +c\left(d\left(\mathcal{S}u,\mathcal{T}v\right)\right)\left[\psi\left(d\left(\mathcal{S}u,\mathcal{B}v\right)\right)\psi\left(d\left(\mathcal{T}v,\mathcal{A}u\right)\right)\right]^{\frac{1}{2}} \\ = & \left[b\left(d\left(\mathcal{A}u,\mathcal{B}v\right)\right)+c\left(d\left(\mathcal{A}u,\mathcal{B}v\right)\right)\right]\psi\left(d\left(\mathcal{A}u,\mathcal{B}v\right)\right) \\ < & \psi\left(d\left(\mathcal{A}u,\mathcal{B}v\right)\right)\end{aligned}$$

which is a contradiction. Hence, $\psi\left(d\left(\mathcal{A}u,\mathcal{B}v\right)\right)=0$ which implies that $\mathcal{A}u=\mathcal{B}v$.
Now, suppose that $\mathcal{A}\mathcal{A}u\neq\mathcal{A}u$. Then, using inequality (3.1), we get

$$\begin{aligned}\psi\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{A}u\right)\right) & = \psi\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{B}v\right)\right) \\ & \leq a\left(d\left(\mathcal{S}\mathcal{A}u,\mathcal{T}v\right)\right)\left[\psi\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{S}\mathcal{A}u\right)\right)+\psi\left(d\left(\mathcal{B}v,\mathcal{T}v\right)\right)\right] \\ & \quad +b\left(d\left(\mathcal{S}\mathcal{A}u,\mathcal{T}v\right)\right)\psi\left(d\left(\mathcal{S}\mathcal{A}u,\mathcal{T}v\right)\right) \\ & \quad +c\left(d\left(\mathcal{S}\mathcal{A}u,\mathcal{T}v\right)\right)\left[\psi\left(d\left(\mathcal{S}\mathcal{A}u,\mathcal{B}v\right)\right)\psi\left(d\left(\mathcal{T}v,\mathcal{A}\mathcal{A}u\right)\right)\right]^{\frac{1}{2}} \\ & = \left[b\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{A}u\right)\right)+c\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{A}u\right)\right)\right]\psi\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{A}u\right)\right) \\ & < \psi\left(d\left(\mathcal{A}\mathcal{A}u,\mathcal{A}u\right)\right)\end{aligned}$$

this contradiction demands that $\mathcal{A}\mathcal{A}u=\mathcal{A}u=\mathcal{S}\mathcal{A}u$.
If $\mathcal{B}\mathcal{B}v\neq\mathcal{B}v$, then inequality (3.1) gives

$$\begin{aligned}\psi\left(d\left(\mathcal{B}v,\mathcal{B}\mathcal{B}v\right)\right) & = \psi\left(d\left(\mathcal{A}u,\mathcal{B}\mathcal{B}v\right)\right) \\ & \leq a\left(d\left(\mathcal{S}u,\mathcal{T}\mathcal{B}v\right)\right)\left[\psi\left(d\left(\mathcal{A}u,\mathcal{S}u\right)\right)+\psi\left(d\left(\mathcal{B}\mathcal{B}v,\mathcal{T}\mathcal{B}v\right)\right)\right] \\ & \quad +b\left(d\left(\mathcal{S}u,\mathcal{T}\mathcal{B}v\right)\right)\psi\left(d\left(\mathcal{S}u,\mathcal{T}\mathcal{B}v\right)\right) \\ & \quad +c\left(d\left(\mathcal{S}u,\mathcal{T}\mathcal{B}v\right)\right)\left[\psi\left(d\left(\mathcal{S}u,\mathcal{B}\mathcal{B}v\right)\right)\psi\left(d\left(\mathcal{T}\mathcal{B}v,\mathcal{A}u\right)\right)\right]^{\frac{1}{2}} \\ & = \left[b\left(d\left(\mathcal{B}v,\mathcal{B}\mathcal{B}v\right)\right)+c\left(d\left(\mathcal{B}v,\mathcal{B}\mathcal{B}v\right)\right)\right]\psi\left(d\left(\mathcal{B}v,\mathcal{B}\mathcal{B}v\right)\right) \\ & < \psi\left(d\left(\mathcal{B}v,\mathcal{B}\mathcal{B}v\right)\right)\end{aligned}$$

a contradiction. Thus, $\mathcal{B}\mathcal{B}v=\mathcal{B}v=\mathcal{T}\mathcal{B}v$. Hence $\mathcal{A}u=\mathcal{B}v=\mathcal{S}u=\mathcal{T}v$ is a common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.
Let $\mathcal{A}u=\mathcal{S}u=\mathcal{B}v=\mathcal{T}v=z$ and let z' be another common fixed point of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ with $z'\neq z$. The use of inequality (3.1) gives

$$\begin{aligned}\psi\left(d\left(z,z'\right)\right) & = \psi\left(d\left(\mathcal{A}z,\mathcal{B}z'\right)\right) \\ & \leq a\left(d\left(\mathcal{S}z,\mathcal{T}z'\right)\right)\left[\psi\left(d\left(\mathcal{A}z,\mathcal{S}z\right)\right)+\psi\left(d\left(\mathcal{B}z',\mathcal{T}z'\right)\right)\right] \\ & \quad +b\left(d\left(\mathcal{S}z,\mathcal{T}z'\right)\right)\psi\left(d\left(\mathcal{S}z,\mathcal{T}z'\right)\right) \\ & \quad +c\left(d\left(\mathcal{S}z,\mathcal{T}z'\right)\right)\left[\psi\left(d\left(\mathcal{S}z,\mathcal{B}z'\right)\right)\psi\left(d\left(\mathcal{T}z',\mathcal{A}z\right)\right)\right]^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned} &= [b(d(z,z')) + c(d(z,z'))]\,\psi(d(z,z')) \\ &< \psi(d(z,z')) \end{aligned}$$

this contradiction implies that $z' = z$.

Now, we give the second result.

Theorem 3.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be four self-mappings of a metric space $(\mathcal{X}, d)$ satisfying the inequality*

$$\begin{aligned} \psi(d(\mathcal{A}x, \mathcal{B}y)) \leq\ & a(d(\mathcal{S}x, \mathcal{T}y))\,\psi(d(\mathcal{S}x, \mathcal{T}y)) \\ & + b(d(\mathcal{S}x, \mathcal{T}y))\,[\psi(d(\mathcal{S}x, \mathcal{B}y))\,\psi(d(\mathcal{T}y, \mathcal{A}x))]^{\frac{1}{2}} \qquad (3.2) \end{aligned}$$

for all x, y in $\mathcal{X}$, where $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = 0$ iff $t = 0$ and $a, b : [0,\infty) \to [0,1)$ with $a(t) + b(t) < 1$ for $t > 0$. Suppose that the pair $(\mathcal{A}, \mathcal{S})$ is owc as well as the pair $(\mathcal{B}, \mathcal{T})$, then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{X}$.

We end this contribution by giving a third result for an infinity mappings.

Theorem 3.3. *Let $\{\mathcal{A}_i\}$, $i = 1, 2, ...$, $\mathcal{S}$ and $\mathcal{T}$ be self-mappings of a metric space $(\mathcal{X}, d)$ satisfying inequality*

$$\begin{aligned} \psi(d(\mathcal{A}_1 x, \mathcal{A}_i y)) \leq\ & a(d(\mathcal{S}x, \mathcal{T}y))\,\psi(d(\mathcal{S}x, \mathcal{T}y)) \\ & + b(d(\mathcal{S}x, \mathcal{T}y))\,[\psi(d(\mathcal{S}x, \mathcal{A}_i y))\,\psi(d(\mathcal{T}y, \mathcal{A}_1 x))]^{\frac{1}{2}} \qquad (3.3) \end{aligned}$$

for all x, y in $\mathcal{X}$, where $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = 0$ iff $t = 0$ and $a, b : [0,\infty) \to [0,1)$ such that $a(t) + b(t) < 1$ for $t > 0$.
If $\mathcal{A}_1$ and $\mathcal{S}$ are owc (resp. $\mathcal{A}_k$ and $\mathcal{T}$ for some $k > 1$), then $\{\mathcal{A}_i\}_{i\geq 1}$, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Since $\mathcal{A}_1$ and $\mathcal{S}$ (resp. $\mathcal{A}_k$ and $\mathcal{T}$, $k > 1$) are owc, as in proof of Theorem 3.1, there exist two points $u, v \in \mathcal{X}$ such that

$$\begin{aligned} \mathcal{A}_1 u &= \mathcal{S}u \text{ and } \mathcal{A}_1\mathcal{S}u = \mathcal{S}\mathcal{A}_1 u; \\ \mathcal{A}_k v &= \mathcal{T}v \text{ and } \mathcal{A}_k\mathcal{T}v = \mathcal{T}\mathcal{A}_k v. \end{aligned}$$

Suppose that $d(\mathcal{A}_1 u, \mathcal{A}_k v) > 0$, then, by using inequality (3.3) we obtain

$$\begin{aligned} \psi(d(\mathcal{A}_1 u, \mathcal{A}_k v)) &\leq a(d(\mathcal{S}u, \mathcal{T}v))\,\psi(d(\mathcal{S}u, \mathcal{T}v)) \\ &\quad + b(d(\mathcal{S}u, \mathcal{T}v))\,[\psi(d(\mathcal{S}u, \mathcal{A}_k v))\,\psi(d(\mathcal{T}v, \mathcal{A}_1 u))]^{\frac{1}{2}} \\ &= [a(d(\mathcal{A}_1 u, \mathcal{A}_k v)) + b(d(\mathcal{A}_1 u, \mathcal{A}_k v))]\,\psi(d(\mathcal{A}_1 u, \mathcal{A}_k v)) \\ &< \psi(d(\mathcal{A}_1 u, \mathcal{A}_k v)) \end{aligned}$$

which is a contradiction, therefore $\psi(d(\mathcal{A}_1u,\mathcal{A}_kv)) = 0$ which implies that $d(\mathcal{A}_1u,\mathcal{A}_kv) = 0$; i.e., $\mathcal{A}_1u = \mathcal{A}_kv$ for $k = 2, 3, \ldots$.
Now, we claim that $\mathcal{A}_1^2u = \mathcal{A}_1u$. If not, then $d(\mathcal{A}_1^2u,\mathcal{A}_1u) > 0$ and the use of inequality (3.3) gives

$$\begin{aligned}
\psi\left(d\left(\mathcal{A}_1^2u,\mathcal{A}_1u\right)\right) &= \psi(d(\mathcal{A}_1\mathcal{A}_1u,\mathcal{A}_kv))\\
&\leq a(d(\mathcal{S}\mathcal{A}_1u,\mathcal{T}v))\,\psi(d(\mathcal{S}\mathcal{A}_1u,\mathcal{T}v))\\
&\quad +b(d(\mathcal{S}\mathcal{A}_1u,\mathcal{T}v))\left[\psi(d(\mathcal{S}\mathcal{A}_1u,\mathcal{A}_kv))\,\psi\left(d\left(\mathcal{T}v,\mathcal{A}_1^2u\right)\right)\right]^{\frac{1}{2}}\\
&= \left[a\left(d\left(\mathcal{A}_1^2u,\mathcal{A}_1u\right)\right)+b\left(d\left(\mathcal{A}_1^2u,\mathcal{A}_1u\right)\right)\right]\psi\left(d\left(\mathcal{A}_1^2u,\mathcal{A}_1u\right)\right)\\
&< \psi\left(d\left(\mathcal{A}_1^2u,\mathcal{A}_1u\right)\right)
\end{aligned}$$

a contradiction. Hence $\psi(d(\mathcal{A}_1^2u,\mathcal{A}_1u)) = 0$ which implies that $\mathcal{A}_1^2u = \mathcal{A}_1u$. Similarly, $\mathcal{A}_k^2v = \mathcal{A}_kv$. Therefore, $\{\mathcal{A}_i\}_{i\geq 1}$, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point $\mathcal{A}_1u = \mathcal{S}u = \mathcal{A}_kv = \mathcal{T}v$.
The uniqueness of the common fixed point follows immediately from inequality (3.3) and the properties of ψ, a and b.

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