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SHARP INEQUALITIES INVOLVING $\psi_P(X)$ FUNCTION

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Abstract. The aim of this paper is to show that for $a \in (0,1)$, the function $f_a = \psi_p(x+a) - \psi_p(x)$ is strictly completely monotonic in $(0,\infty)$. As a direct consequence, a sharp inequality involving the psi function is established.

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1. Introduction and preliminaries

The gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0$$

was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), ... as well as many others.

The gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study

Gamma function is one of the most important special functions with applications in various fields, like analysis, mathematical physics, probability theory and statistics. Many interesting historical information on this function can be found in Davis' survey paper [5].

The gamma function has several representations. From a long list of the representations (see [1],[7],[13],[14]), the following are representative:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)(x+2)\cdots(x+n)}$$

$$(x \neq 0, -1, -2, -3, \ldots)$$

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \{ \Big(1 + \frac{x}{n}\Big) e^{-\frac{x}{n}} \}$$

$$\Gamma(x) = \int_{-\infty}^{\infty} exp(xt - e^t) dt \quad (Re(x) > 0)$$

$$\frac{1}{\Gamma(x)} = x \prod_{n=1}^{\infty} \frac{(1 + \frac{x}{n})}{(1 + \frac{x}{n})^x}$$

where

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \Big(\frac{1}{k} - \ln n \Big) = 0.57721566\ldots,$$

is known as Euler's constant. It is also called Mascheroni's constant and often denoted by the symbol C.

The logarithmic derivative of the gamma function is called the digamma function. It is know as the psi function and is denoted by $\psi(x)$.

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}.$$

The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)} \tag{1}$$

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [8]),

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{2}$$

where

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \tag{3}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [9]), that is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{4}$$

The function ψ_p defined in (4) satisfies the following properties (see [9]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k} = \ln p - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{5}$$

It is increasing on $(0,\infty)$ and it is strictly completely monotonic on $(0,\infty)$. It's derivatives are given by (see [8],[9]):

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p} \frac{(-1)^{n-1}\cdot n!}{(x+k)^{n+1}} = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{6}$$

For $0 < x < y$

$$\psi_p(x) - \psi_p(y) < 0 \tag{7}$$

$$\psi_p(x+1) = \psi_p(x) + \frac{1}{x} - \frac{1}{x+p+1} \tag{8}$$

(see [9])

Recall that a function h is (strictly) completely monotonic on $(0,\infty)$ if

$$(-1)^n f^{(n)}(x) \geq 0, \text{ respective} (-1)^n f^{(n)}(x) > 0$$

for every $x \in (0,\infty)$.

The well-know Hausdorf -Bernstein-Widder theorem states that a function h is completely monotonic if and only if there exists a non-negative measure μ on $[0,\infty)$ such that for every $x \in (0,\infty)$,

$$h(x) = \int_0^\infty e^{-tx} d\mu(t).$$

For proofs and other details, see for example [1, 17].

Cristinel Mortici in paper A sharp inequality involving the psi function (see [11]) proved that the function $f_a = \psi_p(x+a) - \psi_p(x)$ for $a \in (0,1)$, is strictly completely monotonic in $(0,\infty)$. The purpose of this paper is to establish the following result.

2.Main Results

Theorem 1. *For every $a \in (0,1)$, the function $f_a : (0,\infty) \to (0,\infty)$*

$$f_a = \psi_p(x+a) - \psi_p(x)$$

is completely monotonic. In particular, f_a is decreasing and convex.

Proof. We have
$$f_a^{'} = \psi_p^{'}(x+a) - \psi_p^{'}(x)$$
and using the integral representation (6), we obtain

$$f_a^{'} = \int_0^{\infty} \frac{te^{-(x+a)t}}{1-e^{-t}}(1-e^{-pt})dt - \int_0^{\infty} \frac{te^{-xt}}{1-e^{-t}}(1-e^{-pt})dt$$

Straightforward computations lead us to the form

$$f_a^{'} = \int_0^{\infty} \frac{e^{-(x+1)t}(1-e^{-pt})}{1-e^{-t}}\varphi(t)dt$$

where
$$\varphi(t) = te^{(1-a)t} - te^{t}$$

We have
$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(1-a)^k}{k!}t^{k+1} - \sum_{k=0}^{\infty} \frac{1}{k!}t^{k+1}$$

or
$$\varphi(t) = \sum_{k=0}^{\infty} \frac{1}{k!}t^{k+1}((1-a)^k - 1) \leq 0.$$

By the Hausdorff-Bernstein-Widder theorem, it results that $-f_a^{'}$ is completely monotonic.In particular, $f_a^{'} \geq 0$, so f_a is decreasing. We have

$$\lim_{x\to\infty} f_a = 0, \text{ and } f_a > 0$$

(see (7))

As a direct consequence of the fact that f_a is strictly decreasing, we have for every $x \in [1, \infty)$,

$$0 = \lim_{x\to\infty} f_a(x) \leq f_a(1) = \psi_p(a+1) - \psi_p(1)$$

or using (8), we obtain

$$0 < \psi_p(x+a) - \psi_p(x) < \frac{1}{x} - \frac{1}{x+p+1}.$$

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