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ON SOME DOUBLE $\overline{\lambda}(\Delta, F)$ −STATISTICAL CONVERGENCE OF FUZZY NUMBERS

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Abstract. In this paper, we introduce the new concepts of double $\Delta-$statistical convergence, strongly double $\overline{\lambda}(\Delta, F) -$ *summable* sequences and double $\overline{\lambda}(\Delta, F) -$ *statistical* convergence of sequences of fuzzy numbers. We give some inclusion relations related to these concepts..

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1.Introduction

Throughout the paper, a double sequence is denoted by $X = (X_{k,l})$ of fuzzy numbers and denote $w^2(F)$ denote all sequences of fuzzy numbers. Nanda [4] studied single sequence of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space.In [2] , Savaş studied the concept double convergent sequences of fuzzy numbers. Savaş [1] studied the classes of difference sequences of fuzzy numbers $c(\Delta, F)$ and $l_\infty(\Delta, F)$. Later in [3] Savaş studied the concepts of strongly double $\left[V, \overline{\lambda}\right]$ −summable and double $S_{\overline{\lambda}}-$convergent sequences for double sequences of fuzzy numbers.

In this paper, we continue to study the concepts of strongly double $\overline{\lambda}(\Delta, F)$ −summable and $S^2_{\overline{\lambda}}(\Delta)$ −convergence for double sequences of fuzzy numbers.

2.preliminaries

Before continuing with the discussion, we pause to establish some notations. Let D denote the set of all closed bounded intervals $A = \left[\underline{A}, \overline{A}\right]$ on the real line $\mathbb{R}$, where $\underline{A}$ and $\overline{A}$ denote the end points of A. For $A, B \in D$, we define

$$A \leq B \text{ iff } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B},$$

$$\rho(A, B) = \max\left(\left|\underline{A} - \underline{B}\right|, \left|\overline{A} - \overline{B}\right|\right).$$

It is not hard to see that ρ defines a metric on D and $\rho(A, B)$ is called the distance between the intervals A and B. Also, it is easy to see that $\leq$ defined above is a partial relation in D.

A fuzzy number is a fuzzy subset of real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi-continuous and have compact support. In other words if $X \in L(\mathbb{R})$ then for any $\alpha \in [0,1]$, X^{α} is compact set in $\mathbb{R}$, where

$$X^{\alpha} = \begin{matrix} t: \ X(t) \geq \alpha \text{ if } \alpha \in (0,1] \\ t: X(t) > 0 \quad \text{if} \qquad \alpha = 0 \end{matrix}.$$

Define a map $d: L(\mathbb{R})\, x L(\mathbb{R}) \to \mathbb{R}$ by the rule $d(X,Y) = \sup_{\alpha\in[0,1]} \rho(X^{\alpha}, Y^{\alpha})$. It is straighforward to see that d is a metric in $L(\mathbb{R})$. For $X, Y \in L(\mathbb{R})$, define

$$X \leq Y \text{ iff } X^{\alpha} \leq Y^{\alpha} \text{ for any } \alpha \in [0,1].$$

A metric d on $L(\mathbb{R})$ is said to be translation invariant metric if

$$d(X+Z, Y+Z) = d(X,Y) \text{ for } X, Y, Z \in L(\mathbb{R}).$$

Now we give some new definitions.

Definition 2.1. *A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be double $\Delta -$ convergent in the Pringsheim's sense or $P_{\Delta} -$ convergent to a fuzzy number X_o if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$d(\Delta X_{k,l}, X_o) < \varepsilon \textit{ for } k, l > N$$

where $\Delta X_{k,l} = X_{k,l+1} - X_{k,l} - X_{k+1,l} + X_{k+1,l+1}$ and we denote $P - \lim \Delta X = X_o$. The number X_o is called the Pringsheim limit of ΔX. More exactly, we say that a double sequence $(\Delta X_{k,l})$ converges to a finite fuzzy number X_o if ΔX tend to X_o as both k and l tends to infinity independently of one another. Let $c^2(\Delta, F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 2.2. *A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be double $\Delta -$ bounded if there exists a positive number K such that if the set*

$$\{\Delta X_{k,l} : \ k, l \in \mathbb{N}\}$$

We denote the set of all double $\Delta-$bounded sequences of fuzzy numbers by $l^2_{\infty}(\Delta, F)$.

Definition 2.3. *A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be double $\Delta -$ statistically convergent to X_o provided that for each $\varepsilon > 0$,*

$$P - \lim_{m,n} \frac{1}{mn} \left|\{(k,l) : \ k \leq m, l \leq n; d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right| = 0.$$

In this case we write $S^2 - \lim \Delta X = X_o$ *or* $\Delta X_{k,l} \to X_o\left(S^2(\Delta, F)\right)$ *and we denote the set of all double* $\Delta -$ *statistically convergent sequences of fuzzy numbers by* $S^2(\Delta, F)$.

Definition 2.4. *Let* $\beta = (\beta_m)$ *and* $\mu = (\mu_n)$ *be two nondecreasing sequences of positive real numbers such that each tend to infinity and* $\beta_{m+1} \leq \beta_m + 1, \beta_1 = 1$ *and* $\mu_{n+1} \leq \mu_n + 1, \mu_1 = 1$. *A double sequence* $X = (X_{k,l})$ *of fuzzy numbers is said to be strongly double* $\overline{\lambda}(\Delta, F) -$ *summable if there is a fuzzy number* X_o *such that*

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} d(\Delta X_{k,l}, X_o) = 0$$

where $\lambda_{m,n} = \beta_m.\mu_n$ *and* $I_{m,n} = \{(k,l) : \ m - \beta_m + 1 \leq k \leq m, \ n - \mu_n + 1 \leq l \leq n\}$. *We denote the set of strongly double* $\overline{\lambda}(\Delta, F) -$ *summable sequences by* $\left[V_{\overline{\lambda}}\right](\Delta, F)$. *If* $\lambda_{m,n} = mn$ *for all* $m, n \in \mathbb{N}$, *then the class of strongly double* $\overline{\lambda}(\Delta, F) -$ *summable sequences reduce to* $[C, 1, 1](\Delta, F)$, *the class of strongly double Cesaro summable sequences of fuzzy numbers defined as follows:*

$$P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} d(\Delta X_{k,l}, X_o) = 0.$$

Definition 2.5. *A double sequence* $X = (X_{k,l})$ *of fuzzy numbers is said to be double* $\overline{\lambda}(\Delta, F) -$ *statistically convergent or* $S^2_{\overline{\lambda}}(\Delta, F) -$ *convergent to a fuzzy number* X_o *if for every* $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right| = 0.$$

In this case we write $S^2_{\overline{\lambda}} - \lim \Delta X = X_o$ or $\Delta X_{k,l} \to X_o\left(S^2_{\overline{\lambda}}(\Delta, F)\right)$ and we denote the set of all double $\overline{\lambda}(\Delta, F) -$ *statistically convergent* sequences of fuzzy numbers by $S^2_{\overline{\lambda}}(\Delta, F)$ .If $\lambda_{m,n} = mn$ for all $m, n \in \mathbb{N}$, we write $S^2 - \lim \Delta X = X_o$ or $\Delta X_{k,l} \to X_o\left(S^2(\Delta, F)\right)$ and the set $S^2_{\overline{\lambda}}(\Delta, F)$ reduces to $S^2(\Delta, F)$.

We need the following proposition in future.

Proposition 2.1. *If* d *is a translation invariant metric on* $L(\mathbb{R})$ *,then*

$$d\left(\Delta X + \Delta Y, \overline{0}\right) \leq d\left(\Delta X, \overline{0}\right) + d\left(\Delta Y, \overline{0}\right).$$

Proof. The proof is clear so we omitted it.

3. Main results

Theorem 3.1. *A double sequence $X = (X_{k,l})$ of fuzzy numbers is strongly double $\overline{\lambda}(\Delta, F)-$summable to the fuzzy number X_o, then it is double $\overline{\lambda}(\Delta, F)-$statistically convergent to X_o.*

Proof. Given $\varepsilon > 0$.Then

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} d(\Delta X_{k,l}, X_o) \geq \frac{1}{\lambda_{m,n}} \sum_{\substack{(k,l)\in I_{m,n} \\ d\left(\Delta X_{k,l}, X_o\right)\geq\varepsilon}} d(\Delta X_{k,l}, X_o)$$

$$\geq \frac{\varepsilon}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right|.$$

The result follows from this inequality.

Theorem 3.2. *If a double $\Delta-$bounded double sequence of fuzzy numbers $X = (X_{k,l})$ is double $\overline{\lambda}(\Delta, F)-$statistically convergent to the fuzzy number X_o, then it is strongly double $\overline{\lambda}(\Delta, F)-$summable to X_o.*

Proof. Suppose that $X = (X_{k,l})$ is double $\Delta-$*bounded* and double $\overline{\lambda}(\Delta, F)-$*statistically convergent* to X_o.Since $X = (X_{k,l})$ is double $\Delta-$*bounded*, we may write $d(\Delta X_{k,l}, X_o) \leq K$ for all $k, l \in \mathbb{N}$.Also for given $\varepsilon > 0$, we obtain

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} d(\Delta X_{k,l}, X_o) = \frac{1}{\lambda_{m,n}} \sum_{\substack{(k,l)\in I_{m,n} \\ d\left(\Delta X_{k,l}, X_o\right)\geq\varepsilon}} d(\Delta X_{k,l}, X_o)$$

$$+\frac{1}{\lambda_{m,n}} \sum_{\substack{(k,l)\in I_{m,n} \\ d\left(\Delta X_{k,l}, X_o\right)<\varepsilon}} d(\Delta X_{k,l}, X_o)$$

$$\leq \frac{K}{\lambda_{m,n}} \left|\{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right| + \varepsilon$$

which implies that $X = (X_{k,l})$ is strongly double $\overline{\lambda}(\Delta, F)-$*summable* to X_o.

Theorem 3.3. *If a double sequence $X = (X_{k,l})$ of fuzzy numbers is double $\overline{\lambda}(\Delta, F)-$statistically convergent to the fuzzy number X_o, then it is double $\Delta-$statistically convergent to X_o if*

$$P - \liminf_{m,n} \frac{1}{\lambda_{m,n}} > 0.$$

Proof. For given $\varepsilon > 0$, we have

$$\{(k,l) : \; k \leq m, l \leq n; d(\Delta X_{k,l}, X_o) \geq \varepsilon\} \supset \{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}.$$

Therefore

$$\frac{1}{mn}\left|\{(k,l) : \; k \leq m, l \leq n; d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right| \geq \frac{1}{mn}\left|\{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right|$$

$$= \frac{\lambda_{m,n}}{mn}\frac{1}{\lambda_{m,n}}\left|\{(k,l) \in I_{m,n} : d(\Delta X_{k,l}, X_o) \geq \varepsilon\}\right|.$$

Taking the limit as $m, n \to \infty$ in the Pringsheim's sense and using hypothesis, we get $X = (X_{k,l})$ is double $\Delta -$ *statistically convergent* to X_o.

Theorem 3.4. $w^2_{\overline{\lambda},\infty}(\Delta, F) = l^2_\infty(\Delta, F)$, *where*

$$w^2_{\overline{\lambda},\infty}(\Delta, F) = \left\{X = (X_{k,l}) \in w^2(F) : \; \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} d(\Delta X_{k,l}, X_o) < \infty\right\}.$$

Proof. Let $X = (X_{k,l}) \in w^2_{\overline{\lambda},\infty}(\Delta, F)$. Then there exists a constant $K > 0$ such that

$$\frac{1}{\lambda_{1,1}} d(\Delta X_{k,l}, X_o) \leq \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} d(\Delta X_{k,l}, X_o) \leq K$$

and so we have $X = (X_{k,l}) \in l^2_\infty(\Delta, F)$. Conversely, let $X = (X_{k,l}) \in l^2_\infty(\Delta, F)$.Then there exists a constant $H > 0$ such that $d(\Delta X_{k,l}, X_o) \leq H$ for all $k, l \in \mathbb{N}$ and so

$$\frac{1}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} d(\Delta X_{k,l}, X_o) \leq \frac{H}{\lambda_{m,n}}\sum_{(k,l)\in I_{m,n}} 1 = H.$$

Thus $X = (X_{k,l}) \in w^2_{\overline{\lambda},\infty}(\Delta, F)$.This completes the proof.

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