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ON RICCI η-RECURRENT $(LCS)_N$-MANIFOLDS

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Abstract. The object of the present paper is to study $(LCS)_n$-manifolds with η-recurrent Ricci tensor. Several interesting results on $(LCS)_n$-manifolds are obtained. Also the existence of such a manifold is ensured by a non-trivial example.

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1. Introduction

In 2003, A.A. Shaikh [6] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example. An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp. non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (resp. $\leq 0, = 0, > 0$) [1,4].

Recently, A.A. Shaikh and K.K. Baishya [7] introduced the notion of LP-Sasakian manifolds with η-recurrent Ricci tensor which generalizes the notion of η-parallel Ricci tensor introduced by M. Kon [2] for a Sasakian manifold.

In this paper we introduce the same notion on $(LCS)_n$-manifolds and give a non-trivial example. The paper is organized as follows: Section 2 is concerned about basic identities of $(LCS)_n$-manifolds. After section 2, in section 3 we study Ricci η-recurrent $(LCS)_n$-manifolds and prove that in such a manifold if the scalar curvature is constant, then the characteristic vector field ξ and the vector field ρ_1 associated to the 1-form A are co-directional. Since the notion of Ricci η-recurrency is the generalization of Ricci η-parallelity, does there exist a $(LCS)_n$-manifold with η-recurrent but not η-parallel? For this natural question we give a non-trivial example in the last section.

2.Preliminaries

Let M^n be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X) \tag{2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{3}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{4}$$

where ρ being a certain scalar function. By virtue of (2), (3) and (4), it follows that

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{5}$$

where $\beta = -(\xi\rho)$ is a scalar function. Next if we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi. \tag{6}$$

Then from (3) and (6) we have

$$\phi X = X + \eta(X)\xi, \tag{7}$$

from which it follows that ϕ is symmetric $(1,1)$ tensor and is called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefy $(LCS)_n$- manifold) [6]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [3]. In a $(LCS)_n$-manifold, the following relations hold:[5, 6]

$$a)\eta(\xi) = -1, \;\; b)\phi\xi = 0, \;\; c)\eta \circ \phi = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{9}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi], \tag{10}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)X - g(X,Z)Y], \tag{11}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{12}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{14}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

Definition 1: *The Ricci tensor of an $(LCS)_n$-manifold is said to be η-recurrent if its Ricci tensor satisfies the following:*

$$(\nabla_X S)(\phi Y, \phi Z) = A(X)S(\phi Y, \phi Z) \tag{15}$$

for all X, Y, Z where $A(X) = g(X, \rho_1)$, ρ_1 is the associated vector field of the 1-form A. In particular, if the 1-form A vanishes then the Ricci tensor of the $(LCS)_n$-manifold is said to be η-parallel and this notion for Sasakian manifolds was first introduced by Kon [2].

3.Ricci η-recurrent $(LCS)_n$-manifolds

Let us consider a Ricci η-recurrent $(LCS)_n$-manifold. From which it follows that

$$\nabla_Z S(\phi X, \phi Y) - S(\nabla_Z \phi X, \phi Y) - S(\phi X, \nabla_Z \phi Y) = A(Z)S(\phi X, \phi Y). \tag{16}$$

In view of (3), (4), (8), (10), (12) and (14), it can be easily seen that

$$\begin{aligned} & (\nabla_Z S)(X,Y) - \alpha[S(\phi Y, Z)\eta(X) + S(\phi X, Z)\eta(Y)] \\ & +(n-1)[(2\alpha\rho - \beta)\eta(X)\eta(Y)\eta(Z) \\ & +\alpha(\alpha^2 - \rho)\{g(Y,Z)\eta(X) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\}] \\ = \; & A(Z)[S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y)]. \end{aligned} \tag{17}$$

It follows that

$$\begin{aligned}(\nabla_Z S)(X,Y) &= \alpha[S(\phi Y,Z)\eta(X)+S(\phi X,Z)\eta(Y)] \\ &-(n-1)[(2\alpha\rho-\beta)\eta(X)\eta(Y)\eta(Z) \\ &+\alpha(\alpha^2-\rho)\{g(Y,Z)\eta(X)+g(X,Z)\eta(Y) \\ &+2\eta(X)\eta(Y)\eta(Z)\}]+A(Z)[S(X,Y) \\ &+(n-1)(\alpha^2-\rho)\eta(X)\eta(Y)].\end{aligned} \tag{18}$$

Hence we can state the following:

Theorem 1. *In a $(LCS)_n$-manifold M^n, the Ricci tensor is η-recurrent if and only if (18) holds.*

Let $\{e_i, i=1,2,...,n\}$ be an orthonormal frame field at any point of the manifold. then by contracting over Y and Z in (18) we get

$$dr(X)=(n-1)(2\alpha\rho-\beta)\eta(X)+A(X)[r-(n-1)(\alpha^2-\rho)]. \tag{19}$$

If the manifold has constant scalar curvature r, then from (19) we have

$$(n-1)(2\alpha\rho-\beta)\eta(X)=A(X)[(n-1)(\alpha^2-\rho)-r]. \tag{20}$$

For $X=\xi$, the relation (20) yields

$$(n-1)(2\alpha\rho-\beta)=\eta(\rho_1)[r-(n-1)(\alpha^2-\rho)]. \tag{21}$$

In view of (20) and (21) we obtain

$$A(X)=\eta(X)\eta(\rho_1). \tag{22}$$

This leads to the following:

Theorem 2. *In a Ricci η-recurrent $(LCS)_n$-manifold M^n if the scalar curvature r is constant, then the characteristic vector field ξ and the vector field ρ_1 associated to the 1-form A are co-directional and the 1-form A is given by (22).*

Again contracting over X and Z in (18) we obtain

$$\begin{aligned}&\frac{1}{2}dr(X)=\alpha\mu\eta(Y)+(n-1)[(2\alpha\rho-\beta)\eta(Y) \\ &-(n-1)\alpha(\alpha^2-\rho)\eta(Y)]+(n-1)(\alpha^2-\rho)\eta(Y)\eta(\rho_1)+S(Y,\rho_1),\end{aligned} \tag{23}$$

where $\mu = Tr.(Q\phi) = \sum_{i=1}^{n} \epsilon_i S(\phi e_i, e_i)$. By comparing (19) and (23) we have

$$\begin{aligned} & \frac{1}{2}(n-1)(2\alpha\rho-\beta)\eta(Y) + \frac{1}{2}A(Y)[r-(n-1)(\alpha^2-\rho)]. \\ = \; & \alpha\mu\eta(Y) + (n-1)[(2\alpha\rho-\beta)\eta(Y) - (n-1)\alpha(\alpha^2-\rho)\eta(Y)] \\ & +(n-1)(\alpha^2-\rho)\eta(Y)\eta(\rho_1) + S(Y,\rho_1). \end{aligned} \tag{24}$$

Taking $Y = \xi$ in (24) and using(8(a)), we get

$$\begin{aligned} & \frac{1}{2}(n-1)(2\alpha\rho-\beta) + \alpha\mu - (n-1)^2\alpha(\alpha 62-\rho) \\ = \; & \frac{1}{2}[(n-1)(\alpha^2-\rho) - r]\eta(\rho_1). \end{aligned} \tag{25}$$

Considering (25) in (24) we have

$$\begin{aligned} S(Y,\rho_1) \;=\; & \frac{1}{2}[r-(n-1)(\alpha^2-\rho)]g(Y,\rho_1) \\ & +\frac{1}{2}[r-3(n-1)(\alpha^2-\rho_1)]\eta(Y)\eta(\rho_1). \end{aligned} \tag{26}$$

Thus we have the following result:

Theorem 3. *If the Ricci tensor of an $(LCS)_n$-manifold $(M^n, g)(n > 3)$ is η-recurrent, then its Ricci tensor along the associated vector field of the 1-form is given by (26).*

Substituting Y by ϕY in (26) we obtain by virtue of (8) that

$$S(\phi Y, \rho_1) = \frac{1}{2}[r-(n-1)(\alpha^2-\rho)]g(\phi Y, \rho_1) \tag{27}$$

By virtue of $Q\phi = \phi Q$ and the symmetry of ϕ we get from (27) that

$$S(Y, L) = kg(Y, L), \tag{28}$$

where $L = \phi\rho_1$ and $k = \frac{1}{2}[r-(n-1)(\alpha^2-\rho)]$.

From (28) we can state the following:

Theorem 4.*If the Ricci tensor of an $(LCS)-n$-manifold $(M^n, g)(n > 3)$ is η-recurrent, then $k = \frac{1}{2}[r-(n-1)(\alpha^2-\rho)]$ is an Eigen value of the Ricci tensor corresponding to the Eigen vector $\phi\rho_1$ defined by $g(X, L) = D(X) = g(X, \phi\rho_1)$.*

4. Existence of Ricci η-recurrent $(LCS)_n$-manifolds

In this section, first we construct an example of $(LCS)_n$-manifold with global vector fields whose Ricci tensor is η-parallel.

Example 1: We consider a 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3, E_4\}$ be linearly independent global frame on M given by

$$E_1 = z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad E_2 = z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by $g(E_1,E_3) = g(E_2,E_3) = g(E_1,E_2) = 0$ and $g(E_1,E_1) = g(E_2,E_2) = 1, \quad g(E_3,E_3) = -1$.

Let η be the 1-form defined by $\eta(U) = g(U,E_3)$ for any $U \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_1$, $\phi E_2 = E_2$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ,ξ,η,g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1,E_2] = -zE_2, \quad [E_1,E_3] = -\frac{1}{z}E_1, \quad [E_2,E_3] = -\frac{1}{z}E_2.$$

Taking $E_3 = \xi$ and using Koszula formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned}
\nabla_{E_1}E_3 &= -\frac{1}{z}E_1, \quad \nabla_{E_3}E_3 = 0, \quad \nabla_{E_2}E_3 = -\frac{1}{z}E_2,\\
\nabla_{E_1}E_1 &= -\frac{1}{z}E_3, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_2}E_1 = zE_2,\\
\nabla_{E_2}E_2 &= -\frac{1}{z}E_3 - zE_1, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_1 = 0.
\end{aligned}$$

From the above it can be easily seen that (ϕ,ξ,η,g) is a $(LCS)_3$ structure on M. Consequently $M^3(\phi,\xi,\eta,g)$ is a $(LCS)_3$-manifold with $\alpha = -\frac{1}{z} \neq 0$ and $\rho = -\frac{1}{z^2}$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned}
R(E_1,E_3)E_1 &= -\frac{2}{z^2}E_3, \quad R(E_1,E_3)E_3 = -\frac{2}{z^2}E_1,\\
R(E_1,E_2)E_2 &= \frac{1}{z^2}E_1 - z^2E_1, \quad R(E_1,E_2)E_1 = z^2E_2 - \frac{1}{z^2}E_2,\\
R(E_2,E_3)E_2 &= -\frac{2}{z^2}E_3, \quad R(E_2,E_3)E_3 = -\frac{2}{z^2}E_2
\end{aligned}$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(E_1, E_1) = -\left(z^2 + \frac{1}{z^2}\right), \quad S(E_2, E_2) = -\left(z^2 + \frac{1}{z^2}\right), \quad S(E_3, E_3) = -\frac{4}{z^2}.$$

Since $\{E_1, E_2, E_3\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3 \quad \text{and} \quad Y = a_2E_1 + b_2E_2 + c_2E_3,$$

where $a_i, b_i, c_i \in R^+$(the set of all positive real numbers), $i = 1, 2$. This implies that

$$\phi X = a_1E_1 + b_1E_2 \quad \text{and} \quad \phi Y = a_2E_1 + b_2E_2.$$

Hence

$$S(\phi X, \phi Y) = (a_1a_2 + b_1b_2)\left(z^2 + \frac{1}{z^2}\right) \neq 0.$$

By the virtue of the above we have the following:

$$(\nabla_{E_i} S)(\phi X, \phi Y) \;=\; 0 \quad \text{for} \quad i = 1, 2, 3.$$

This implies that the manifold under consideration is an $(LCS)_3$-manifold with η-parallel Ricci tensor. This leads to the following:

Theorem 5. *There exists a $(LCS)_3$-manifold (M^3, g) with η-parallel Ricci tensor.*

Now we construct an example of $(LCS)_n$-manifolds with η-recurrent but not η-parallel Ricci tensor.

Example 2: We consider a 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad E_3 = e^z \frac{\partial}{\partial z},$$

where a is non-zero constant.

Let g be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$, $\phi E_3 = 0$. then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -e^z E_1, \quad [E_2, E_3] = -e^z E_2.$$

Taking $E_3 = \xi$ and using Koszula formula for the Lorentzian metric g, we can easily calculate

$$\begin{aligned} &\nabla_{E_1} E_3 = -e^z E_1, \quad \nabla_{E_1} E_1 = e^z E_3, \quad \nabla_{E_1} E_2 = 0, \\ &\nabla_{E_2} E_3 = -e^z E_2, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_2} E_1 = ae^z E_2, \\ &\nabla_{E_3} E_3 = 0, \quad \nabla_{E_1} E_2 = -ae^z E_1 + e^z E_3, \quad \nabla_{E_3} E_1 = 0. \end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a $(LCS)_3$ structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$-manifold with $\alpha = e^z \neq 0$ such that $(X\alpha) = \rho\eta(X)$, where $\rho = 2e^{2z}$. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$\begin{aligned} &R(E_2, E_3)E_3 = -e^{2z} E_2, \quad R(E_1, E_3)E_3 = -e^{2z} E_1, \\ &R(E_1, E_2)E_2 = -(1 + a^2)e^{2z} E_1, \quad R(E_2, E_3)E_2 = e^{2z} E_3, \\ &R(E_1, E_3)E_1 = e^{2z} E_3, \quad R(E_1, E_2)E_1 = (1 + a^2)e^{2z} E_2. \end{aligned}$$

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor S as follows:

$$S(E_1, E_1) = -(2 + a^2)e^{2z}, \quad S(E_2, E_2) = -(2 + a^2)e^{2z}, \quad S(E_3, E_3) = -2e^{2z}.$$

Since $\{E_1, E_2, E_3\}$ forms a basis, any vector field $X, Y \in \chi(M)$ can be written as

$$X = a_1 E_1 + b_1 E_2 + c_1 E_3 \quad \text{and} \quad Y = a_2 E_1 + b_2 E_2 + c_2 E_3,$$

where $a_i, b_i, c_i \in R^+$(the set of all positive real numbers), $i = 1, 2$. This implies that

$$\phi X = -a_1 E_1 - b_1 E_2 \quad \text{and} \quad \phi Y = -a_2 E_1 - b_2 E_2.$$

Hence

$$S(\phi X, \phi Y) = -(a_1 a_2 + b_1 b_2)(2 + a^2)e^{2z} \neq 0.$$

By the virtue of the above we have the following:

$$\begin{aligned}(\nabla_{E_1}S)(\phi X,\phi Y) &= -a_1b_2(2a+a^3)e^{2z}\\ (\nabla_{E_2}S)(\phi X,\phi Y) &= (a_1b_2+a_2b_1)(2+a^2)e^{2z}\\ (\nabla_{E_3}S)(\phi X,\phi Y) &= 0.\end{aligned}$$

Let us now consider the 1-forms

$$\begin{aligned}A(E_1) &= \frac{(a_1b_2)}{(a_1a_2+b_1b_2)}a,\\ A(E_2) &= -\frac{(a_1b_2+a_2b_1)}{(a_1a_2+b_1b_2)},\\ A(E_3) &= 0,\end{aligned}$$

at any point $p \in M$. In our M^3, (15) reduces with these 1-forms to the following equation:

$$(\nabla_{E_i}S)(\phi X,\phi Y) = A(E_i)S(\phi X,\phi Y), \quad \text{i} = 1,2,3.$$

This implies that the manifold under consideration is an $(LCS)_3$-manifold with η-recurrent but not η-parallel Ricci tensor. This leads to the following:

Theorem 6. *There exists a $(LCS)_3$-manifold (M^3,g) with η-recurrent but not η-parallel Ricci tensor.*

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References

[1] J.K.Beem and P.E.Ehrlich, Global Lorentzian Geometry, Marcel Dekker, New York, 1981.

[2] M.Kon, Invariant submanifolds in Sasakian manifolds, Mathematische Annalen, 219 (1976), 277-290.

[3] K.Matsumoto, On Lorentzian paracontact manifolds, Bull. Yamagata Univ. Natur. Sci. 12(2) (1989), 151-156.

[4] B. O' Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.

[5] D.G. Prakasha, On generalized recurrent $(LCS)_n$-manifolds, J. Tensor Soc., 4, (2010), 33-40.

[6] A.A. Shaikh, *On Lorentzian almost paracontact manifolds with a structure of the concircular type,* Kyungpook Math.J., 43, 2, (2003), 305-314.

[7] A.A. Shaikh and K.K. Baishya, Some results on LP-Sasakian manifolds, Bull. Math. Soc. Sci. Math. Rommanie Tome 49, 97, 2, (2006), 197-205.

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