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# A CAUCHY PROBLEM FOR HELMHOLTZ EQUATION: REGULARIZATION AND ERROR ESTIMATES

Nguyen Huy Tuan and Pham Hoang Quan

Abstract. In this paper, the Cauchy problem for the Helmholtz equation is investigated. It is known that such problem is severely ill-posed. We propose a new regularization method to solve it based on the solution given by the method of separation of variables. Error estimation and convergence analysis have been given. Finally, we present numerical results for several examples and show the effectiveness of the proposed method.

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## 1. Introduction

The Helmholtz equation arises in many physical applications (see, e.g., [1, 2, 4, 9, 12] and the references therein). The direct problem for Helmholtz equation, i.e., Dirichlet, Neumann or mixed boundary value problems have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary of the concerning domain, which will lead to some inverse problems. The Cauchy problem for the Helmholtz equation is an inverse problem and is severely ill-posed [3]. That means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution. In recent years, the Cauchy problems associated with the Helmholtz equation have been studied by using different numerical methods, such as the Landweber method with boundary element method (BEM) [8], the conjugate gradient method [7], the method of fundamental solutions (MFS) [14] and so on. However, most of numerical methods are short of stability analysis and error estimate.

Although there exists a vast literature on the Cauchy problem for the Helmholtz equation, to the authors knowledge, there are much fewer papers devoted to the error estimates. Recently, in [6], the authors give a quasi-reversibility method for

solving a Cauchy problem of modified Helmhotlz equation where they consider a homogenous Neumann boundary condition, the results are less encouraging. The main aim of this paper is to present a new regularization method, and investigate the error estimate between the regularization solution and the exact one.
The paper is organized as follows. In Section 2, the regularization method is introduced; in Section 3, some stability estimates are proved under some priori conditions; in Section 4, some numerical results are reported.

## 2. Mathematical Problem And Regularization.

We consider the following Cauchy problem for the Helmholtz equation with nonhomogeneous Neumman boundary condition

$$\begin{cases} \Delta u + k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = f(x), (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{1}$$

where $g(x), f(x)$ is a given vector in $L^2(0,\pi)$ and $0 < k < 1$ is the wave number.
By the method of separation of variables, the solution of problem (1) is as follows

$$u(x,y) = \sum_{n=1}^{\infty} \left[ \left( \frac{e^{\sqrt{n^2-k^2}y} + e^{-\sqrt{n^2-k^2}y}}{2} \right) g_n + \left( \frac{e^{\sqrt{n^2-k^2}y} - e^{-\sqrt{n^2-k^2}y}}{2\sqrt{n^2-k^2}} \right) f_n \right] \sin nx \tag{2}$$

where

$$f(x) = \sum_{n=1}^{\infty} f_n \sin nx, g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

Physically, $g$ can only be measured, there will be measurement errors, and we would actually have as data some function $g^\epsilon \in L^2(0,\pi)$, for which

$$\|g^\epsilon - g\| \le \epsilon$$

where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the $L^2$-norm. Denote $\beta$ is the regularization parameter depend on $\epsilon$.
The case $f = 0$, the problem (1) becomes

$$\begin{cases} \Delta u + k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{3}$$

Very recently, in [6], H.H.Quin and T.Wei considered (2) by the quasi-reversibility method. They established the following problem for a fourth-order equation

$$\begin{cases} \Delta u^\epsilon + k^2 u^\epsilon - \beta^2 u^\beta_{xxyy} = 0, (x,y) \in (0,\pi)\times(0,1) \\ u^\epsilon(0,y) = u^\epsilon(\pi,y) = 0, y \in (0,1) \\ u^\epsilon_y(x,0) = 0, (x,y) \in (0,\pi)\times(0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{4}$$

Separation of variables leads to the solution of problem (4) as follows

$$u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left( \frac{e^{\sqrt{\frac{n^2-k^2}{1+\beta^2 n^2}}y} + e^{-\sqrt{\frac{n^2-k^2}{1+\beta^2 n^2}}y}}{2} \right) g_n \sin nx. \tag{5}$$

We note the reader that the term $e^{\sqrt{n^2-k^2}y}$ in (2) increase rather quickly when $n$ become large, so it is the unstability cause. To regularization the problem (2), we should replace it by the better terms. In (4), the authors replaced $e^{\sqrt{n^2-k^2}y}$ and $e^{-ny}$ by two better terms $e^{\sqrt{\frac{n^2-k^2}{1+\beta n^2}}y}$ and $e^{-\sqrt{\frac{n^2-k^2}{1+\beta n^2}}y}$ respectively. Notice the reader that in the case $k = 0$, the problem (4) is also considered in [10](See page 481).
To the author's knowledge, although the problem (4) is investigated by some recent paper but there are rarely results of regularize method for treating the problem (3) until now. In this paper, we shall replace $e^{\sqrt{n^2-k^2}y}$ by the different better terms $e^{\left(\sqrt{n^2-k^2}-\beta(n^2-k^2)\right)y}$ and modify the exact solution $u$ as follows

$$u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[ \left( \frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2} \right) g_n + \left( \frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2-k^2}y}}{2\sqrt{n^2-k^2}} \right) f_n \right] \sin nx. \tag{6}$$

where $A(\beta,n,k) = \sqrt{n^2-k^2} - \beta(n^2-k^2)$. Let $v^\epsilon$ be the regularized solution corresponding to the noisy data $g^\epsilon$

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[ \left( \frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2} \right) g^\epsilon_n + \left( \frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2-k^2}y}}{2\sqrt{n^2-k^2}} \right) f_n \right] \sin nx. \tag{7}$$

where $g^\epsilon_n = \frac{2}{\pi}\int_0^\pi g^\epsilon(x)\sin(nx)dx$.

## 3. Main Results

The following theorem proves that the solution of problem (7) depends continuously on the given Cauchy data $g^\epsilon$.

**Theorem 1***Let $v^\epsilon$ and $w^\epsilon$ be the solution of problem (7) and $v^\epsilon(x,0) = g^\epsilon(x), \;\; w^\epsilon(x,0) = h^\epsilon(x)$. Assume that $\|g^\epsilon - h^\epsilon\| \le \epsilon$, then we have*

$$\|v^\epsilon(.,y) - w^\epsilon(.,y)\| \le e^{\frac{1}{4\beta}}\epsilon. \tag{8}$$

**Proof.**

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2}\right)g_n^\epsilon + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2-k^2}y}}{2\sqrt{n^2-k^2}}\right)f_n\right]\sin nx. \tag{9}$$

and

$$w^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2}\right)h_n^\epsilon + \left(\frac{e^{A(\beta,n,k)y} - e^{-\sqrt{n^2-k^2}y}}{2\sqrt{n^2-k^2}}\right)f_n\right]\sin nx. \tag{10}$$

where

$$h_n^\epsilon = \frac{2}{\pi}\int_0^\pi h^\epsilon(x)\sin(nx)dx.$$

It follows from (9) and (10) that

$$v^\epsilon(x,y) - w^\epsilon(x,y) = \sum_{n=1}^{\infty}\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2}\right)(g_n^\epsilon - h_n^\epsilon)\sin nx.$$

Using the inequality $A(\beta,n,k) \le \frac{1}{4\beta}$ and $(a+b)^2 \le 2a^2 + 2b^2$, we have

$$\begin{aligned}
\|v^\epsilon(.,y) - w^\epsilon(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty}\left(\frac{e^{A(\beta,n,k)y} + e^{-\sqrt{n^2-k^2}y}}{2}\right)^2 |g_n^\epsilon - h_n^\epsilon|^2 \\
&\le \frac{\pi}{4}(e^{\frac{1}{2\epsilon}} + 1)\sum_{n=1}^{\infty}|g_n^\epsilon - h_n^\epsilon|^2 \\
&\le \frac{1}{2}(e^{\frac{1}{2\beta}} + 1)\|g - h\|^2 \\
&\le e^{\frac{1}{2\beta}}\epsilon^2. \qquad (11)
\end{aligned}$$

This completes the proof of Theorem 1.

**Theorem 2.***Let $\|u(.,1)\| \le A_1$. Let $f$ be a function such that*

$$\sum_{n=1}^{\infty}(n^2-k^2)^2 e^{2\sqrt{n^2-k^2}a} f_n^2 < A_2. \tag{12}$$

*Let* $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ *then one has*

$$\|u(x,y) - v^\epsilon(x,y)\| \le \sqrt{\epsilon} + \left(\ln \frac{1}{\epsilon}\right)^{-1} \sqrt{\frac{2}{(1-y)^4} A_1 + \frac{\pi}{4} A_2} \tag{13}$$

*for every* $y \in [0,1)$, *where* $v^\epsilon$ *is the unique solution of Problem* (7) *.*
**Remark 1.** 1. If $f = 0$ then the error (13) becomes

$$\|u(x,y) - v^\epsilon(x,y)\| \le \sqrt{\epsilon} + \left(\ln \frac{1}{\epsilon}\right)^{-1} \frac{\sqrt{2\pi A_1}}{(1-y)^2}. \tag{14}$$

This error order is the same in the Theorem 3.1, in [6].
*Proof.* We have

$$u(x,y) - u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[ \left( \frac{e^{\sqrt{n^2-k^2}y} - e^{A(\beta,n,k)y}}{2} \right) \varphi_n + \left( \frac{e^{\sqrt{n^2-k^2}y} - e^{A(\beta,n,k)y}}{2\sqrt{n^2-k^2}} \right) f_n \right] \sin nx.$$

We also have

$$< u(x,1), \sin nx >= \left( \frac{e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}}}{2} \right) \varphi_n + \left( \frac{e^{\sqrt{n^2-k^2}} - e^{-\sqrt{n^2-k^2}}}{2\sqrt{n^2-k^2}} \right) f_n.$$

It implies that

$$\varphi_n = \frac{2}{e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}}} < u(x,1), \sin nx > - \frac{e^{\sqrt{n^2-k^2}} - e^{-\sqrt{n^2-k^2}}}{(e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}})\sqrt{n^2-k^2}} f_n.$$

Hence,

$$\begin{aligned} < u(x,y) - u^\epsilon(x,y), \sin nx > &= \frac{e^{\sqrt{n^2-k^2}y} - e^{A(\beta,n,k)y}}{e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}}} < u(x,1), \sin nx > \\ &+ \frac{e^{\sqrt{n^2-k^2}y} - e^{A(\beta,n,k)y}}{(e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}})2\sqrt{n^2-k^2}} (e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}} - 1) f_n \end{aligned}$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we have

$$\begin{aligned} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 &\le 2 \left( \frac{e^{\sqrt{n^2-k^2}y}(1 - e^{-\beta(n^2-k^2)y})}{e^{\sqrt{n^2-k^2}} + e^{-\sqrt{n^2-k^2}}} \right)^2 | < u(x,1), \sin nx > |^2 + \\ &\quad \frac{1}{2} \left( \frac{e^{\sqrt{n^2-k^2}y} - e^{A(\beta,n,k)y}}{\sqrt{n^2-k^2}} \right)^2 f_n^2 \\ &\le e^{2(y-1)\sqrt{n^2-k^2}} \beta^2 (n^2-k^2)^2 y^2 | < u(x,1), \sin nx > |^2 + \\ &\quad + \frac{1}{2} \beta^2 y^2 (n^2-k^2) e^{2\sqrt{n^2-k^2}y} f_n^2. \end{aligned} \tag{15}$$

For $k, n > 0$, it is easy to prove that $\frac{(n^2-k^2)^2}{e^{2k\sqrt{n^2-k^2)}}} \le \frac{4}{k^4}$. Thus, for $y < 1$

$$e^{2(y-1)\sqrt{n^2-k^2)}}\beta^2(n^2-k^2)^2 \le \frac{4\beta^2}{(1-y)^4}.$$

This follows that

$$| < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \le \frac{4\beta^2}{(1-y)^4} < u(x,1), \sin nx > |^2 + \frac{1}{2}\beta^2(n^2-k^2)^2 e^{2\sqrt{n^2-k^2)}y} f_n^2.$$

Thus

$$\begin{aligned} \|u(x,y) - u^\epsilon(x,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \\ &\le \frac{2\pi\beta^2}{(1-y)^4}\sum_{n=1}^{\infty} < u(x,1), \sin nx > |^2 + \frac{\pi}{4}\beta^2\sum_{n=1}^{\infty}(n^2-k^2)^2 e^{2\sqrt{n^2-k^2)}y} f_n^2 \\ &\le \frac{2\beta^2}{(1-y)^4}\|u(.,1)\|^2 + \frac{\pi}{4}\beta^2 A_2. \end{aligned} \tag{16}$$

From Theorem 1, we get

$$\|v^\epsilon(.,y) - u^\epsilon(.,y)\| \le e^{\frac{1}{4\beta}}\epsilon. \tag{17}$$

From $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ and combining (11), (16) and (17), we obtain

$$\begin{aligned} \|u(x,y) - v^\epsilon(x,y)\| &\le \|u(x,y) - u^\epsilon(x,y)\| + \|u^\epsilon(x,y) - v^\epsilon(x,y)\| \\ &\le e^{\frac{1}{2\beta}}\epsilon + \beta\sqrt{\frac{2}{(1-y)^4}A_1 + \frac{\pi}{4}A_2} \\ &\le \epsilon^{\frac{3}{4}} + \left(\ln\frac{1}{\epsilon}\right)^{-1}\sqrt{\frac{2}{(1-y)^4}A_1 + \frac{\pi}{4}A_2}. \end{aligned}$$

**Theorem 3.***Let $f$ be as Theorem 2. Suppose that $u(.,1)$ satisfy the condition*

$$\sum_{n=1}^{\infty}(n^2-k^2)^2| < u(x,1), \sin nx > |^2 < A_3.$$

*Let $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ then one has*

$$\|u(x,y) - v^\epsilon(x,y)\| \le \left(\ln\frac{1}{\epsilon}\right)^{-1}\sqrt{\frac{\pi}{2}A_3 + \frac{\pi}{4}A_2} + \epsilon^{\frac{3}{4}}$$

*for every* $y \in [0,1]$*, where* $v^\epsilon$ *is the unique solution of Problem* (7) .
*Proof.*It follows from (15) that

$$\begin{aligned} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 &\leq \beta^2 (n^2-k^2)^2 y^2 | < u(x,1), \sin nx > |^2 + \\ &+ \frac{1}{2}\beta^2 y^2 (n^2-k^2) e^{2\sqrt{n^2-k^2}y} f_n^2. \end{aligned}$$

Then

$$\begin{aligned} \|u(x,y) - u^\epsilon(x,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \\ &\leq \frac{\pi}{2}\beta^2 (n^2-k^2)^2 y^2 | < u(x,1), \sin nx > |^2 \\ &\quad + \frac{\pi}{4}\beta^2 y^2 (n^2-k^2) e^{2\sqrt{n^2-k^2}y} f_n^2 \\ &\leq \frac{\pi}{2}\beta^2 A_3 + \frac{\pi}{4}\beta^2 A_2. \end{aligned}$$

Therefore we get

$$\|u(x,y) - u^\epsilon(x,y)\| \leq \beta\sqrt{\frac{\pi}{2}A_3 + \frac{\pi}{4}A_2}. \tag{18}$$

From $\beta = \left(\ln \frac{1}{\epsilon}\right)^{-1}$ and combining (11), (18), we obtain

$$\begin{aligned} \|u(x,y) - v^\epsilon(x,y)\| &\leq \|u(x,y) - u^\epsilon(x,y)\| + \|u^\epsilon(x,y) - v^\epsilon(x,y)\| \\ &\leq \beta\sqrt{\frac{\pi}{2}A_3 + \frac{\pi}{4}A_2} + e^{\frac{1}{4\beta}}\epsilon \\ &\leq \left(\ln\frac{1}{\epsilon}\right)^{-1}\sqrt{\frac{\pi}{2}A_3 + \frac{\pi}{4}A_2} + \epsilon^{\frac{3}{4}}. \end{aligned}$$

## 4. Numerical Results

In this section, a simple example is devised for verifying the validity of the proposed method. For the reader can make a comparison between this paper with [6] by using same example with same parameters, we consider the problem

$$\begin{cases} u_{xx} + u_{yy} + \frac{1}{4}u = 0, (x,y) \in (0,\pi)\times(0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = 0, (x,y) \in (0,\pi)\times(0,1) \\ u(x,0) = \sin(x), 0 < x < \pi \end{cases} \tag{19}$$

The exact solution to this problem is

$$u(x,y) = \frac{e^{\frac{\sqrt{3}}{2}y} + e^{-\frac{\sqrt{3}}{2}y}}{2}\sin x.$$

Let $y = 1$, we get $u(x,1) = 1.39903135064514\sin x$.
Let $g_m$ be the measured data

$$g_m(x) = \sin(x) + \frac{1}{m}\sin(mx).$$

So that the data error, at the $t = 0$ is

$$F(m) = \|g_m - g\| = \sqrt{\int_0^\pi \frac{1}{m^2}\sin^2(mx)dx} = \sqrt{\frac{\pi}{2}}\frac{1}{m} \le \epsilon.$$

The solution of (19), corresponding the $g_m$, is

$$u^m(x,t) = \frac{e^{\frac{\sqrt{3}}{2}y} + e^{-\frac{\sqrt{3}}{2}y}}{2}\sin x + \frac{e^{\sqrt{m^2-\frac{1}{4}}y} + e^{-\sqrt{m^2-\frac{1}{4}}y}}{2m}\sin mx$$

The error in $y = 1$ is

$$\begin{aligned} O(n) := \|u^m(.,1) - u(.,1)\| &= \sqrt{\int_0^\pi \frac{(e^{\sqrt{m^2-\frac{1}{4}}} + e^{-\sqrt{\sqrt{m^2-\frac{1}{4}}}})^2}{4m^2}\sin^2(mx)\,dx} \\ &= \frac{(e^{2\sqrt{m^2-\frac{1}{4}}} + e^{-2\sqrt{m^2-\frac{1}{4}}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Then, we notice that

$$\lim_{m\to\infty} F(m) = \lim_{m\to\infty}\frac{1}{m}\sqrt{\frac{\pi}{2}} = 0, \tag{20}$$

$$\lim_{m\to\infty} O(m) = \lim_{m\to\infty}\frac{(e^{2\sqrt{m^2-\frac{1}{4}}} + e^{-2\sqrt{m^2-\frac{1}{4}}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}} = \infty. \tag{21}$$

From the two equalities above, we see that (19) is an ill-posed problem. Hence, the Cauchy problem (19) cannot be solved by using classical numerical methods and it needs regularization techniques.
Let $\epsilon = \sqrt{\frac{\pi}{2}}\frac{1}{m}$. By approximating the problem as in (15), the regularized solution is

$$v^\epsilon(x,y) = \sum_{n=1}^\infty \left[\left(\frac{e^{\left(\sqrt{n^2-\frac{1}{4}}-\epsilon(n^2-\frac{1}{4})\right)y} + e^{-\sqrt{n^2-\frac{1}{4}}y}}{2}\right) < g_m(x), \sin nx > \right]\sin nx. \tag{22}$$

Table 1: The error of the method in this paper.

| $\epsilon$ | $v_\epsilon$ | $a_\epsilon = \|v_\epsilon(.,1) - u(.,1)\|$ |
|---|---|---|
| $\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$ | $1.38790989314992\sin(x)$ $+4,994531108 \times 10^{-14}\sin(100x)$ | 0.0139386799063127 |
| $\epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.39891961780226\sin(x)$ $+3.712424644 \times 10^{-1105}\sin(10^4 x)$ | 0.000140036351583956 |
| $\epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}$ | $1.39903135053340\sin(x)$ $+6.716243945 \times 10^{-1100129330}\sin(10^{10}x)$ | $1.40045321703634 \times 10^{-10}$ |

Let $y = 1$, the solution is written as

$$v^\epsilon(x,1) = \frac{e^{\frac{\sqrt{3}}{2}-\frac{3}{4}\epsilon} + e^{-\frac{\sqrt{3}}{2}}}{2}\sin x + \frac{e^{(\sqrt{m^2-\frac{1}{4}}-\epsilon(m^2-\frac{1}{4}))} + e^{-\sqrt{m^2-\frac{1}{4}}}}{2m}\sin mx.$$

The error in $y = 1$ is

$$\|v^\epsilon(.,1) - u(.,1)\| = \frac{\pi}{2}\left[\left(\frac{e^{\frac{\sqrt{3}}{2}-\frac{3}{4}\epsilon} - e^{\frac{\sqrt{3}}{2}}}{2}\right)^2 + \left(\frac{e^{(\sqrt{m^2-\frac{1}{4}}-\epsilon(m^2-\frac{1}{4}))} + e^{-\sqrt{m^2-\frac{1}{4}}}}{2m}\right)^2\right].$$

Table 1 shows the the error between the regularization solution $v^\epsilon$ and the exact solution $u$, for three values of $\epsilon$. We have the table numerical test by choose some values as follows
1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^2$.
2. $\epsilon = 10^{-4}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^4$.
3. $\epsilon = 10^{-10}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{10}$.

By applying the method in [6], we have the approximated solution

$$w^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{\sqrt{\frac{n^2-\frac{1}{4}}{1+\epsilon n^2}}y} + e^{-\sqrt{\frac{n^2-\frac{1}{4}}{1+\epsilon n^2}}y}}{2}\right) < g_m(x), \sin nx >\right]\sin nx. \quad (23)$$

Let $y = 1$, we have

$$\begin{aligned} w^\epsilon(x,1) &= \sum_{n=1}^{\infty}\left[\left(\frac{e^{\sqrt{\frac{n^2-\frac{1}{4}}{1+\epsilon n^2}}y} + e^{-\sqrt{\frac{n^2-\frac{1}{4}}{1+\epsilon n^2}}y}}{2}\right) < g_m(x), \sin nx >\right]\sin nx \\ &= \frac{e^{\sqrt{\frac{3}{4+4\epsilon}}} + e^{-\sqrt{\frac{3}{4+4\epsilon}}}}{2}\sin x + \frac{e^{\sqrt{\frac{m^2-\frac{1}{4}}{1+\epsilon m^2}}} + e^{-\sqrt{\frac{m^2-\frac{1}{4}}{1+\epsilon m^2}}}}{2m}\sin mx \end{aligned}$$

$$\|w^\epsilon(.,1) - u(.,1)\| =$$
$$\frac{\pi}{2}\left[\left(\frac{e^{\sqrt{\frac{3}{4+4\epsilon}}} + e^{-\sqrt{\frac{3}{4+4\epsilon}}}}{2} - \frac{e^{\frac{\sqrt{3}}{2}} + e^{-\frac{\sqrt{3}}{2}}}{2}\right)^2 + \left(\frac{e^{\sqrt{\frac{m^2-\frac{1}{4}}{1+\epsilon m^2}}} + e^{-\sqrt{\frac{m^2-\frac{1}{4}}{1+\epsilon m^2}}}}{2m}\right)^2\right].$$

We note that if we choose $\epsilon$ and $m$ such that $\epsilon = \sqrt{\frac{\pi}{2}}\frac{1}{m}$ then $\|w^\epsilon(.,1) - u(.,1)\|$ does not converges to zero. Thus, to compare the error of two method, we choose some same parameter values to get the table numerical test as follows
1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^4$.
2. $\epsilon = 10^{-3}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{15}$.
3. $\epsilon = 10^{-4}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.

Table 2: The error of the method in the paper [6]

| $\epsilon$ | $w^\epsilon$ | $\|w^\epsilon - u\|$ |
|---|---|---|
| $10^{-2}\sqrt{\frac{\pi}{2}}$ | $v_1 = 1.39379109494861\sin(x) + 0.3786841911\sin(10^4 x)$ | 0.474655690138409 |
| $10^{-3}\sqrt{\frac{\pi}{2}}$ | $1.39850107010763\sin(x) + 0.0009255956190\sin(10^{15}x)$ | 0.00133695472487849 |
| $10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.39850107010763\sin(x) + 9.255956190 \times 10^{-10}\sin(10^{20}x)$ | 0.000664608094405560 |

Looking at Tables 1,2,3 a comparison between the two methods, we can see the error results of in Table 3 are smaller than the errors in Tables 2. In the same parameter regularization, the error is Table 1 and 3 converges to zero more quickly many times than the Table 2 . This shows that our approach has a nice regularizing effect and give a better approximation with comparison to the many previous results, such as [5, 6, 14].

Table 3: The different error of the method in this paper.

| $\epsilon$ | $v_\epsilon$ | $a_\epsilon = \|v_\epsilon(.,1) - u(.,1)\|$ |
|---|---|---|
| $\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$ | $1.38790989314992\sin(x)$ $+5,667504490 \times 10^{-4348}\sin(10^4 \times x)$ | 0.0139386799063127 |
| $\epsilon_2 = 10^{-3}\sqrt{\frac{\pi}{2}}$ | $1.39891961780226\sin(x)$ $+7.548683905 \times 10^{-4342953}\sin(10^{15}x)$ | 0.000140036351583956 |
| $\epsilon_3 = 10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.39902017688822\sin(x)$ $+6.247671360 \times 10^{-434294492}\sin(10^{20}x)$ | 0.0000140042275147629 |

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Nguyen Huy Tuan
Department of Mathematics and Application
SaiGon University
273 An Duong vuong street, HoChiMinh city, VietNam
email:*tuanhuy bs@yahoo.com*

Pham Hoang Quan
Department of Mathematics and Application
SaiGon University
273 An Duong vuong street, HoChiMinh city, VietNam
email:*quan.ph@cb.sgu.edu.vn*