Acta Universitatis Apulensis No. 24/2010
ISSN: 1582-5329 pp. 255-270

ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

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Abstract. In this paper, we prove the endpoint estimates for the multilinear commutator of Marcinkiewicz operator.

2000 *Mathematics Subject Classification*: 42B20, 42B25.

1. Introduction

As the development of singular integral operators, their commutators have been well studied. Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [3]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [2][5], the boundedness properties of the commutators for the extreme values of p are obtained. And note that $[b, T]$ is not bounded for the end point boundedness (that is $p = 1$ and $p = \infty$). In this paper, we will introduce the multilinear commutator of Marcinkiewicz operator and prove the boundedness properties of the operator for the extreme cases.

First let us introduce some notations (see [1][4][7][8]). In this paper, $Q = Q(x, r)$ will denote a cube of R^n with sides parallel to the axes and center at x and edge is r. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f(Q) = \int_Q f(x)dx$, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

f is said to belong to $BMO(R^n)$ if $f^\# \in L^\infty(R^n)$ and define $||f||_{BMO} = ||f^\#||_{L^\infty}$. We have $|f_{2Q} - f_Q| \le C||f||_{BMO}$ and $||f - f_{2^kQ}||_{BMO} \le Ck||f||_{BMO}$ for $k \ge 1$ (see [4][8]). We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

It is well-known that

$$||f||_{CMO} \approx \sup_{r>1} \inf_{c\in C} |Q(0,r)|^{-1} \int_Q |f(x) - c|dx.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

Definition 1. *A function a is called a $H^1(R^n)-$atom, if there exists a cube Q, such that*

1) supp $a \subset Q = Q(x_0, r)$,

2) $||a||_{L^\infty} \le |Q|^{-1}$,

3) $\int_{R^n} a(x)dx = 0$.

It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [4][8]).

The A_p weight is defined by (see [4])

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

Definition 2. *Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that*

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$.

Definition 3. *Let b_j $(j = 1, \cdots, m)$ be the fixed locally integrable functions on R^n, $0 < \delta < n$ and $0 < \gamma \le 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \ge 2)$*

equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1}, i.e.

$$|\Omega(x') - \Omega(y')| \le M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$;

The Marcinkiewicz multilinear commutator is defined by

$$\mu_{s,\delta}^{\vec{b}}(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] f(z)dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)dz.$$

We also define that

$$\mu_{s,\delta}(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [5],[6] and [10]).

Remark. Fixed $\lambda > \max(1, 2n/(n+2-2\delta))$. Another Marcinkiewicz multilinear operators is defined by

$$\mu_{\lambda,\delta}^{\vec{b}}(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy.$$

We also define

$$\mu_\lambda(f)(x)=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|F_t(f)(y)|^2\frac{dydt}{t^{n+3}}\right)^{1/2},$$

which is another Marcinkiewicz operators.

Let H be the Hilbert space $H=\left\{h:||h||=\left(\int\int_{R_+^{n+1}}|h(y,t)|^2dydt/t^{n+3}\right)^{1/2}<\infty\right\}$. Then for each fixed $x\in R^n$, $F_t^{\tilde{b}}(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_{s,\delta}^{\vec{b}}(f)(x)=\left|\left|\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)\right|\right|,\ \mu_{s,\delta}(f)(x)=\left|\left|\chi_{\Gamma(x)}F_t(f)(y)\right|\right|.$$

Note that when $b_1=\cdots=b_m$, $\mu_{s,\delta}^{\tilde{b}}$ and $\mu_\lambda^{\tilde{b}}$ are just the m order commutators. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-8] and [10]).

Given a positive integer m and $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\cdots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\cdots,b_m)$ and $\sigma=\{\sigma(1),\cdots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO}=||b_{\sigma(1)}||_{BMO}\cdots||b_{\sigma(j)}||_{BMO}$.

2. Theorems and Proofs

We begin with some preliminary lemmas.

Lemma 1.(see [8]) *Let $1<r<\infty$, $b_j\in BMO(R^n)$ for $j=1,\cdots,k$ and $k\in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q\prod_{j=1}^k|b_j(y)-(b_j)_Q|dy\le C\prod_{j=1}^k||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q\prod_{j=1}^k|b_j(y)-(b_j)_Q|^rdy\right)^{1/r}\le C\prod_{j=1}^k||b_j||_{BMO}.$$

Lemma 2.(see [6]) *Let $w\in A_1$, $0<\delta<n$, $1<p<n/\delta$ and $1/q=1/p-\delta/n$. Then $\mu_{s,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.*

Lemma 3. *Let $w\in A_1$. Then $w\chi_{Q'}\in A_p$ for any cube Q', where $\chi_{Q'}$ denotes the characteristic function of the cube Q'.*

Proof. By definition, we have

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

thus

$$\begin{aligned} & \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\chi_{Q'}(x)dx\right)\left(\frac{1}{|Q|}\int_Q (w(x)\chi_{Q'}(x))^{-1/(p-1)}dx\right)^{p-1} \\ \leq & \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty, \end{aligned}$$

that is $w\chi_{Q'} \in A_p$.

Theorem 1. *Let $0 < \delta < n$ and $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \leq j \leq m$. Then $\mu_{s,\delta}^{\vec{b}}$ is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$.*

Proof. It is only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|}\int_Q |\mu_{s,\delta}^{\vec{b}}(f)(x) - C_Q|dx \leq C||f||_{L^{n/\delta}}.$$

Fix a cube Q, $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{(R^n\setminus 2Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1}\int_Q b_1(y)dy$, we have

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_Q)F_t(f)(x) - F_t((b_1 - (b_1)_Q)f_1)(x) - F_t((b_1 - (b_1)_Q)f_2)(x),$$

so

$$\begin{aligned} & |\mu_{s,\delta}^{b_1}(f)(x) - \mu_{s,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)| \\ = & \left| ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y)|| - ||\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)|| \right| \\ \leq & ||\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y) - \chi_{\Gamma(x_0)}F_t(((b_1)_{2Q} - b_1)f_2)(y)|| \\ \leq & ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_{2Q})F_t(f)(y)|| + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_1)(y)|| \\ & + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_{2Q})f_2)(y) - \chi_{\Gamma(x_0)}F_t((b_1 - (b_1)_{2Q})f_2)(y)|| \\ = & A(x) + B(x) + C(x). \end{aligned}$$

For $A(x)$, set $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $1/q + 1/q' = 1$, by the Hölder's inequality and Lemma 2,3, we get

$$\begin{aligned}
\frac{1}{Q}\int_Q |A(x)|dx &\le \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_{R^n}|\mu_{s,\delta}(f)(x)|^q\chi_Q(x)dx\right)^{1/q}\\
&\le C||b_1||_{BMO}\frac{1}{|Q|^q}\left(\int_{R^n}|f(x)|^p\chi_Q(x)dx\right)^{1/p}\\
&\le C||b_1||_{BMO}\frac{1}{|Q|^q}||f||_{L^{n/\delta}}|Q|^{(1-(\delta p/n))/p}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $B(x)$, taking $1<r<n/\delta$ and $1/s=1/r-\delta/n$, by the Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |B(x)|dx &\le \left(\frac{1}{|Q|}\int_{R^n}(\mu_{s,\delta}((b_1(x)-(b_1)_Q)f_1)(x))^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}||(b_1(x)-(b_1)_Q)f\chi_{2Q}||_{L^r}\\
&\le C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_Q|^s dx\right)^{1/s}||f||_{L^{n/\delta}}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $C(x)$, by the Minkowski's inequality, we obtain

$$\begin{aligned}
C(x) &\le \left(\int\int_{R_+^{n+1}}||(\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)})F_t((b_1-(b_1)_{2Q})f_2(y)||)^2\frac{dydt}{t^{n+3}}\right)^{1/2}\\
&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\\
&\quad\times\left|\int\int_{|x-y|\le t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2-2\delta}t^{n+3}}-\int\int_{|x_0-y|\le t}\frac{\chi_{\Gamma(z)}(y,t)dydt}{|y-z|^{2n-2-2\delta}t^{n+3}}\right|^{1/2}dz\\
&\le \int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\\
&\quad\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\left|\frac{1}{|x+y-z|^{2n-2-2\delta}}-\frac{1}{|x_0+y-z|^{2n-2-2\delta}}\right|\frac{dydt}{t^{n+3}}\right)^{1/2}dz\\
&\le \int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\\
&\quad\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|}{|x+y-z|^{2n-1-2\delta}}t^{-n-3}dydt\right)^{1/2}dz,
\end{aligned}$$

note that $|x-z|\le 2t$, $|x+y-z|\ge |x-z|-t\ge |x-z|-3t$ when $|y|\le t$, $|x+y-z|\le t$, then, for $x\in Q$,

$$\begin{aligned}
C(x) &\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2}\\
&\quad\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{t^{-n}dydt}{|x+y-z|^{2n+2-2\delta}}\right)^{1/2}dz\\
&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2}\\
&\quad\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{t^{-n}dydt}{(|x-z|-3t)^{2n+2-2\delta}}\right)^{1/2}dz\\
&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{1/2}\left(\int_{|x-z|/2}^{\infty}\frac{dt}{(|x-z|-3t)^{2n+2-2\delta}}\right)^{1/2}dz\\
&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(z)-(b_1)_{2Q}|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}\\
&\quad\times\left(\int_{2^{k+1}Q}|f(z)|^{n/\delta}dz\right)^{\delta/n}\\
&\le C\sum_{k=1}^{\infty}k2^{-k/2}||b_1||_{BMO}||f||_{L^{n/\delta}}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

This completes the proof of the case $m=1$.

When $m>1$, set $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy$, $1\le$

$j \le m$, we have

$$\begin{aligned}
& F_t^{\vec{b}}(f)(x,y) = (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y) \\
& +(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{|y-z|\le t}(b(z)-b(x))_{\sigma^c}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}f(z)dz \\
= \; & (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y) \\
& +(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y) \\
& +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y),
\end{aligned}$$

thus,

$$\begin{aligned}
& |\mu_{s,\delta}^{\vec{b}}(f)(x)-\mu_{s,\delta}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)| \\
\le \; & ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m)f_2)(y)|| \\
\le \; & ||\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)|| \\
+ \; & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\tilde{b}(x)-(b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y)|| \\
+ \; & ||\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(y)|| \\
+ \; & ||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(y)|| \\
= \; & S_1(x)+S_2(x)+S_3(x)+S_4(x).
\end{aligned}$$

For $S_1(x)$, taking $1<p<n/\delta$, and $1/q=1/p-\delta/n$, by the Hölder's inequality and Lemma 1,2,3, we have

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q S_1(x)dx \le \left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|\mu_{s,\delta}(f)(x)|^q dx\right)^{1/q} \\
\le \; & C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q|f(x)|^p dx\right)^{1/p}|Q|^{(1-(\delta p/n))/p} \\
\le \; & C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $S_2(x)$, taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q S_2(x)dx \\
\le\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q dx\right)^{1/q} \\
\le\ & C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{1/q}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^p\chi_Q(x)dx\right)^{1/p} \\
\le\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^q dx\right)^{1/q}||f||_{L^{n/\delta}} \\
\le\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^{n/\delta}} \\
\le\ & C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $S_3(x)$, taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q S_3(x)dx \\
\le\ & \left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q dx\right)^{1/q} \\
\le\ & C|Q|^{-1/q}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)||_{L^p} \\
\le\ & C\left(\frac{1}{|2Q|}\int_{2Q}|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^q dx\right)^{1/q}||f||_{L^{n/\delta}} \\
\le\ & C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $S_4(x)$, similar to the proof of $C(x)$ in Case $m=1$, we obtain

$$\begin{aligned}
S_4(x) &\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x-x_0|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})\right||f(z)|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})\right|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}||f||_{L^{n/\delta}}\\
&\leq C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

This completes the total proof of Theorem 1.

Theorem 2. *Let $0<\delta<n$, $1<p<n/\delta$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\leq j\leq m$. Then $\mu_{s,\delta}^{\vec{b}}$ is bounded from $B_p^{\delta}(R^n)$ to $CMO(R^n)$.*

Proof. It suffices to prove that there exist constant C_Q, such that

$$\frac{1}{|Q|}\int_Q|\mu_{s,\delta}^{\vec{b}}(f)(x)-C_Q|dx\leq C||f||_{B_p^{\delta}}$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Fix a cube $Q=Q(0,d)$ with $d>1$. Set $f_1=f\chi_{2Q}$, $f_2=f\chi_{R^n\setminus 2Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$, $1\leq j\leq m$, we have

$$\begin{aligned}
&|\mu_{s,\delta}^{\vec{b}}(f)(x)-\mu_{s,\delta}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)|\\
\leq\ &||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m)f_2)(y)||\\
\leq\ &||\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)||\\
+\ &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\chi_{\Gamma(x)}(\tilde{b}(x)-(b_m)_{2Q})_{\sigma}F_t^{\vec{b}_{\sigma^c}}(f)(x,y)||\\
+\ &||\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(y)||\\
+\ &||\chi_{\Gamma(x)}F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})f_2)(y)||\\
=\ &H_1(x)+H_2(x)+H_3(x)+H_4(x).
\end{aligned}$$

Taking $1<p<n/\delta$, $1/s=1/r-\delta/n$, by the Hölder's inequality and Lemma 1,2,3,

we have

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q H_1(x)dx \\
\leq\ & \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}(f)(x)|^q dx\right)^{1/q} \\
\leq\ & C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p} \\
\leq\ & C||\vec{b}||_{BMO}d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p} \\
\leq\ & C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

For $H_2(x)$, taking $1<p<n/\delta$, $1/s=1/r-\delta/n$, and $1/s'+1/s=1$, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q H_2(x)dx \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^s dx\right)^{1/s} \\
\leq\ & C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{-1/s}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^r\chi_Q(x)dx\right)^{1/r} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{pr/(p-r)}dx\right)^{(p-r)/pr}|Q|^{(\delta/n-1/p)}||f\chi_Q||_{L^p} \\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p} \\
\leq\ & C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

For $H_3(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$ and $1/s' + 1/s = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_3(x)dx \\
\le\ & \left(\frac{1}{|Q|}\int_Q |\mu_{s,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s} \\
\le\ & C|Q|^{-1/s}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_{2Q}||_{L^r} \\
\le\ & C\left(\frac{1}{|2Q|}\int_{2Q}|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^{pr/(p-r)}dx\right)^{(p-r)/pr} d^{-n(1/p-\delta/n)}||f\chi_{2Q}||_{L^p} \\
\le\ & C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $H_4(x)$, we have

$$\begin{aligned}
S_4(x) \le\ & C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x-x_0|^{1/2}|x_0-z|^{-(n+1/2-\delta)}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)|dz \\
\le\ & C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right||f(z)|dz \\
\le\ & C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right|^{p/(p-1)}dz\right)^{(p-1)/p} \\
& \times\left(\int_{2^{k+1}Q}|f(z)|^p dz\right)^{1/p} \\
\le\ & C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right|^{p/(p-1)}dz\right)^{(p-1)/p} \\
& \times|2^{k+1}Q|^{-(1/p-\delta/n)}||f\chi_{2^{k+1}Q}||_{L^p} \\
\le\ & C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

This completes the total proof of Theorem 2.

Theorem 3. *Let* $0 < \delta < n$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in BMO(R^n)$ *for* $1 \le j \le m$. *If for any* $H^1(R^n)-$*atom* a *supported on certain cube* Q *and* $u \in Q$, *there is*

$$\sum_{j=1}^m\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left|\left|\int_Q(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\right|\right|\right)^{n/(n-\delta)}dx \le C,$$

then $\mu^{\vec{b}}_{s,\delta}$ *is bounded from* $H^1(R^n)$ *to* $L^{n/(n-\delta)}(R^n)$.

Proof. Let a be an atom supported in some cube Q. We write

$$\int_{R^n}|\mu^{\vec{b}}_{s,\delta}(a)(x)|^{n/(n-\delta)}dx=\int_{2Q}|\mu^{\vec{b}}_{s,\delta}(a)(x)|^{n/(n-\delta)}dx+\int_{(2Q)^c}|\mu^{\vec{b}}_{s,\delta}(a)(x)|^{n/(n-\delta)}dx=I+II.$$

For I, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we have

$$I\le||\mu^{\vec{b}}_{s,\delta}(a)||^{n/(n-\delta)}_{L^q}|2Q|^{1-n/((n-\delta)q)}\le C||a||^{n/(n-\delta)}_{L^p}|Q|^{1-n/((n-\delta)q)}\le C.$$

For II, we first calculate $F^{\vec{b}}_t(a)(x)$, we have

$$\begin{aligned}|F^{\vec{b}}_t(a)(x)| &\le \left|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}a(z)dz\right|\\ &+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|y-z|\le t}\left(\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}-\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\right)(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|\\ &+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|y-z|\le t}\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|\\ &= \nu_1+\nu_2+\nu_3,\end{aligned}$$

$$\begin{aligned}\mu^{\vec{b}}_\delta(a)(x) &= ||F^{\vec{b}}_t(a)(x)||\le\left(\int\int_{\Gamma(x)}|\nu_1|^2\frac{dydt}{t^{n+3}}\right)^{1/2}+\left(\int\int_{\Gamma(x)}|\nu_2|^2\frac{dydt}{t^{n+3}}\right)^{1/2}\\ &+\left(\int\int_{\Gamma(x)}|\nu_3|^2\frac{dydt}{t^{n+3}}\right)^{1/2}=A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, we have

$$\begin{aligned}
A(x) &\leq C\left(\int\int_{R^n}\left|\frac{|\chi_{\Gamma(x)}-\chi_{\Gamma(y)}|}{|y-z|^{n-1-\delta}}|a(z)|dz\right|^2\frac{dydt}{t^{n+3}}\right)^{1/2}\prod_{j=1}^m|b_j(x)-(b_j)_Q| \\
&\leq C\int_{R^n}\left|\int_{|x-z|\leq t}\frac{1}{|x-y|^{2n-2-2\delta}}\frac{dt}{t^3}-\int_{|u-z|\leq t}\frac{1}{|x-u|^{2n-2-2\delta}}\frac{dt}{t^3}\right|^{1/2}|a(z)|dz \\
&\quad\times\prod_{j=1}^m|b_j(x)-(b_j)_Q| \\
&\leq C\int_{R^n}\left(\int_{|x|\leq t,|x+y-u|\leq t}\left|\frac{1}{|x+y-u|^{2n-2-2\delta}}-\frac{1}{|x|^{2n-2-2\delta}}\right|\frac{dt}{t^3}\right)^{1/2}|a(z)|dz \\
&\quad\times\prod_{j=1}^m|b_j(x)-(b_j)_Q| \\
&\leq C\int_{R^n}\left(\int_{|x|\leq t,|x+y-u|\leq t}\frac{|y-u|}{|x+y-u|^{2n-1-2\delta}}\frac{dt}{t^3}\right)^{1/2}|a(z)|dz\prod_{j=1}^m|b_j(x)-(b_j)_Q| \\
&\leq C\int_{R^n}\frac{|y-u|^{1/2}}{|x+y-u|^{n+1/2-\delta}}|a(z)|dz\prod_{j=1}^m|b_j(x)-(b_j)_Q| \\
&\leq C|Q|^{1/2n}|x-u|^{-(n+1/2-\delta)}\prod_{j=1}^m|b_j(x)-(b_j)_Q|,
\end{aligned}$$

thus

$$\begin{aligned}
&\left(\int_{(2Q)^c}(A(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n} \\
&\leq C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left(\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n} \\
&\leq C||\vec{b}||_{BMO}.
\end{aligned}$$

For $B(x)$, we have

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\left(\int\int_{\Gamma(x)}\left|\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}}a(z)(\vec{b}(z)-\vec{b}_Q)_{\sigma}dz\right|\frac{dydt}{t^{n+3}}\right)^{1/2}$$
$$\leq \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\left(\int_{R^n}\left(\int_{R^n}\chi_{\Gamma(y)}\left(\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x-u)|}{|x-u|^{n-1-\delta}}\right)\right.\right.$$
$$\left.\left.\times\ a(z)(\vec{b}(z)-\vec{b}_Q)_{\sigma}dz\right)^2\frac{dt}{t^3}\right)^{1/2},$$

similarly, we get

$$|B(x)|\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\frac{|Q|^{1/2n}}{|x-u|^{n+1/2-\delta}}||\vec{b}_{\sigma}||_{BMO},$$

thus

$$\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$
$$\leq C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right|^{n/(n-\delta)}dx\right)^{(n-\delta)/n}||\vec{b}_{\sigma}||_{BMO}$$
$$\leq C||\vec{b}||_{BMO}.$$

So, if

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left|\left|\int_Q(\vec{b}(z)-\vec{b}_Q)_{\sigma}a(z)dz\frac{\Omega(y-u)}{|y-u|^{n-1-\delta}}\right|\right|\right)^{n/(n-\delta)}dx\leq C,$$

then

$$\int_{R^n}|\mu_{s,\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx\leq C.$$

This completes the proof of the Theorem 3.

Remark. Theorem 1, 2 and 3 also hold for $\mu_{\lambda}^{\tilde{b}}$, we omit the details.

References

[1] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J., 31(1982), 7–16.

[2] W. G. Chen and G. E. Hu, *Weak type (H^1, L^1) estimate for a multilinear singular integral operator*, Adv. in Math.(China), 30(2001), 63–69.

[3] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., 103(1976), 611–635.

[4] J. Garcia-Cuerva, J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math., 116, Amsterdam, 1985.

[5] E. Harboure, C. Segovia, J. L. Torrea, *Boundedness of commutators of fractional and singular integrals for the extreme values of p*, Illinois J. Math., 41(1997), 676–700.

[6] L. Z. Liu and B. S. Wu, *Weighted boundedness for commutator of Marcinkiewicz integral on some Hardy spaces*, Southeast Asian Bulletin of Math., 28(2004), 643–650.

[7] C. Pérez, R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., 65(2002), 672–692.

[8] E. M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.

[9] W. X. Yu, L. Z. Liu, *Weighted endpoint inequalities for multilinear Marcinkiewicz integral operator*, Bull. Korean Math. Soc., 45(2008), 1–10

[10] A. Torchinsky, S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math., 60/61(1990), 235–243.

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