Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 87-98

DIFFERENTIAL SANDWICH THEOREMS FOR MULTIVALENT FUNCTIONS INVOLVING CERTAIN OPERATOR

M. K. Aouf, A. Shamandy, A. O. Mostafa and F. Z. El-Emam

Abstract. In this paper, we give some results for differential subordination and superordination for multivalent functions involving the integral operator I_p^α.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let $H = H(U)$ denotes the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ denotes the subclass of the functions $f \in H$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots \quad (a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, \ldots\}).$$

Also, let $A(p)$ be the subclass of the functions $f \in H$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}). \tag{1}$$

If f , $g \in H$ are analytic in U, we say that f is subordinate to g, or g is superordinate to f, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$ In such a case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true (cf., e.g.,[4] and [6]):

$$f(z) \prec g(z)\ (z \in U) \Leftrightarrow f(0) = g(0) \quad \textit{and} \quad f(U) \subset g(U).$$

Supposing that φ and h are two analytic functions in U, let

$$\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If φ and $\psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z)$ are univalent functions in U and if φ satisfies the second-order superordination

$$h(z) \prec \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z), \tag{2}$$

then h is called to be a solution of the differential superordination (2). A function $q \in H$ is called a subordinant of (2), if $q(z) \prec \varphi(z)$ for all the functions φ satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant.

Recently, Miller and Mocanu [7] obtained sufficient conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z); z) \Rightarrow q(z) \prec \varphi(z).$$

Using these results, the second author considered certain classes of first-order differential superordinations [3], as well as superordination-preserving integral operators [2]. Ali et al. [1], using the results from [3], obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. ([11], [12] and [13]) obtained the such called sandwich results for certain classes of analytic functions. Further subordination results can be found in [8], [9], [10], [14], [15] and [16].

Motivated essentially by Jung et al. [5], Shams et al. [10] introduced the operator $I_p^\alpha : A(p) \to A(p)$ as follows:

$$(i) I_p^\alpha f(z) = \frac{(p+1)^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt \qquad (\alpha > 0; p \in \mathbb{N}; z \in U)$$

and

$$(ii) I_p^0 f(z) = f(z), \quad (\alpha = 0; p \in \mathbb{N}).$$

Note that the one-parameter family of integral operator $I^\alpha \equiv I_1^\alpha$ was defined by Jung et al. [5].

For $f \in A(p)$ given by (1), it was shown that (see [10])

$$I_p^\alpha f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+1}{k+1}\right)^\alpha a_k z^k \quad (\alpha \geq 0; p \in \mathbb{N}). \tag{3}$$

Using (3), it is easily verified that (see [10])

$$z\left(I_p^\alpha f(z)\right)' = (p+1) I_p^{\alpha-1} f(z) - I_p^\alpha f(z) \quad (\alpha \geq 0). \tag{4}$$

2. Preliminaries

To prove our results we shall need the following definition and lemmas.

Definition 1 [7]. *Let Q be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1 [4]. *Let q be an univalent function in U and $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ such that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -Re\frac{1}{\gamma}\right\}.$$

If φ is analytic in U, with $\varphi(0) = q(0)$ and

$$\varphi(z) + \gamma z\varphi'(z) \prec q(z) + \gamma z q'(z), \tag{5}$$

then $\varphi(z) \prec q(z)$ and q is the best dominant of (5).

Lemma 2 [4]. *Let q be convex function in U, with $q(0) = a$ and $\gamma \in \mathbb{C}$ such that $Re\gamma > 0$. If $\varphi \in H[a, 1] \cap Q$ and $\varphi(z) + \gamma z\varphi'(z)$ is univalent in U, then*

$$q(z) + \gamma z q'(z) \prec \varphi(z) + \gamma z\varphi'(z) \Rightarrow q(z) \prec \varphi(z)$$

and q is the best subordinant.

In this paper we will determine some properties on admissible functions defined with the integral operator I_p^α.

3. Main Results

Theorem 1. *Let q be univalent function in U with $q(0) = 1$ such that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -Re\frac{1}{\gamma}\right\} \quad (\gamma \in \mathbb{C}^*). \tag{6}$$

If $f \in A(p)$ and

$$\frac{I_p^{\alpha+1} f(z)}{I_p^\alpha f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^\alpha f(z)]^2}\right\} \prec q(z) + \gamma z q'(z), \tag{7}$$

then

$$\frac{I_p^{\alpha+1} f(z)}{I_p^\alpha f(z)} \prec q(z) \quad (z \in U) \tag{8}$$

and q is the best dominant of (7).

Proof: Let

$$\varphi(z)=\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} \quad (z\in U). \tag{9}$$

Differentiating (9) logarithmically with respect to z and using the identity (4) in the resulting equation, we have

$$\frac{z\varphi^{'}(z)}{\varphi(z)}=(p+1)\left[\frac{I_p^{\alpha}f(z)}{I_p^{\alpha+1}f(z)}-\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}\right]=(p+1)\left[\frac{1}{\varphi(z)}-\frac{I_p^{\alpha-1}f(z)}{I_p^{\alpha}f(z)}\right].$$

It follows that

$$\varphi(z)+\gamma z\varphi^{'}(z)=\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)}+\gamma(p+1)\left\{1-\frac{I_p^{\alpha-1}f(z)I_p^{\alpha+1}f(z)}{[I_p^{\alpha}f(z)]^2}\right\}. \tag{10}$$

Hence the subordination (7) is equivalent to

$$\varphi(z)+\gamma z\varphi^{'}(z)\prec q(z)+\gamma zq^{'}(z).$$

Combining this last relation together with Lemma 1, we obtain our result.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 1, we have the following result.

Corollary 1. *Let $-1\le B<A\le 1$ and*

$$Re\left\{\frac{1-Bz}{1+Bz}\right\}>\max\left\{0,-Re\frac{1}{\gamma}\right\} \quad (\gamma\in\mathbb{C}^*;z\in U).$$

If $f\in A(p)$ and

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)}+\gamma(p+1)\left\{1-\frac{I_p^{\alpha-1}f(z)I_p^{\alpha+1}f(z)}{[I_p^{\alpha}f(z)]^2}\right\}\prec\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}, \tag{11}$$

then

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)}\prec\frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (11).

In particular, if we take $q(z)=\frac{1+z}{1-z}$ in Theorem 1, we have the following result.

Corollary 2. *Let*

$$Re\left\{\frac{1+z}{1-z}\right\} > \max\left\{0, -Re\frac{1}{\gamma}\right\} \quad (\gamma \in \mathbb{C}^*; z \in U).$$

If $f \in A(p)$ *and*

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1}f(z)I_p^{\alpha+1}f(z)}{[I_p^{\alpha}f(z)]^2}\right\} \prec \frac{1+z}{1-z} + \frac{2\gamma z}{(1-z)^2}, \tag{12}$$

then

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant of (12).*

Theorem 2. *Let* q *be a convex function in* U, *with* $q(0) = 1$ *and* $\gamma \in \mathbb{C}$ *such that* $Re\gamma > 0$. *If* $f \in A(p)$,

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} \in H[q(0), 1] \cap Q,$$

$$\frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1}f(z)I_p^{\alpha+1}f(z)}{[I_p^{\alpha}f(z)]^2}\right\}$$

is univalent in U *and*

$$q(z) + \gamma z q^{'}(z) \prec \frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1}f(z)I_p^{\alpha+1}f(z)}{[I_p^{\alpha}f(z)]^2}\right\}, \tag{13}$$

then

$$q(z) \prec \frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} \tag{14}$$

and q *is the best subordinant of (13).*

Proof: Let

$$\varphi(z) = \frac{I_p^{\alpha+1}f(z)}{I_p^{\alpha}f(z)} \quad (z \in U). \tag{15}$$

Differentiating (15) logarithmically with respect to z and using the identity (4) in the resulting equation, we have (10) holds. Hence the subordination (13) is equivalent to

$$q(z) + \gamma z q^{'}(z) \prec \varphi(z) + \gamma z \varphi^{'}(z).$$

Combining this last relation together with Lemma 2, we obtain our result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2, we have the following result.

Corollary 3. *Let $-1 \leq B < A \leq 1$ and $\gamma \in \mathbb{C}$ such that $Re\gamma > 0$. If $f \in A(p)$,*

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} \in H[q(0), 1] \cap Q,$$

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1) \left\{ 1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2} \right\}$$

is univalent in U and

$$\frac{1+Az}{1+Bz} + \gamma \frac{(A-B)z}{(1+Bz)^2}$$
$$\prec \quad \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1) \left\{ 1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2} \right\},$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

In particular, if we take $q(z) = \frac{1+z}{1-z}$ in Theorem 2, we have the following result.

Corollary 4. *Let $\gamma \in \mathbb{C}$ such that $Re\gamma > 0$. If $f \in A(p)$,*

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} \in H[q(0), 1] \cap Q,$$

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1) \left\{ 1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2} \right\}$$

is univalent in U and

$$\frac{1+z}{1-z} + \frac{2\gamma z}{(1-z)^2}$$
$$\prec \quad \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1) \left\{ 1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2} \right\},$$

then

$$\frac{1+z}{1-z} \prec \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)}$$

and $\frac{1+z}{1-z}$ *is the best subordinant.*

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.
Theorem 3. *Let* q_1 *be convex function with* $q_1(0) = 1$ *in* U *and* q_2 *be univalent function with* $q_2(0) = 1$ *in* U, $q_2(z)$ satisfies (6). *Let* $\gamma \in \mathbb{C}$ *such that* $Re\gamma > 0$. *If* $f \in A(p)$,

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} \in H[q(0), 1] \cap Q,$$

$$\frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2}\right\}$$

is univalent in U *and*

$$\begin{aligned} q_1(z) + \gamma z q_1^{'}(z) \quad &\prec \quad \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} + \gamma(p+1)\left\{1 - \frac{I_p^{\alpha-1} f(z) I_p^{\alpha+1} f(z)}{[I_p^{\alpha} f(z)]^2}\right\} \\ &\prec \quad q_2(z) + \gamma z q_2^{'}(z), \end{aligned} \tag{16}$$

then

$$q_1(z) \prec \frac{I_p^{\alpha+1} f(z)}{I_p^{\alpha} f(z)} \prec q_2(z),$$

and q_1 *and* q_2 *are the best subordinant and the best dominant respectively of (16).*
Theorem 4. *Let* q *be an univalent function in* U *with* $q(0) = 1$ *and (6) holds. If* $f \in A(p)$ *and*

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} + \gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2} - 2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3} \prec q(z) + \gamma z q^{'}(z), \tag{17}$$

then

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \prec q(z) \tag{18}$$

and q *is the best dominant of subordination (17).*
Proof: Let

$$\varphi(z) = \frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \quad (z \in U). \tag{19}$$

Differentiating (19) logarithmically with respect to z and using the identity (4) in the resulting equation, we have

$$\frac{z\varphi^{'}(z)}{\varphi(z)} = p + 1 + (p+1)\frac{I_p^{\alpha-1} f(z)}{I_p^{\alpha} f(z)} - 2(p+1)\frac{I_p^{\alpha} f(z)}{I_p^{\alpha+1} f(z)}.$$

It follows that

$$\varphi(z)+\gamma z\varphi^{'}(z) = (1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p[I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}. \tag{20}$$

Hence the subordination (17) is equivalent to

$$\varphi(z) + \gamma z\varphi^{'}(z) \prec q(z) + \gamma z q^{'}(z).$$

Combining this last relation together with Lemma 1, we obtain our result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 4, we have the following result.

Corollary 5. *Let* $-1 \le B < A \le 1$ *and*

$$Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -Re\frac{1}{\gamma}\right\} \quad (\gamma \in \mathbb{C}^*; z \in U).$$

If $f \in A(p)$ *and*

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p[I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$
$$\prec \frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}, \tag{21}$$

then

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (21).*

In particular, if we take $q(z) = \frac{1+z}{1-z}$ in Theorem 4, we have the following result.

Corollary 6. *Let*

$$Re\left\{\frac{1+z}{1-z}\right\} > \max\left\{0, -Re\frac{1}{\gamma}\right\} \quad (\gamma \in \mathbb{C}^*; z \in U).$$

If $f \in A(p)$ *and*

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p[I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$
$$\prec \frac{1+z}{1-z}+\frac{2\gamma z}{(1-z)^2}, \tag{22}$$

then

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ *is the best dominant of (22).*

Theorem 5. *Let q be a convex function in U, with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $Re\gamma > 0$. If $f \in A(p)$,*

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \in H[q(0), 1] \cap Q,$$

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} + \gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2} - 2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$

is univalent in U and

$$q(z) + \gamma z q^{'}(z) \prec (1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} + \gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2} - 2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}, \quad (23)$$

then

$$q(z) \prec \frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \quad (24)$$

and q is the best subordinant of superordination (23).

Proof: Let

$$\varphi(z) = \frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \quad (z \in U). \quad (25)$$

Differentiating (25) logarithmically with respect to z and using the identity (4) in the resulting equation, we have (20) holds. Hence the subordination (23) is equivalent to

$$q(z) + \gamma z q^{'}(z) \prec \varphi(z) + \gamma z \varphi^{'}(z).$$

Combining this last relation together with Lemma 2, we obtain our result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 5, we have the following result.

Corollary 7. *Let $-1 \le B < A \le 1$ and $\gamma \in \mathbb{C}$ such that $Re\gamma > 0$. If $f \in A(p)$,*

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \in H[q(0), 1] \cap Q,$$

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$

is univalent in U *and*

$$\frac{1+Az}{1+Bz}+\gamma\frac{(A-B)z}{(1+Bz)^2}$$
$$\prec \quad (1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3} \quad (26)$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant of superordination of (26).*

In particular, if we take $q(z)=\frac{1+z}{1-z}$ in Theorem 5, we have the following result.
Corollary 8. *Let* $\gamma\in\mathbb{C}$ *such that* $Re\gamma>0$. *If* $f\in A(p)$,

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}\in H[q(0),1]\cap Q,$$

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$

is univalent in U *and*

$$\frac{1+z}{1-z}+\frac{2\gamma z}{(1-z)^2}$$
$$\prec \quad (1+2\gamma)\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}, \quad (27)$$

then

$$\frac{1+z}{1-z}\prec\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}$$

and $\frac{1+z}{1-z}$ *is the best subordinant of superordination of (27)*

Combining Theorem 4 and Theorem 5, we get the following sandwich theorem.
Theorem 6. *Let* q_1 *be convex function with* $q_1(0)=1$ *in* U *and* q_2 *be univalent function with* $q_2(0)=1$ *in* U, $q_2(z)$ satisfies (6). *Let* $\gamma\in\mathbb{C}$ *such that* $Re\gamma>0$. *If* $f\in A(p)$,

$$\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}\in H[q(0),1]\cap Q,$$

$$(1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3}$$

is univalent in U and

$$q_1(z)+\gamma z q_1^{'}(z) \prec (1+\gamma(p+1))\frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2}+\gamma(p+1)\frac{z^p I_p^{\alpha-1} f(z)}{[I_p^{\alpha+1} f(z)]^2}$$

$$-2\gamma(p+1)\frac{z^p [I_p^{\alpha} f(z)]^2}{[I_p^{\alpha+1} f(z)]^3} \prec q_2(z)+\gamma z q_2^{'}(z), \tag{28}$$

then

$$q_1(z) \prec \frac{z^p I_p^{\alpha} f(z)}{[I_p^{\alpha+1} f(z)]^2} \prec q_2(z),$$

and q_1 and q_2 are the best subordinant and the best dominant respectively of (28).

References

[1] R.M. Ali, V. Ravichandran, M.H. Khan and K.G. Subramaniam, *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci., 15 (2004), no. 1, 87–94.

[2] T. Bulboacă, *A class of superordination preserving integral operators*, Indag-Math. (New Series), 13 (2002), no. 3, 301–311.

[3] T. Bulboacă, *Classes of first-order differential subordinations*, Demonstratio Math., 35 (2002), no. 2, 287–392.

[4] T. Bulboaca, *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[5] I.B. Jung, Y.C. Kim and H.M. Srivastava, *The Hardy space of analytic functions associated with certain one parameter families of integeral operators*, J. Math. Anal. Appl., 176 (1993), 138-147.

[6] S.S. Miller and P.T. Mocanu, *Differential Subordination : Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[7] S.S. Miller and P.T. Mocanu, *Subordinates of differential superordinations*, Complex Variables, 48 (2003), no. 10, 815-826.

[8] M. Obradović, M.K. Aouf and S. Owa, *On some results for starlike functions of complex order*, Publ. Inst. Math. Belgrade, 46 (60)(1989), 79–85.

[9] M. Obradović and S. Owa, *On certain properties for some classes of starlike functions*, J. Math. Anal. Appl., 145 (1990), no.2, 357–364.

[10] S. Shams, S.R. Kulkarni, and J.M. Jahangiri, *Subordination properties of p-valent functions defned by integral operators*, Internat. J. Math. Math. Sci., Vol. 2006, Article ID 94572, 1-3.
[11] T.N. Shanmugan, C. Ramachandran, M. Darus and S. Sivasubbramanian, *Differential sandwich theorems for some subclasses of analytic functions involving a linear operator*, Acta Math. Univ. Comenianae, 74 (2007), no.2, 287–294.
[12] T.N. Shanmugam, V. Ravichandran and S. Sivasubramanian, *Differantial sandwich theorems for some subclasses of analytic functions*, Austral. J. Math. Anal. Appl., 3 (2006), no. 1, Art. 8, 1-11.
[13] T.N. Shanmugam, S. Sivasubramanian and H. Silverman, *On sandwich theorems for some classes of analytic functions*, Internat. J. Math. Math. Sci., Vol. 2006, Article ID 29684, 1–13.
[14] V. Singh, *On some criteria for univalence and starlikeness*, Indian J. Pure. Appl. Math., 34 (2003), no.4, 569–577.
[15] H.M. Srivastava and A.Y. Lashin, *Some applications of the Briot-Bouquet differential subordination*, J. Ineq. Pure Appl. Math., 6 (2005), no.2, article 41, 1–7.
[16] Z. Wang, C. Gao and M. Liao, *On certain generalized class of non-Bazilević functions*, Acta Math. Acad. Paed. Nyireg. New Series, 21(2005), no.2, 147–154.

M. K. Aouf, A. Shamandy, A. O. Mostafa and F. Z. El-Emam
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt.
email:*mkaouf127@yahoo.com, shamandy16@hotmail.com*
adelaeg254@yahoo.com, fatma_elemam@yahoo.com