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SPECIAL SEMI MAGIC SQUARES

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Abstract. In this paper we introduce and study a special type of semi magic squares of order six and order seven. We list some enumerations of these squares.

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1.Introduction and definitions

A magic square is a square matrix, where the sum of all entries in each row or column and both main diagonals yields the same number. This number is called the magic constant. A natural semi magic square of order n is a matrix of size $n \times n$ such that its entries consists of all integers from one to n^2. The magic constant in this case is $\frac{n(n^2+1)}{2}$. By performing suitable permutation of the rows and columns of any natural semi magic square the number 1 can be transferred to the entry in the top row and left hand column. The other rows and columns can then be permuted so that the entries in the top row and left hand column are arranged in increasing order of magnitude. Also by an interchange of rows and columns (transpose of the square), if necessary, the element in the top row next to the element one may be made larger than the element in the left hand column which is next below the entry of the element one. Hence, we can transform any natural semi magic square of order 5 uniquely into a square of the structure

$$\begin{bmatrix} A & B & C & D & E \\ F & * & * & * & * \\ G & * & * & * & * \\ H & * & * & * & * \\ I & * & * & * & * \end{bmatrix}$$

where
1) $A = 1$
2) $B < C < D < E$
3) $F < G < H < I$
4) $B < F$

These squares are called fundamental squares. The combinations of all possible permutations of rows and columns together with the transpose operation are called the semi magic preserving transformations since they transfer a semi magic square into a semi magic square. There are semi magic preserving transformations for a semi magic square of order n. The application of the semi magic preserving transformations on the set of fundamental squares leads to the whole set of semi magic squares. Each selection of the variables A, B, C, D, E, F, G, H and I which satisfies 1) - 4) is called a semi magic 5×5 pattern. It was calculated that there are 67 semi magic 5×5 patterns. There are only 60 of these patterns which generate fundamental semi magic squares. The number of semi magic fundamental squares is 477 (see [1], [2]). Thus, we have the total number of

$$477*2*(4!)^2 = 549\ 504$$

semi magic squares of order 4.

In the case of squares of order 5 we have 12 043 semi magic 5×5 patterns. They generate 160 845 292 fundamental squares of order 5. Thus, we have the total number of

$$160\ 845\ 292*2*(5!)^2 = 4\ 632\ 344\ 409\ 600$$

semi magic squares of order 5. By using the computer we calculated the number of semi magic 6×6 patterns to be 4 531 580. But, the number of semi magic squares of order 6 is still unknown (see [4])

2.Four corner semi magic squares 6 by 6

We call squares of the following structure

$$\begin{bmatrix} x & f & D & t & c & y \\ z & j & n & 2s-n-q-j & q & s-z \\ A & c+f-m & e & a & m & J \\ r & B & E & 2s-r-t-x & 2s-m-q-k & K \\ d & 2s-k-q-j & d-n+z & H & k & s-d \\ g & s-f & F & I & s-c & h \end{bmatrix}$$

where
$A = 3s - g - r - d - x - z, B = k - f - c + m + q, D = 3s - f - c - t - x - y,$
$E = g - a + h + r - 2s + t + 2x + y - e, F = a + c - d + f - g - h - r + 2s - x - z,$
$H = j - d + n + q - z, I = d - a + r - s + x + z, J = d - c - a - f + g + r + x + z - e,$
$K = d - h + s - J + z - y$
four corner semi magic squares 6 by 6. It is a semi magic square with the property

$$a_{ij} + a_{(i+3)(j+3)} + a_{i(j+3)} + a_{(i+3)j} = 2s \quad \text{for } i, j = 1, 2, 3$$

Now, we see from the structure of the square that the 2 by 2 square in the center depends only on the variables, which appear in the outer frame. We have 128 transformations, which transform a four corner semi magic squares 6 by 6 into another one. These transformations are compositions of twelve basic transformations. The first basic transformation is the interchange of the second and fifth entry in the first and last row (res. column). The third basic transformation is the interchange of the first and last row of the 4 by 4 square in the center. In other wards, the third basic transformation will transform the previous square into

$$\begin{bmatrix} x & f & D & t & c & y \\ z & q & n & 2s-n-q-j & j & s-z \\ A & m & e & a & c+f-m & J \\ r & 2s-m-q-k & E & 2s-r-t-x & B & K \\ d & k & d-n+z & H & 2s-k-q-j & s-d \\ g & s-f & F & I & s-c & h \end{bmatrix}$$

We obtain a new transformation by doing the same to the columns. The fourth basic transformation is the interchange of the first and last column of the 4 by 4 square in the center. The remaining basic transformations are the eight classical transformations, i. e. rotation with angles 0, $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ (see [3]).

We will compute the number of some natural four corner semi magic squares 6 by 6. In order to avoid long computations we will take the mentioned transformations in consideration. This is done by requiring that

$$z < d, f < c, q < j, q < k, q < 2s - k - q - j, x < h, x < y < g \tag{1}$$

We compute two subclasses of the natural four corner semi magic squares 6 by 6. We count first the squares with symmetric corners. By this subclass we mean that the following holds

$$h = s - x \text{ and } g = s - y$$

In this case the conditions (1) will be translated into

$$z < d,\ f < c,\ q < j,\ q < k,\ q < 2s - k - q - j,\ 1 \leq x < y \leq 18$$

The number of the squares is shown in the following tables

x	number	x	number	x	number	x	number
1	9270363	5	8139574	9	5899081	13	3773111
2	9496185	6	7575885	10	5487324	14	3349655
3	8942201	7	7110499	11	4962236	15	2570921
4	8507031	8	6666887	12	4382607	16	1952371

x	number
17	1027950

We count now the squares with nonsymmetric corners having the sum condition. By this subclass we mean that the following holds:

$$h \neq s - x \text{ and } h = 2s - x - y - g$$

In this case the conditions in (1) will be translated into

$$z < d,\ f < c,\ q < j,\ q < k,\ q < 2s - k - q - j,\ 1 \leq x < y \leq 35$$

The number of the squares is shown in the following tables

x	y	number	x	y	number	x	y	number	x	y	number
1	2	689320	1	9	2672350	1	16	5187158	1	23	4875295
1	3	893072	1	10	3034504	1	17	5498590	1	24	5046289
1	4	1284806	1	11	3228562	1	18	5833992	1	25	5065163
1	5	1309503	1	12	3721939	1	19	6427179	1	26	4734037
1	6	1817844	1	13	3720272	1	20	5691260	1	27	4935683
1	7	1969001	1	14	4262810	1	21	5400578	1	28	4295235
1	8	2384537	1	15	4948572	1	22	5136157	1	29	4351436

x	y	number
1	30	4117285
1	31	4005243
1	32	3467736
1	33	3422164
1	34	2754516
1	35	2346744

x	y	number	x	y	number	x	y	number	x	y	number
2	3	1255078	2	11	3640376	2	19	5544102	2	27	4297204
2	4	1539175	2	12	3785503	2	20	5531712	2	28	4662613
2	5	1851348	2	13	4399749	2	21	5137833	2	29	4193050
2	6	1951188	2	14	4528398	2	22	4971331	2	30	4030667
2	7	2380448	2	15	4901113	2	23	4901113	2	31	3605371
2	8	2676803	2	16	5417504	2	24	4895951	2	32	3684136
2	9	2777327	2	17	5825393	2	25	4624475	2	33	3220581
2	10	3148062	2	18	5467439	2	26	5055652	2	34	2025674

x	y	number	x	y	number	x	y	number
3	4	1834126	3	12	4031128	3	20	5471698
3	5	1845699	3	13	4461283	3	21	5046012
3	6	2412423	3	14	4538529	3	22	5101447
3	7	2689605	3	15	5280160	3	23	4726552
3	8	2904183	3	16	5554491	3	24	4978293
3	9	3092335	3	17	5965462	3	25	5276324
3	10	3482359	3	18	5251635	3	26	4572916
3	11	3621871	3	19	5889214	3	27	4752882

x	y	number
3	28	4333774
3	29	4052045
3	30	3848473
3	31	3924792
3	32	2662671
3	33	1842872

x	y	number	x	y	number	x	y	number	x	y	number
4	5	2454898	4	12	4299296	4	19	5678468	4	26	4483742
4	6	2605388	4	13	4431829	4	20	5143758	4	27	4275457
4	7	2782885	4	14	4645774	4	21	5331958	4	28	4125759
4	8	2898902	4	15	5281452	4	22	4951082	4	29	4086632
4	9	3633540	4	16	5649967	4	23	4758193	4	30	3271415
4	10	3711760	4	17	5449356	4	24	5162866	4	31	2329705
4	11	3957548	4	18	5336846	4	25	4441780	4	32	1573908

x	y	number	x	y	number	x	y	number	x	y	number
5	6	2917555	5	12	4292476	5	18	5224052	5	24	4792739
5	7	3034341	5	13	4900051	5	19	5409285	5	25	5027415
5	8	3488750	5	14	5856236	5	20	5181590	5	26	4609222
5	9	3394348	5	15	5783581	5	21	4959222	5	27	4503980
5	10	3934380	5	16	6089170	5	22	4768440	5	28	3687639
5	11	4317849	5	17	5465145	5	23	4712734	5	29	3030049

x	y	number
5	30	2312262
5	31	1577404

x	y	number	x	y	number	x	y	number	x	y	number
6	7	3651144	6	12	4915554	6	17	5586624	6	22	5133979
6	8	3645902	6	13	5129501	6	18	5406795	6	23	5123514
6	9	3884506	6	14	5214634	6	19	5443433	6	24	5022216
6	10	4080883	6	15	5867843	6	20	4941422	6	25	4592014
6	11	4306889	6	16	5809333	6	21	4974910	6	26	4060252

x	y	number
6	27	3352601
6	28	2866843
6	29	2114359
6	30	1396250

x	y	number	x	y	number	x	y	number	x	y	number
7	8	4003153	7	13	5428614	7	18	5392066	7	23	5198359
7	9	4417629	7	14	5683774	7	19	5572207	7	24	4417643
7	10	4532736	7	15	5996915	7	20	5495201	7	25	3875163
7	11	4752311	7	16	5685810	7	21	5167266	7	26	3257497
7	12	5068488	7	17	5149791	7	22	5201767	7	27	2768986

x	y	number
7	28	2071598
7	29	1445944

x	y	number	x	y	number	x	y	number	x	y	number
8	9	4670164	8	14	5814304	8	19	5211793	8	24	3648421
8	10	4804558	8	15	5838977	8	20	5339196	8	25	3040498
8	11	5319588	8	16	5691800	8	21	4997833	8	26	2670054
8	12	5529494	8	17	5681260	8	22	4818679	8	27	2045097
8	13	5671944	8	18	5572776	8	23	4163053	8	28	1370962

x	y	number	x	y	number	x	y	number
9	10	5271921	9	15	5730875	9	20	5097563
9	11	5713944	9	16	5662201	9	21	4166984
9	12	5756000	9	17	5346378	9	22	4204087
9	13	5969094	9	18	5307857	9	23	3618566
9	14	6133690	9	19	5874254	9	24	3014193

x	y	number
9	25	2659566
9	26	1952437
9	27	1404296

x	y	number	x	y	number	x	y	number	x	y	number
10	11	6284762	10	15	5931617	10	19	4656964	10	23	2751248
10	12	6301709	10	16	5611391	10	20	4064597	10	24	2462527
10	13	6534298	10	17	5414834	10	21	3804323	10	25	1898046
10	14	6201895	10	18	5324441	10	22	3540679	10	26	1296320

x	y	number	x	y	number
11	12	6245175	11	19	4203981
11	13	6570197	11	20	3769697
11	14	6321894	11	21	3624499
11	15	5777130	11	22	2980986
11	16	5510311	11	23	2397337
11	17	5081419	11	24	1897123
11	18	4618688	11	25	1378746

x	y	number	x	y	number
12	13	7037475	12	19	3884563
12	14	5945958	12	20	2807139
12	15	5347475	12	21	2972146
12	16	4805283	12	22	2381369
12	17	4469172	12	23	1741584
12	18	3800258	12	24	1087254

x	y	number	x	y	number
13	14	5354949	13	19	2974650
13	15	4875471	13	20	2818028
13	16	4454915	13	21	2458501
13	17	3566262	13	22	1897214
13	18	3825576	13	23	1286182

x	y	number	x	y	number
14	15	4602952	14	19	2809005
14	16	3881998	14	20	2511380
14	17	4033770	14	21	1893634
14	18	2737733	14	22	1142784

x	y	number	x	y	number
15	16	3895092	15	19	1958926
15	17	2658580	15	20	2003359
15	18	2919429	15	21	1381548

x	y	number
16	17	2942170
16	18	1844601
16	19	1674510
16	20	1237474

x	y	number
17	18	2192170
17	19	1550296

We note that the squares with nonsymmetric corners having the sum condition together with the squares with symmetric corners form the subclass of squares having the sum condition, namely squares for which the following holds:

$$h = 2s - x - y - g$$

3.Concentric semi magic squares

We mean by this kind the following matrices

$$\begin{bmatrix} a & b & c & d & e & f & F \\ g & l & m & n & o & B & 25-g \\ h & p & t & u & 75-t-u & 25-p & 25-h \\ i & q & w & 25 & 50-w & 25-q & 25-i \\ j & r & 75-t-w & 50-u & A & 25-r & 25-j \\ k & D & 25-m & 25-n & 25-o & E & 25-k \\ G & 25-b & 25-c & 25-d & 25-e & 25-f & L \end{bmatrix}$$

where
$A = t+u+w-50, B = 125-l-m-n-o, D = 125-l-p-q-r$
$E = l+m+n+o+p+q+r-75, F = 175-a-b-c-d-e-f$
$G = 175-a-g-h-i-j-k, L = a+b+c+d+e+f+g+h+i+j+k-125$

We see that the values for the outer frame

$$a, b, c, d, e, f, g, h, i, j, k, F, G, L, 25-b, 25-c, 25-d, 25-e, 25-f, 25-g,$$
$$25-h, 25-i, 25-j, 25-k$$

can not be always odd. We can generate natural squares if we require that the corners a, F, G, L are even, while the other values for the outer frame are odd. We require that the multiples of five comes in the positions a, d, i, F, G, L. This leads us

to a subclass of the concentric semi magic squares. We call this class the M-class. It consists of all squares satisfying the following conditions:

$$a, F, G, L \in \{10, 20, 30, 40\} \quad , \quad d, i \in \{5, 15, 35, 45\}$$

$$b, c, d, e, f, g, h, i, j, k \text{ are odd}$$

There are several transformations which takes a square from the M-class into itself. We focus on the some transformations. They are three rotation operations, three reflection operations and two permutation operations. Each operation effects one frame of the squares. We can imagine that the square consist of three frames: the outer frame, the middle and the inside frame. For example, the inside frame consists of the entries

$$t, u, w, 75 - t - w, 75 - t - u, 50 - u, A$$

The three rotation operations are the rotations with angles $\frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ of each of the three frames independently. The three reflection operations corresponds to a reflection operation for each of the three frames. We reflect each frame about the main diagonal, which is similar to the operation of the transpose of a matrix. The permutation of the entries b, c, e, f together with the entries $25-b, 25-c, 25-e, 25-f$ does not change the property of belonging to the M-class. The same is true for the permutation of the entries g, h, j, k together with the entries $25 - g$,

$25-h, 25-j, 25-k$. These permutations change entries in the outer frame. There are similar permutations, which change entries in the middle frame. We permute the entries m, n, o together with the entries $25 - m, 25 - n, 25 - o$. Or, regarding the columns we permute the entries p, q, r together with the entries $25-p, 25-q, 25-r$.

In order to count the number of squares, which belongs to the M-class, we count the fundamental squares in the M-class, i. e. squares satisfying

$$a < F, a < G, a < L, \;\; l < B, l < D, l < E, \;\; t < 75 - t - u, t < 75 - t - w, t < A$$
$$b < g, \quad m < p, u < w$$
$$b < c < e < f, \;\; g < h < j < k, \;\; m < n < o, \;\; p < q < r$$

We generate then all squares in the M-class from the fundamental squares by means of the previous mentioned transformations. We found that there are 5760 possible values for the outer frame $(a, b, c, d, e, f, g, h, i, j, k)$. These values match into 148 650 fundamental squares in the M-class. The total number of squares in the M-class will be

$$148\ 650 \times 4^3 \times 2^3 \times 24^2 \times 6^2 = 1\ 578\ 192\ 076\ 800$$

different squares.

References

[1] S. Al-ashhab, The Magic and Semi Magic 4x4 Squares Problem, Dirasat, October 1998.

[2] J. Bellew, Counting the Number of Compound and Nasik Magic Squares, Mathematics Today, August 1997.

[3] B. Rosser and J. Walker, On the Transformation Group for Diabolic Magic Squares of Order Four, The American Mathematical Society Bulletin, 1938, vol. XLIV.

[4] W. W. Rouse Ball, Mathematical Recreations and Essays, 1911.

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